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SOME MEREOLOGICAL MODELS

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In this paper we show that the non-empty regular sets of any topological space form a Boolean algebra with zero deleted. In [1] it is shown that any Boolean algebra with zero deleted gives rise to a model of mereology which is "isomorphic" to it in the sense that

$$[AB] : A \leqq B \ .\equiv.\ \chi\{A\}\ \eta\ \mathbf{el}\langle \chi\{B\}\rangle,$$

where $\chi\{A\}$ and $\chi\{B\}$ correspond to A and B, and η is an analog of ontological ε. This paper will thus furnish us with a variety of mereological models. For example, Euclidean 3-space with the usual topology yields a model of atomless mereology.[1]

First we give some ontological preliminaries.

DO1 $[Aa] : A\ \varepsilon \sim(a)\ .\equiv.\ A\ \varepsilon A\ .\ \sim(A\ \varepsilon\ a)$

DO2 $[A\sigma] : A\ \varepsilon \bigcup\langle\sigma\rangle\ .\equiv.\ [\exists a]\ .\ A\ \varepsilon\ a\ .\ \sigma\{a\}.$

DO3 $[A\sigma] \therefore A\ \varepsilon \bigcap\langle\sigma\rangle\ .\equiv: A\ \varepsilon A : [a] : \sigma\{a\}\ .\supset.\ A\ \varepsilon\ a\ .$

We shall usually write $\bigcup\sigma$ instead of $\bigcup\langle\sigma\rangle$.

DO4 $[ab] \therefore a \subset b\ .\equiv: [A] : A\ \varepsilon\ a\ .\supset.\ A\ \varepsilon\ b$
DO5 $[ab] : a \circ b\ .\equiv.\ a \subset b\ .\ b \subset a$
DO6 $[ab] : \circ\{a\}\{b\}\ .\equiv.\ a \circ b$
DO7 $[A] : A\ \varepsilon \vee\ .\equiv.\ A\ \varepsilon A$
DO8 $[A] : A\ \varepsilon \wedge\ .\equiv.\ A\ \varepsilon A\ .\ \sim(A\ \varepsilon A)$
DO9 $[a] : !\{a\}\ .\equiv.\ [\exists A]\ .\ A\ \varepsilon\ a$

O1 $[\sigma a] : \sigma\{a\}\ .\supset.\ a \subset \bigcup\sigma$

O2 $[\sigma a] : \sigma\{a\}\ .\supset.\ \bigcap\sigma \subset a$

Now we introduce the notion of topological space into Leśniewski's

1. Concerning atomless mereology, *cf.* e.g., [2].

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logic. Let $\vee$ be the underlying name. Let $\boldsymbol{O}$ be the proposition-forming functor of one name argument which gives the open names. The following two axioms give a topological space.

P1 $[\sigma]:\sigma\subset\boldsymbol{O}.\supset.\boldsymbol{O}\{\bigcup\sigma\}$

P2 $[\sigma]:\sigma\subset\boldsymbol{O}.\,\text{Finite}\,\langle\sigma\rangle.\supset.\boldsymbol{O}\{\bigcap\sigma\}$

DP1 $[a]:\boldsymbol{C}\{a\}.\equiv.\boldsymbol{O}\{\sim(a)\}.$ ($\boldsymbol{C}$ gives the closed names)

DP2 $[Aa]:A\,\varepsilon\,a^{\circ}.\equiv.[\exists b].\boldsymbol{O}\{b\}.b\subset a.A\,\varepsilon\,b.$ (the interior of a)

DP3 $[Aa]\therefore A\,\varepsilon\,a^{-}.\equiv:A\,\varepsilon\,A:[b]:\boldsymbol{C}\{b\}.a\subset b.\supset.A\,\varepsilon\,b.$ (the closure of a)

DP4 $[Aa]:A\,\varepsilon\,\mathbf{ext}(a).\equiv.A\,\varepsilon\,(\sim(a))^{\circ}$ (the exterior of a)

We now state without proof some basic properties that follow from these definitions.

P3 $[a].a^{\circ}\subset a$

P4 $[a].a\subset a^{-}$

P5 $[ab]:a\subset b.\supset.a^{\circ}\subset b^{\circ}$

P6 $[ab]:a\subset b.\supset.a^{-}\subset b^{-}$

P7 $[a].\boldsymbol{C}\{a^{-}\}.$

P8 $[a].\boldsymbol{O}\{a^{\circ}\}.$

P9 $[a].a^{\circ\circ}\circ a^{\circ}$

P10 $[a].a^{--}\circ a^{-}$

P11 $[a].a^{\circ}\subset a^{-\circ}$

P12 $[a].a\cap\mathbf{ext}(a)\circ\Lambda$

P13 $[a]:\boldsymbol{O}\{a\}.\supset.a\subset a^{-\circ}$

P14 $[a].\sim(a^{-})\circ(\sim(a))^{\circ}$

P15 $[a].\sim(a^{\circ})\circ(\sim(a))^{-}$

P16 $[a]\therefore\boldsymbol{O}\{a\}.\supset:[b]:!\{a\cap b^{-}\}.\supset.!\{a\cap b\}$

DP5 $[a]:\boldsymbol{R}\{a\}.\equiv.a\circ a^{-\circ}$

P17 $\boldsymbol{R}\subset\boldsymbol{O}$ [*DP5*; *P8*]

P18 $[a].\boldsymbol{R}\{a^{-\circ}\}.$

PR $[a].$

1. $a^{-\circ}\subset a^{-\circ-}.$ [*P4*]
2. $a^{-\circ\circ}\subset a^{-\circ-\circ}.$ [*P5*; 1]
3. $a^{-\circ}\subset a^{-\circ-\circ}.$ [*P9*; 2]
4. $a^{-\circ}\subset a^{-}.$ [*P3*]
5. $a^{-\circ-}\subset a^{--}.$ [*P6*; 4]
6. $a^{-\circ-}\subset a^{-}.$ [*P10*; 5]
7. $a^{-\circ-\circ}\subset a^{-\circ}.$ [*P5*; 6]
8. $a^{-\circ}\circ a^{-\circ-\circ}.$ [3; 7]

$\boldsymbol{R}\{a^{-\circ}\}$ [*DP5*; 8]

P19 $[a]:\boldsymbol{R}\{a\}.\supset.\boldsymbol{R}\{\mathbf{ext}(a)\}.$

PR $[a]:\text{Hp}(1).\supset.$

2. $\boldsymbol{O}\{a\}.$ [*P17*; 1]
3. $\boldsymbol{C}\{\sim(a)\}.$ [*DP1*; 1; DO1]
4. $\sim(a)\circ(\sim(a))^{-}.$ [*P7*; 3; *DP3*]
5. $\mathbf{ext}(a)\circ(\sim(a))^{-\circ}.$ [*DP4*; 4; *DP2*]

$\boldsymbol{R}\{\mathbf{ext}(a)\}.$ [*P18*; 5]

$P20$ $[ab]:\boldsymbol{R}\{a\}.b\subset a.\supset.b^{-\circ}\subset a$

PR $[ab]:\mathrm{Hp}(2).\supset.$

3. $b^{-}\subset a^{-}$ [$P6$; 2]

4. $b^{-\circ}\subset a^{-\circ}.$ [$P5$; 3]

$b^{-\circ}\subset a$ [4; $DP5$; 1]

$P21$ $[ab]:\boldsymbol{R}\{a\}.\boldsymbol{R}\{b\}.\supset.\boldsymbol{R}\{a\cap b\}.$

PR $[ab]:\mathrm{Hp}(2).\supset.$

3. $\boldsymbol{O}\{a\cap b\}.$ [$P17$; 1; 2; $P2$]

4. $a\cap b\subset(a\cap b)^{-\circ}.$ [$P13$; 3]

5. $(a\cap b)^{-\circ}\subset a.$ [$P20$; 1]

6. $(a\cap b)^{-\circ}\subset b.$ [$P20$; 2]

7. $(a\cap b)^{-\circ}\subset a\cap b.$ [5; 6]

8. $a\cap b\circ(a\cap b)^{-\circ}.$ [4; 7]

$\boldsymbol{R}\{a\cap b\}$ [$DP5$; 8]

$P22$ $[a]:\boldsymbol{R}\{a\}.\supset.\mathcal{N}(a)\circ(\mathcal{N}(a))^{\circ-}.$

PR $[a]:\mathrm{Hp}(1).\supset.$

2. $\mathcal{N}(a)\circ\mathcal{N}(a^{-\circ}).$ [$DP5$; 1]

3. $\mathcal{N}(a)\circ(\mathcal{N}(a^{-}))^{-}.$ [$P15$; 2]

$\mathcal{N}(a)\circ(\mathcal{N}(a))^{\circ-}$ [$P14$; 3]

$P23$ $[ab]:\boldsymbol{R}\{a\}.\boldsymbol{R}\{b\}.b\subset a.\sim(b\circ a).\supset.[\exists c].\boldsymbol{R}\{c\}.!\{c\}.c\subset a.c\cap b\circ\wedge.$

PR $[ab]:\mathrm{Hp}(4).\supset.$

5. $!\{a\cap\mathcal{N}(b)\}.$ [3; 4]

6. $!\{a\cap(\mathcal{N}(b))^{\circ-}\}.$ [$P22$; 2; 5]

7. $\boldsymbol{O}\{a\}.$ [$P17$; 1]

8. $!\{a\cap(\mathcal{N}(b))^{\circ}\}.$ [$P16$; 7; 6]

9. $!\{a\cap\mathbf{ext}(b)\}.$ [$DP4$; 8]

10. $\boldsymbol{R}\{\mathbf{ext}(b)\}.$ [$P19$; 2]

11. $\boldsymbol{R}\{a\cap\mathbf{ext}(b)\}.$ [$P21$; 1; 11]

$[\exists c].\boldsymbol{R}\{c\}.!\{c\}.c\subset a.c\cap b\circ\wedge$ [11; 9; $P12$]

$P24$ $[ab]::\boldsymbol{R}\{a\}.\boldsymbol{R}\{b\}.b\subset a.\supset\therefore[c]\therefore\boldsymbol{R}\{c\}.!\{c\}.c\subset a.\supset.!\{c\cap b\}:\supset.a\circ b$ [$P23$]

$DP6$ $[a]:\boldsymbol{U}\{a\}.\equiv.!\{a\}.\boldsymbol{R}\{a\}.$

$P25$ $\boldsymbol{U}\subset\boldsymbol{R}$ [$DP6$]

$DP7$ $[ab]:a\leqq b.\equiv.\boldsymbol{U}\{a\}.\boldsymbol{U}\{b\}.a\subset b$

$P26$ $[\sigma d]:\sigma\subset\boldsymbol{U}.\sigma\{d\}.\supset.d\leqq\left(\bigcup\sigma\right)^{-\circ}$

PR $[\sigma d]:\mathrm{Hp}.(2).\supset.$

3. $\sigma\subset\boldsymbol{R}.$ [$P25$; 1]

4. $\sigma\subset\boldsymbol{O}.$ [$P17$; 3]

5. $\boldsymbol{O}\left\{\bigcup\sigma\right\}.$ [$P1$; 4]

6. $d\subset\bigcup\sigma.$ [$O1$; 2]

7. $d\subset\left(\bigcup\sigma\right)^{-\circ}.$ [6; $P13$; 5]

8. $\boldsymbol{U}\{d\}.$ [1; 2]

9. $!\{d\}.$ [$DP6$; 8]

10. $!\left\{\left(\bigcup\sigma\right)^{-\circ}\right\}.$ [9; 7]

11.	$\boldsymbol{U}\{(\bigcup\sigma)^{-\circ}\}$.	[*DP6*; 10; *P18*]
	$d \leqq (\bigcup\sigma)^{-\circ}$	[*DP7*; 8; 11; 7]
P27	$[\sigma d] : \sigma \subset \boldsymbol{U} . d \leqq (\bigcup\sigma)^{-\circ} . \supset . [\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e$.	
PR	$[\sigma d] :: \mathrm{Hp}(2) . \supset \therefore$	
3.	$\boldsymbol{U}\{d\}$.	[*DP7*; 2]
4.	$d \subset (\bigcup\sigma)^{-\circ}$.	
5.	$d \subset (\bigcup\sigma)^{-}$.	[*P3*; 4]
6.	$\boldsymbol{R}\{d\}$.	[*P25*; 3]
7.	$\boldsymbol{O}\{d\}$.	[*P17*; 6]
8.	$!\{d\}$.	[*DP6*; 3]
9.	$!\{d \cap (\bigcup\sigma)^{-}\}$.	[8; 5]
10.	$!\{d \cap \bigcup\sigma\}$:	[*P16*; 7; 9]
	$[\exists A]$:	
11.	$A \varepsilon d$.	[10]
12.	$A \varepsilon \bigcup\sigma$.	
	$[\exists e]$.	
13.	$\sigma\{e\}$.	[DO2; 12]
14.	$A \varepsilon e$.	
15.	$!\{d \cap e\}$.	[11; 14]
16.	$\boldsymbol{U}\{e\}$.	[1; 13]
17.	$\boldsymbol{R}\{e\}$.	[*P25*; 16]
18.	$\boldsymbol{R}\{d \cap e\}$.	[*P21*; 6; 17]
19.	$\boldsymbol{U}\{d \cap e\} \therefore$	[*DP6*; 15; 18]
	$[\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e$	[13; *DP7*; 19; 3; 16]
P28	$[ab] :: b \leqq a . \supset \therefore [c] \therefore c \leqq a . \supset . !\{c \cap b\} : \supset . a \circ b$	[*P24*; *DP7*; *DP6*]
P29	$[\sigma bc] :: \sigma \subset \boldsymbol{U} . \sigma\{b\} \therefore [d] : \sigma\{d\} . \supset . d \leqq c : [d] : d \leqq c . \supset . [\exists ef] . \sigma\{e\}$. $f \leqq d . f \leqq e \therefore \supset . c \circ (\bigcup\sigma)^{-\circ}$	
PR	$[\sigma bc] :: \mathrm{Hp}(4) \therefore \supset \therefore$	
5.	$[d] : \sigma\{d\} . \supset . d \subset c$:	[*DP7*; 3]
6.	$\bigcup\sigma \subset c$.	[DO2; 5]
7.	$b \leqq c$.	[2; 3]
8.	$\boldsymbol{R}\{c\}$.	[*DP7*; 7; *P25*]
9.	$(\bigcup\sigma)^{-\circ} \subset c$.	[*P20*; 8; 6]
10.	$\sigma \subset \boldsymbol{O}$.	[1; *P25*; *P17*]
11.	$\boldsymbol{O}\{\bigcup\sigma\}$:	[*P1*; 10]
12.	$[d] : d \leqq c . \supset . [\exists f] . \boldsymbol{U}\{f\} . f \subset d . f \subset \bigcup\sigma$:	[4; *DP7*; DO2]
13.	$[d] : d \leqq c . \supset . [\exists f] . !\{f\} . f \subset d . f \subset (\bigcup\sigma)^{-\circ}$:	[12; *DP6*; *P13*; 11]
14.	$[d] : d \leqq c . \supset . !\{d \cap (\bigcup\sigma)^{-\circ}\}$:	[13]
15.	$c \leqq c$.	[7; *DP7*]
16.	$!\{c \cap (\bigcup\sigma)^{-\circ}\}$.	[14; 15]

17. $!\{(\bigcup\sigma)^{-\circ}\}$. [16]

18. $\boldsymbol{U}\{(\bigcup\sigma)^{-\circ}\}$. [*DP6*; 17; *P18*]

19. $(\bigcup\sigma)^{-\circ} \leqq c$. [*DP7*; 17; 15; 19]

$c \circ (\bigcup\sigma)^{-\circ}$ [*P28*; 19; 14]

P30 $[\sigma b] :: \sigma \subset \boldsymbol{U} . \sigma\{b\} . \supset \therefore [c] \therefore c \circ (\bigcup\sigma)^{-\circ} . \equiv : [d] : \sigma\{d\} . \supset . d \leqq c :$
$[d] : d \leqq c . \supset . [\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e$. [*P26*; *P27*; *P29*]

P31 $[abc] : a \leqq b . b \leqq c . \supset . a \leqq c$ [*DP7*]

P32 $[\sigma\tau ab] :: a \leqq b . \sigma \subset \boldsymbol{U} . \sigma\{b\} \therefore [c] \therefore \tau\{c\} . \equiv : [d] : \sigma\{d\} . \supset . d \leqq c :$
$[d] : d \leqq c . \supset . [\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e \therefore \supset . \tau \circ \circ\{\!\!\{(\bigcup\sigma)^{-\circ}\}\!\!\} . a \leqq (\bigcup\sigma)^{-\circ}$

PR $[\sigma\tau ab] :: \mathrm{Hp}(4) \therefore \supset :$

5. $[c] : \tau\{c\} . \equiv . c \circ (\bigcup\sigma)^{-\circ} :$ [*P30*; 2; 3; 4]

6. $[c] : \tau\{c\} . \equiv \circ\{\!\!\{(\bigcup\sigma)^{-\circ}\}\!\!\}\{c\} :$ [DO6; 3]

7. $\tau \circ \circ\{\!\!\{(\bigcup\sigma)^{-\circ}\}\!\!\}$. [6]

8. $\tau\{(\bigcup\sigma)^{-\circ}\} :$ [7]

9. $[d] : \sigma\{d\} . \supset . d \leqq (\bigcup\sigma)^{-\circ} :$ [4; 8]

10. $b \leqq (\bigcup\sigma)^{-\circ}$. [9; 3]

11. $a \leqq (\bigcup\sigma)^{-\circ}$. [*P31*; 1; 10]

$\tau \circ \circ\{\!\!\{(\bigcup\sigma)^{-\circ}\}\!\!\} . a \leqq (\bigcup\sigma)^{-\circ}$ [7; 11]

P33 $[a] : \boldsymbol{U}\{a\} . \equiv . a \leqq a$ [*DP7*]

P34 $[ab] \because\!\because \boldsymbol{U}\{b\} :: b \leqq b . \supset \because\!\because [\sigma\tau] :\cdot: \sigma \subset \boldsymbol{U} . \tau \subset \boldsymbol{U} . \sigma\{b\} \therefore [c] \therefore \tau\{c\} . \equiv :$
$[d] : \sigma\{d\} . \supset . d \leqq c : [d] : d \leqq c . \supset . [\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e :: \supset .$
$[\exists g] . \tau \circ \circ\{\!\!\{g\}\!\!\} . a \leqq g \because\!\because \supset . a \leqq b$

PR $[ab] \because\!\because \mathrm{Hp}(2) \because\!\because \supset \therefore$

3. $b \leqq b$. [*P33*; 1]

4. $\circ\{\!\!\{b\}\!\!\}\{b\}$. [DO6]

5. $\circ\{\!\!\{b\}\!\!\} \subset \boldsymbol{U}$. [1; DO6]

6. $\bigcup\langle\circ\{\!\!\{b\}\!\!\}\rangle \circ b$. [DO2; DO6]

7. $\boldsymbol{R}\{b\}$. [*P25*; 1]

8. $(\bigcup\langle\circ\{\!\!\{b\}\!\!\}\rangle)^{-\circ} \circ b \therefore$ [*DP5*; 7; 6]

9. $[c] \therefore \circ\{\!\!\{b\}\!\!\}\{c\} . \equiv : [d] : \circ\{\!\!\{b\}\!\!\}\{d\} . \supset d \leqq c : [d] : d \leqq c . \supset .$
$[\exists ef] . \circ\{\!\!\{b\}\!\!\}\{e\} . f \leqq d . f \leqq e \therefore$ [*P30*; 5; 4; 8; DO6]
$[\exists g]$.

10. $\circ\{\!\!\{b\}\!\!\} \circ \circ\{\!\!\{g\}\!\!\}$.

11. $a \leqq g$. [2; 3; 5; 5; 4; 10]

12. $b \circ g$. [DO6]

$a \leqq b$ [11; 12]

P35 $[ab] \because\!\because a \leqq b . \equiv \because\!\because \boldsymbol{U}\{a\} . \boldsymbol{U}\{b\} :: b \leqq b . \supset \because\!\because [\sigma\tau] :\cdot: \sigma \subset \boldsymbol{U} . \tau \subset \boldsymbol{U} . \sigma\{b\} \therefore [c] \therefore$
$\tau\{c\} . \equiv : [d] : \sigma\{d\} . \supset . d \leqq c : [d] : d \leqq c . \supset . [\exists ef] . \sigma\{e\} . f \leqq d . f \leqq e :: \supset .$
$[\exists g] . \tau \circ \circ\{\!\!\{g\}\!\!\} . a \leqq g$ [*DP7*; *P32*; *P34*]

P35 is a replica of the single axiom for Boolean algebra with zero deleted which is given in [1].

REFERENCES

[1] Clay, R. E., "The relation of Leśniewski's mereology to Boolean algebra," to appear.

[2] Lejewski, C., "A contribution to the study of extended mereology," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 55-97.

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