UNIVERSITY OF CAPE COAST

STABILITY OF TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH MULTIPLE TIME-VARYING DELAYS

MICHAEL ASANTE KWARTENG

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BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Michael Asante Kwarteng

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

This thesis is concerned with the stability properties of solutions of nonlinear neutral differential equations with multiple time varying delays. Fixed point theory is used in this thesis to investigate the stability properties of solutions of nonlinear neutral differential equations with multiple time varying delays. In particular, the contraction mapping principle is used in this thesis. The nonlinear neutral differential equation is inverted to obtain an equivalent integral equation. The result of the inversion is used to define a suitable mapping which is then used to discuss the stability properties of solutions of nonlinear neutral differential equations with multiple time varying delays. Sufficient conditions that guarantee that the zero solutions of nonlinear neutral differential equations with multiple time varying delays are asymptotically stable are derived.

KEY WORDS

Asymptotic stability

Contraction mapping principle

Differential equation

Fixed point theory

Integral equation

Neutral differential equation

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Finally, this work wouldn't have been complete without my family. I therefore wish to appreciate my mother Grace and my siblings for their understanding and support. God bless you all.

DEDICATION

To my father, Kwadwo Asante Darko(deceased), my mother, Grace Kwartemaa and my sister, Abigail Agyeiwaa Asante.

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LIST OF ABBREVIATIONS

NDE	Neutral Differential Equation
ODE	Ordinary Differential Equations

CHAPTER ONE

INTRODUCTION

Background to the Study

Differential Equations have been a major branch of mathematics since their discovery around 17 century. The beginning of differential equation was attributed to Leibniz, the Bernouli brothers and others from 1680s. Even before that, in the early 1671, an English mathematician, Isaac Newton in his unpublished notes at that time brought out the following three kinds of differential equations;

$$\frac{dy}{dx} = f(x),$$

$$\frac{dy}{dx} = f(x, y),$$

and

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

Presently, the first and second equations are termed as ordinary differential equations because they have only ordinary derivatives of one or more dependent variables. The third equation is also having partial derivatives of dependent variables which is now called partial differential equations. Newton dubbed these equations as "Fluxions". Around 1675, the German mathematician Gottfreid Wilhelm Leibniz, also in an unpublished notes, came out with two different ideas. This include; his own differential and the very first recorded instance of the integral symbol

$$\int x dx = \frac{1}{2}x^2.$$

Both Lebniz and Newton finally published and handed out the solutions to their differential equations in 1963, marking 1963 as the beginning of differential equations as a specific field of mathematics.

During the 18th century, Euler and Lagrange made a very good impact on the theory of differential equations. Euler created the notion of function and he also initiated the form of writing function and arguments as $f(x)$ meaning that f is applied to argument x. Euler combined the works of Newton and Leibniz together to create tools such as the numerical approximation of integrals and Euler-muscheroni constant that to a great extent simplified the use of their calculus in physics. The approach to solving differential equations known as ”Variation of Parameters” which was basically invented by Euler was developed by Lagrange.

In recent years, mathematicians have gained interest in differential equations relying on past events. Introducing delays in mathematical models gives a good narration of concrete phenomena and also help us to make accurate guesses of their behaviour in the near future. Such delay models can also be termed as systems with after effect or time delay systems.

In the 19th century, Poincaré (1882) and Lyapunov (1893) laid down the basis of qualitative theory of differential equations. Poincaré (1882) made a comprehensive use of geometric methods in regard to the solutions of systems of differential equations as curves in appropriate space. On this ground he produced a general theory of the nature of solutions of second-order differential equations and solved some basic problems on the dependence of elementary solutions on parameters. Lyapunov (1893) on the other hand studied the behaviour of solutions in a neighborhood of an equilibrium position. He also founded the modern theory of stability of motion. Birkhoff (1927) in the $(1920s)$ used the geometric approach developed by Poincaré (1882) to invent many important information in the qualitative theory of higher-dimensional systems of differential equations.

The most effective and reliable tool for investigating the stability properties of ordinary, partial, functional differential and integro-differential equations is Lyapunov's direct method. The importance of this approach is that we can attain the stability without any previous idea of solutions. Burton (2003) pinpointed some challenges to encounter when using Lyapunov's direct method to solve problems in stability especially when the equation has unbounded terms or the delays are unbounded.

Since the Lyapunov's direct method has some difficulties when it is applied to stability problems, Mathematicians have diverted their attention to the fixed point techniques. This technique has solved some problems that the Lyapunov's direct Method did not.

Statement of the Problem

Recently, several mathematicians have obtained so many results on stability properties of nonlinear neutral differential equations.

In particular, Ardjourni, Derrardjia and Djoudi (2014) obtained sufficient conditions for which the zero solution of the nonlinear neutral differential equation

$$\frac{d}{dt} = -a(t)g\big(x(t-\tau(t))\big)+\frac{d}{dt}G\big(t,x(t-\tau(t))\big)$$

was asymptotically stable.

Ardjouni and Djoudi (2015) also studied the asymptotic stability of a nonlinear neutral differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -\sum_{j=1}^{N} b_j(t)x\left(t-\tau_j(t)\right)\\ &\quad + \frac{d}{dt}Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)\end{aligned}$$

$$+ G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

by applying fixed point theory.

Also, Akbulut and Tunç (2019) obtained sufficient conditions for which the zero solution of the neutral differential equation

$$\frac{d}{dt} = \sum_{i=1}^{2} -a(t)g_i\big(x(t-\tau_i(t))\big) + \frac{d}{dt}\sum_{i=1}^{2} G_i\big(t, x(t-\tau_i(t))\big)$$

is stable.

However, the stability results obtained by the above authors cannot be applied to the totally nonlinear neutral differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -\sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big) \\ &\quad + \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \\ &\quad + Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big).\end{aligned}$$

Purpose of the Study

The purpose of this study is to obtained sufficient conditions for the zero solution of a nonlinear neutral differential equation with multiple time varying delays is asymptotically stable.

Research Objectives

The objective of this thesis is to obtain sufficient conditions under which the zero solution of the totally nonlinear neutral differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -\sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)\\ &\quad + \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)\\ &\quad + Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \qquad (1.1)\end{aligned}$$

is asymptotically stable .

Significance of the Study

The study generalizes some results in stability of a neutral differential equation with multiple time varying delays and also add to literature which can be used by researchers in the area of stability of differential equations.

Delimitations

The study considers a totally nonlinear neutral differential equation with multiple time varying delays and also the study determines sufficient conditions for which the zero solution of the nonlinear neutral differential equation is asymptotically stable. The study does not take into consideration nonlinear differential equations with finite or variable delay and also all the terms on the right hand side of Equation (1.1) are nonlinear, so the results obtained in this study cannot be generalized for all neutral differential equations.

Limitations

The fixed point theory is the main tool used in this research because it is less complex in obtaining the stability solution of neutral differential equation with delays than the construction of Lyapunov Direct Method which is tedious and difficult to use.

Organization of the Study

Chapter One talks about the background to the study, statement of the problem, purpose of the study, objectives of the study, significance of the study and organization of the thesis. Chapter Two talks about the previous work related to the study. Chapter Three also talks about various theorems and methodology. Chapter four deals with results and discussion of the study. Chapter Five, which is the final part of the work talks about the summary and conclusions to the study.

CHAPTER TWO

LITERATURE REVIEW

Introduction

Many mathematicians have studied stability theory of differential equations. This chapter focuses on reviewing some related articles relevant to this work.

Relevant Literature

Many mathematicians have worked on and are still working on the stability properties of differential equations which include delays or without delays and this has resulted in the achievement of several results on stability of solutions of differential equations.

For example, Raffoul (2004) determined some sufficient conditions for which the zero solution of a scaler neutral differential equation

$$x^{/}(t) = -a(t)x(t) + c(t)x^{/}\big(t - g(t)\big) + q\big(x(t), x(t - g(t))\big),$$

with functional delays is stable.

Becker and Burton (2006) also investigated the scaler equation

$$x^{/}(t) = -\int_{t-\tau}^{t} a(t,s)g(x(s))ds \tag{2.1}$$

for $t \geq 0$, where $\tau : [0, \infty) \to [0, \infty), a : [0, \infty) \times [-\tau(0), \infty) \to R$, and $g : R \to R$ are continuous functions. They did not only obtain sufficient conditions for the existence and uniqueness of solutions of Equation (2.1) but also established some sufficient conditions for which the zero solution of Equation (2.1) is asymptotically stable.

Jin and Luo (2008) obtained sufficient conditions for which the zero solution of the neutral differential equation

$$x^{/}(t) \quad = \quad -a(t)x(t) - b(t)(x - \tau(t)) + c(t)x^{/}(t - \tau(t))$$

where $a, b, c \in (R^{+}, R^{+})$ with $t - \tau(t) \to \infty$ as $t \to \infty$ is asymptotically stable. The authors did not use Lyapunov direct method but they got interesting results even when the delay is unbounded.

Ardjourni, Djoudi and Soualhia (2012) also obtained sufficient conditions for which the zero solution of the linear neutral integro-differential equation

$$x^{/}(t) = -\sum_{i=1}^{N} \int_{t-\tau}^{t} a_i(t,s)x(s)ds + \sum_{i=1}^{N} C_i(t)x^{/}(t - \tau_i(t))$$

is asymptotically stable by fixed point technique.

The work by these authors improved and generalized the work of Becker and Burton (2006) and Jin and Luo (2008).

Ardjouni et al. (2014) further obtained sufficient conditions for the zero solution of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) \quad = \quad -a(t)g(x(t - \tau(t)) + \frac{d}{dt}G(t, x(t - \tau(t))$$

to be asymptotically stable.

Moreover, Ardjouni and Djoudi (2015) also studied the asymptotic stability of the generalized nonlinear neutral differential equation

$$\frac{d}{dt}x(t) \quad = \quad -\sum_{i=1}^{N} b_i(t)x(t - \tau_i(t)) + \frac{d}{dt}Q\big(t, x(t - \tau_1(t)), \ldots, x(t - \tau_N(t))\big)$$

$$+ G\big(t, x(t - \tau_1(t)), \ldots, x(t - \tau_N(t))\big).$$

Their results generalized the results of Ardjourni, Djoudi and Soualhia (2012).

The works of Ardjouni and Djoudi (2015) motivated Akbulut and Tunç (2019) and they obtained sufficient conditions for which the zero solution of the neutral differential equation of the first order

$$\frac{d}{dt}x(t) = -\sum_{i=1}^{2} a_i(t)g_i(x(t-\tau_i(t)) + \frac{d}{dt}\sum_{i=1}^{2} G_i\big(t, x(t-\tau_i(t))\big)$$

is stable.

Basic Concepts of Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving derivatives of an unknown function with one variable.

Examples are

$$\frac{dx}{dt} + x = e^t$$

and

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0.$$

In symbols, an nth order ordinary differential equation can be expressed in a general form as

$$F\Big(x, y, y^{/}.y^{//}, ..., y^{(n)}\Big) = 0. \tag{2.2}$$

The differential equation

$$\frac{d^n y}{dx^n} = f\Big(x, y, y^{/}.y^{//}, ..., y^{(n-1)}\Big)$$

is referred to as the normal form of Equation (2.2).

Delay differential equations

In mathematics, a differential equation in which the derivative of an unknown function at an unperturbed time is given in terms of values of the function at earlier times is called a Delay differential equation. Delay differential equations are also called time-delay systems, systems with after effect, hereditary systems, or equations with past arguments.

An example is the equation

$$x^{/}(t) = F\Big(t, x(t), x(t-\tau(t))\Big), \tag{2.3}$$

for $t \geq 0$ where $\tau > 0$ is called the delay with

$$x(t) = \psi_0(t),\ t_0 - \tau \leq t \leq t_0. \tag{2.4}$$

The function, $\psi_0 : R \to R^n$ is known and continuous and is called the initiatory history function, for t_0 is the initiatory constant and $[t_0 - \tau, t_0]$ the initiatory set. Delays might sometimes be a constant or a function.

Neutral delay differential equation

Delay equations containing delays of the derivatives are termed as Neutral delay differential equations. Neutral delay differential equations depend on past and present values of the function. An example is

$$x^{/}(t) = a(t)x(t) + x^{/}(t - \tau(t)).$$

Chapter Summary

In this chapter, review was done on relevant literature and some basic concepts of ordinary differential equations.

CHAPTER THREE

RESEARCH METHODS

Introduction

This chapter talks about the methods, definitions and some theorems that will be use to accomplish the objectives of this research.

Fixed Point Theory

The fixed point theory is one of the most important tools used to investigate nonlinear equations, such as algebraic equations and differential equations. One of the main backbone of the theory of metric spaces is the fixed point theory. A lot of mathematicians have investigated the Banach fixed point theory in different directions and have presented generalizations, extensions and applications of their findings.

Generally, the investigation of the stability of an equation using fixed point technique involves the construction of a suitable fixed point mapping.

This will be done by transforming Equation (1.1) to a more tractable, but equivalent, equation which will then be inverted to obtain an equivalent integral equation.

Next, a suitable complete metric space depending on the initial condition will be defined.

The Banach Theorem will then be applied to establish the sufficient conditions.

Definition 1 (Metric space)

Let S be a non-empty set, then a mapping $d : S \times S \to R$ is a metric if $\forall x, y, z \in S$, the following properties are satisfied:

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$, if and only if $x = y$

(iii) $d(x, y) = d(y, x)$ and

(iv) $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2 (Complete metric space)

Let (S, d) be a metric space. This metric space is complete if every Cauchy Sequence in it converges to an element of it. A sequence $\{X_p\}_{p\geq 1} \subset S$ is a Cauchy sequence if for every $\epsilon \geq 0$, there exist a positive integer N for all $p, q, \geq N$ imply $d(X_P, X_q) \leq \epsilon$.

Contraction mapping principle occurred clearly in Banach (1922) thesis where it was used to well-establish the presence and uniqueness of solutions of an integral equation. Due to this, the contraction mapping principle was named after Stephen Banach Caccioppoli. The theorem is now called Banach Contraction mapping or Banach Cacciopoli theorem. This theorem has gained more recognition recently. The effectiveness of the results lies in the application with intelligently selected metrics.

Theorem 1(Contraction Mapping Principle)

The contraction mapping principle states that if (X, d) is a complete metric space and $T : X \rightarrow X$ is a mapping such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y, \in X$, where $0 \leq k < 1$, then there exists a unique $x \in X$ such that $T(x) = x$.

Definition 3 (Stability)

The zero solution of an ordinary differential equation is stable if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x_0| \leq \delta$ implies that

$$|x(t, t_0, x_0)| \leq \epsilon$$

for $t \geq t_0$.

Definition 4 (Asymptotic stability)

The zero solution of an ordinary differential equation is asymptotically stable if it is stable and in addition to this there exists $r(t_0) \geq 0$ such that $|x_0| \leq r(t_0)$ implies that

$$\lim_{t \to \infty} |x(t, t_0, x_0)| = 0$$

for $t \geq t_0$.

Chapter Summary

This chapter talks about some theorems and definitions as well as the method used to obtain sufficient conditions for the zero solution to be asymptotically stable. Some definitions and theorems include; metric space, complete metric space, contraction mapping principle, stability and asymptotic stability.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter covers the results of stability properties of totally non-linear neutral differential equations with multiple time varying delays .

Results

Consider the totally nonlinear neutral differential equation

$$\begin{aligned} \frac{d}{dt}x(t) &= -\sum_{i=1}^{N} a_i(t)g_i\big(x\,(t-\tau_i(t))\,\big) \\ &\quad + \frac{d}{dt}G\big(t, x\,(t-\tau_1(t))\,, x\,(t-\tau_2(t))\,, ..., x\,(t-\tau_N(t))\,\big) \\ &\quad + Q\big(t, x\,(t-\tau_1(t))\,, x\,(t-\tau_2(t))\,, ..., x\,(t-\tau_N(t))\,\big) \qquad (4.1) \end{aligned}$$

with an initial condition

$$x(t) = \psi(t)\ on\ t \in C\big([m(0), 0], R\big) \qquad (4.2)$$

where

$$m_i(0) = \Big\{\inf\{\ t - \tau_i(t), t \geq 0\}\Big\}\ and\ m(0) = min\Big\{m_i(0), 1 \leq i \leq N\Big\}.$$

In this thesis the following assumptions are made. Assume that the functions $Q\,(t, x_1, ..., x_N)$ and $G\,(t, x_1, ..., x_N)$ are globally lipschitz continuous in $x_1, ..., x_N$, that is, there exists positive constants $M_1, ..., M_N$ and $E_1, ..., E_N$

such that

$$\Big|Q(t,x_1,...,x_N)-Q(t,y_1,...,y_N)\Big|\leq\sum_{i=1}^{N}M_i\Big\|x_i-y_i\Big\| \tag{4.3}$$

and

$$\Big|G(t,x_1,...,x_N)-G(t,y_1,...,y_N)\Big|\leq\sum_{i=1}^{N}E_i\Big\|x_i-y_i\Big\|. \tag{4.4}$$

Also assume that

$$Q\big(t,0,0,...,0\big)=G\big(t,0,0,...,0\big)=0.,\ and\ g_i(0)=0, i=1,...,N. \tag{4.5}$$

Let

$$E=\max\{E_1,E_2,...,E_N\}$$

and

$$M=\max\{M_1,M_2,...,M_N\}.$$

Since there is no linear term in the NDE (4.1), it makes it difficult to obtain a fixed point mapping for NDE (4.1). So, to make NDE (4.1) more tractable, there is a need to transform it.

Lemma 4.1

If $H:[m(0),\infty)\to$ R is an arbitrary continuous function then Equation (4.1) is equivalent to the equation

$$\frac{d}{dt}\Big[x(t)-G\big(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right),...,x\left(t-\tau_N(t)\right)\big)\Big]$$

$$= -H(t)\Big[x(t) - G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right)$$

$$, ..., x\left(t-\tau_N(t)\right)\big)\Big]$$

$$+ \sum_{i=1}^{N} \frac{d}{dt} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds - \sum_{i=1}^{N} a_i(t) g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\left(1-\tau_i^{/}(t)\right) g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$- H(t) G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+ H(t)\left[x(t) - \sum_{i=1}^{N} g_i\left(x(t)\right)\right]. \qquad (4.6)$$

Proof.

From Equation (4.6),

$$\frac{d}{dt}x(t) = \frac{d}{dt} G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$- H(t)\Big[x(t) - G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right)$$

$$, ..., x\left(t-\tau_N(t)\right)\big)\Big]$$

$$+\sum_{i=1}^{N}\frac{d}{dt}\int_{t-\tau_i(t)}^{t}H(s)g_i\left(x(s)\right)ds-\sum_{i=1}^{N}a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+\sum_{i=1}^{N}H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\,g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+Q\big(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right),...,x\left(t-\tau_N(t)\right)\big)$$

$$-H(t)G\big(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right),...,x\left(t-\tau_N(t)\right)\big)$$

$$+H(t)\left[x(t)-\sum_{i=1}^{N}g_i\left(x(t)\right)\right]. \tag{4.7}$$

Differentiating the integral term in Equation (4.7) gives

$$\sum_{i=1}^{N}\frac{d}{dt}\int_{t-\tau_i(t)}^{t}H(s)g_i\left(x(s)\right)ds$$

$$=\sum_{i=1}^{N}\left[H(t)g_i\left(x(t)\right)\frac{dt}{dt}-H\big(t-\tau_i(t)\big)g_i\big(x\left(t-\tau_i(t)\right)\big)\frac{d}{dt}\left(t-\tau_i(t)\right)\right]$$

$$+\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}\frac{\partial}{\partial t}H(s)g_i\left(x(s)\right)ds$$

$$=\sum_{i=1}^{N}\left[H(t)g_i\left(x(t)\right)-H\left(t-\tau_i(t)\right)g_i\big(x\left(t-\tau_i(t)\right)\big)\Big(1-\tau_i^{/}(t)\Big)\right]+0.$$

$$=\sum_{i=1}^{N}\left[H(t)g_i\left(x(t)\right)-H\left(t-\tau_i(t)\right)g_i\left(x\left(t-\tau_i(t)\right)\right)\Big(1-\tau_i^{/}(t)\Big)\right].$$

$$=\sum_{i=1}^{N}H(t)g_i\left(x(t)\right)-\sum_{i=1}^{N}H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)$$

$$\times\, g_i\big(x\left(t-\tau_i(t)\right)\big). \tag{4.8}$$

Substituting Equation (4.8) into Equation (4.7) gives

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \\
&- H(t)\Big[x(t) - G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right) \\
&\quad , ..., x\left(t-\tau_N(t)\right)\big)\Big] \\
&+ \sum_{i=1}^{N} H(t)g_i\left(x(t)\right) - \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big) \\
&\times g_i\big(x\left(t-\tau_i(t)\right)\big) \\
&- \sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big) + \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big) \\
&\times g_i\big(x\left(t-\tau_i(t)\right)\big) \\
&+ Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \\
&- H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \\
&+ H(t)\left[x(t) - \sum_{i=1}^{N} g_i\left(x(t)\right)\right]. \\
&= \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big) \\
&+ H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)
\end{aligned}
$$

$$+\sum_{i=1}^{N} H(t)g_i\left(x(t)\right)-\sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)$$

$$\times\, g_i\big(x\left(t-\tau_i(t)\right)\big)-\sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+\sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\, g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+\,Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$-\,H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+\,H(t)x(t)-\ h(t)\sum_{i=1}^{N} g_i\left(x(t)\right)-H(t)x(t).$$

$$=\ \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+\,H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$-\,H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+\sum_{i=1}^{N} H(t)g_i\left(x(t)\right)-H(t)\sum_{i=1}^{N} g_i\left(x(t)\right)+H(t)x(t)$$

$$-\sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\, g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$-\sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)-H(t)x(t)$$

$$+\sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\, g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big).$$

$$= \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+ H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$- H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+ \sum_{i=1}^{N} H(t)g_i\left(x(t)\right) - \sum_{i=1}^{N} H(t)g_i\left(x(t)\right)$$

$$- \sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right) - H(t)x(t) + H(t)x(t)\big)$$

$$- \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\left(1-\tau_i^{/}(t)\right) g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\left(1-\tau_i^{/}(t)\right) g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big).$$

$$= - \sum_{i=1}^{N} a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ \frac{d}{dt}G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big)$$

$$+ Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\big).$$

This completes the proof.

In Lemma 4.2, there is going to be a derivation of an equivalent integral equation to Equation (4.1).

Lemma 4.2

Suppose that the hypothesis of Lemma 4.1 hold. Then $x(t)$ is a solution of equation (4.1) if

$$
\begin{aligned}
x(t) \;=\; & \Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right) \\
& - \sum_{i=1}^{N} \int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds\Big] \exp\left(-\int_0^t H(v)dv\right) \\
& + \Big[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\Big] \\
& + \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds \\
& - \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) du ds \\
& + \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big] \\
& \times g_i\big(x\left(s-\tau_i(s)\right)\big) \exp\left(-\int_s^t H(v)dv\right) ds \\
& + \int_0^t \Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big) \\
& - H(s) G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)\Big]
\end{aligned}
$$

$$\times \ \exp\left(-\int_s^t H(v)dv\right) ds$$

$$+ \ \int_0^t H(s)\left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right] \exp\left(-\int_s^t H(v)dv\right) ds. \quad (4.9)$$

Proof.

Multiplying the terms on both sides of the Equation (4.6) by $\exp\left(\int_0^t H(v)dv\right)$, gives

$$\frac{d}{dt}\left[\left(x(t) - G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\right)\right.$$

$$\left.\times \exp\left(\int_0^t H(v)dv\right)\right]$$

$$= \ -H(t)\left[x(t) - G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\right]$$

$$\times \exp\left(\int_0^t H(v)dv\right)$$

$$+ \ \exp\left(\int_0^t H(v)dv\right) \sum_{i=1}^{N} \frac{d}{dt} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds$$

$$- \ \exp\left(\int_0^t H(v)dv\right) \sum_{i=1}^{N} a_i(t) g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ \ \exp\left(\int_0^t H(v)dv\right) \sum_{i=1}^{N} H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\, g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+ \ \exp\left(\int_0^t H(v)dv\right) Q\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right)\big)$$

$$,...,x\left(t-\tau_N(t)\right)\big)$$

$$-\ \exp\left(\int_0^t H(v)dv\right)H(t)G\big(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right)$$

$$,...,x\left(t-\tau_N(t)\right)\big)$$

$$+\ \exp\left(\int_0^t H(v)dv\right)H(t)\left[x(t)-\sum_{i=1}^{N}g_i\left(x(t)\right)\right].$$

$$\frac{d}{dt}\Bigg[\Big(x(t)-G\left(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right),...,x\left(t-\tau_N(t)\right)\right)\Big)$$

$$\times\exp\left(\int_0^t H(v)dv\right)\Bigg]$$

$$+H(t)\Big[x(t)-G\left(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right),...,x\left(t-\tau_N(t)\right)\right)\Big]$$

$$\times\exp\left(\int_0^t H(v)dv\right)$$

$$=\ \exp\left(\int_0^t H(v)dv\right)\sum_{i=1}^{N}\frac{d}{dt}\int_{t-\tau_i(t)}^{t}H(s)g_i\left(x(s)\right)ds$$

$$-\ \exp\left(\int_0^t H(v)dv\right)\sum_{i=1}^{N}a_i(t)g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+\ \exp\left(\int_0^t H(v)dv\right)\sum_{i=1}^{N}H\big(t-\tau_i(t)\big)\Big(1-\tau_i^{/}(t)\Big)\,g_i\big(x\left(t-\tau_i(t)\right)\big)$$

$$+\ \exp\left(\int_0^t H(v)dv\right)Q\big(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right)$$

$$, ..., x\left(t-\tau_N(t)\right)\Big)$$

$$- \exp\left(\int_0^t H(v)dv\right) H(t)G\big(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right)$$

$$, ..., x\left(t-\tau_N(t)\right)\Big)$$

$$+ \exp\left(\int_0^t H(v)dv\right) H(t)\left[x(t) - \sum_{i=1}^{N} g_i\left(x(t)\right)\right]. \tag{4.10}$$

Integrating Equation (4.10) from 0 to t gives

$$\int_0^t \frac{d}{ds}\bigg[\Big(x(s) - G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\big)\Big) \exp\left(\int_0^s H(v)dv\right)\bigg] ds$$

$$= \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right) duds$$

$$- \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} a_i(s)g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} H\big(s-\tau_i(s)\big)\big(1-\tau_i^{/}(s)\big)$$

$$\times\ g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right)\Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\Big)\Big]ds$$

$$-\int_0^t \exp\left(\int_0^s H(v)dv\right) H(s)G\Big[\Big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\Big)\Big]$$

$$+\int_0^t \exp\left(\int_0^s H(v)dv\right) H(s)\left[x(s)-\sum_{i=1}^{N} g_i\left(x(s)\right)\right] ds.$$

$$\Big[x(s)-G\Big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\Big)\Big]$$

$$\times \exp\left(\int_0^s H(v)dv\right)\Big|_0^t$$

$$= \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right) du ds$$

$$-\int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} a_i(s)g_i\Big(x\left(s-\tau_i(s)\right)\Big)ds$$

$$+\int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} H\Big(s-\tau_i(s)\Big)\Big(1-\tau_i^{/}(s)\Big)$$

$$\times\ g_i\Big(x\left(s-\tau_i(s)\right)\Big)ds$$

$$+\int_0^t Q\Big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\Big)ds$$

$$\times\ \exp\left(\int_0^s H(v)dv\right)$$

$$- \int_0^t H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)ds$$

$$\times \exp\left(\int_0^s H(v)dv\right)$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) H(s)\left[x(s) - \sum_{i=1}^{N} g_i\big(x(s)\big)\right] ds.$$

$$\Big[x(t) - G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\Big]$$

$$\times \exp\left(\int_0^t H(v)dv\right)$$

$$-\Big[x(0) - G\left(0, x\left(0-\tau_1(0)\right), x\left(0-\tau_2(0)\right), ..., x\left(0-\tau_N(0)\right)\right)\Big]$$

$$\times \exp\left(\int_0^0 H(v)dv\right)$$

$$= \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right) duds$$

$$- \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} a_i(s)g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)$$

$$\times \; g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)ds$$

$$\times \exp\left(\int_0^s H(v)dv\right)$$

$$- \int_0^t H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\big)\exp\left(\int_0^s H(v)dv\right) ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) H(s)\left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right] ds.$$

$$\left[x(t) - G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\right]$$

$$\times \exp\left(\int_0^t H(v)dv\right)$$

$$= \left[\psi(0) - G\left(0, \psi\left(0-\tau_1(0)\right), \psi\left(0-\tau_2(0)\right), ..., \psi\left(0-\tau_N(0)\right)\right)\right]$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right) duds$$

$$- \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} a_i(s)g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} H\big(s-\tau_i(s)\big)\left(1-\tau_i^{/}(s)\right)$$

$$\times g_i\big(x\left(s-\tau_i(s)\right)\big)ds$$

$$+ \int_0^t \exp\left(\int_0^s H(v)dv\right) Q\Big[\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\big)\Big] ds$$

$$- \int_0^t \exp\left(\int_0^s H(v)dv\right) H(s)G\Big[(s, x(s-\tau_1(s)), x(s-\tau_2(s))$$

$$, ..., x(s-\tau_N(s)))\Big]ds$$

$$+ \int_0^t \exp\left(\int_0^s H(s)dv\right) H(s)\left[x(s) - \sum_{i=1}^{N} g_i\big(x(s)\big)\right] ds. \qquad (4.11)$$

Dividing through Equation (4.11) by $\exp\left(\int_0^t H(v)dv\right)$ gives

$$\Big[x(t) - G(t, x(t-\tau_1(t)), x(t-\tau_2(t)), ..., x(t-\tau_N(t)))\Big]$$

$$= \Big[\psi(0) - G(0, \psi(0-\tau_1(0)), \psi(0-\tau_2(0)), ..., \psi(0-\tau_N(0)))\Big]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \left[\int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds}\int_{s-\tau_i(s)}^{s} H(u)g_i(x(u))\,duds\right.$$

$$\left.\times \exp\left(-\int_0^t H(v)dv\right)\right]$$

$$- \left[\int_0^t \exp\left(\int_0^s H(v)dv\right) \sum_{i=1}^{N} a_i(s)g_i\big(x(s-\tau_i(s))\big)\right.$$

$$\left.\times \exp\left(-\int_0^t H(v)dv\right)\right]ds$$

$$+ \int_0^t \left[\sum_{i=1}^{N} H\big(s-\tau_i(s)\big)\left(1-\tau_i^{\prime}(s)\right) g_i(x(s-\tau_i(s)))\right]$$

$$\times \exp\left(\int_0^s H(v)dv\right) \exp\left(-\int_0^t H(v)dv\right) ds$$

$$+ \int_0^t \Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)$$

$$\times \exp\left(\int_0^s H(v)dv\right) \exp\left(-\int_0^t H(v)dv\right)\Big] ds$$

$$- \int_0^t \Big[H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)$$

$$\times \exp\left(\int_0^s H(v)dv\right) \exp\left(-\int_0^t H(v)dv\right)\Big] ds$$

$$+ \int_0^t \exp\left(\int_0^s H(s)dv\right) H(s) \left[x(s) - \sum_{i=1}^N g_i\left(x(s)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right) ds.$$

$$= \Big[\psi(0) - G\big(0, \psi\left(0-\tau_1(0)\right), \psi\left(0-\tau_2(0)\right), ..., \psi\left(0-\tau_N(0)\right)\big)\Big]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \int_0^t \left[\sum_{i=1}^N \frac{d}{ds} \int_{s-\tau_i(s)}^s H(u) g_i\left(x(u)\right) du ds\right]$$

$$\times \left[\exp\left(\int_0^s H(v)dv\right) \exp\left(-\int_0^t H(v)dv\right)\right]$$

$$- \int_0^t \left[\sum_{i=1}^N a_i(s) g_i\big(x\left(s-\tau_i(s)\right)\big) ds\right]$$

$$\times \left[\exp\left(\int_0^s H(v)dv\right)\exp\left(-\int_0^t H(v)dv\right)\right]$$

$$+ \int_0^t \left[\sum_{i=1}^{N} H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\, g_i\big(x\left(s-\tau_i(s)\right)\big)ds\right]$$

$$\times \left[\exp\left(\int_0^s H(v)dv\right)\exp\left(-\int_0^t H(v)dv\right)\right]$$

$$+ \int_0^t \left[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)ds\right]$$

$$\times \left[\exp\left(\int_0^s H(v)dv\right)\exp\left(-\int_0^t H(v)dv\right)\right]$$

$$- \int_0^t \left[H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)\right]$$

$$\times \left[\exp\left(\int_0^s H(v)dv\right)\exp\left(-\int_0^t H(v)dv\right)\right] ds$$

$$+ \int_0^t H(s)\left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right]$$

$$\times \left[\exp\left(\int_0^s H(s)dv\right)\exp\left(-\int_0^t H(v)dv\right)\right] ds.$$

$$= \left[\psi(0) - G\left(0, \psi\left(0-\tau_1(0)\right), \psi\left(0-\tau_2(0)\right), ..., \psi\left(0-\tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \int_0^t \left[\sum_{i=1}^{N} \frac{d}{ds}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right)duds\right]$$

$$\times \left[\exp\left(-\int_0^t H(v)dv + \int_0^s H(v)dv\right)\right]$$

$$- \int_0^t \left[\sum_{i=1}^{N} a_i(s) g_i\left(x\left(s - \tau_i(s)\right)\right) ds\right]$$

$$\times \left[\exp\left(-\int_0^t H(v)dv + \int_0^s H(v)dv\right)\right]$$

$$+ \int_0^t \left[\sum_{i=1}^{N} H\left(s - \tau_i(s)\right)\left(1 - \tau_i^{/}(s)\right) g_i\left(x\left(s - \tau_i(s)\right)\right) ds\right]$$

$$\times \left[\exp\left(-\int_0^t H(v)dv + \int_0^s H(v)dv\right)\right]$$

$$+ \int_0^t \left[Q\left(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right), ..., x\left(s - \tau_N(s)\right)\right)\right]$$

$$\times \left[\exp\left(-\int_0^t H(v)dv + \int_0^s H(v)dv\right)\right] ds$$

$$- \int_0^t \left[H(s) G\left(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right), ..., x\left(s - \tau_N(s)\right)\right)\right]$$

$$\times \left[\exp\left(-\int_0^t H(v)dv + \int_0^s H(v)dv\right)\right] ds$$

$$+ \int_0^t H(s)\left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right]$$

$$\times \left[\exp\left(-\int_0^t H(s)dv + \int_0^s H(v)dv\right)\right] ds.$$

$$= \left[\psi(0) - G\left(0, \psi\left(0 - \tau_1(0)\right), \psi\left(0 - \tau_2(0)\right), ..., \psi\left(0 - \tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \int_0^t \left[\sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) du ds \right]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right)$$

$$- \int_0^t \left[\sum_{i=1}^{N} a_i(s) g_i\big(x\left(s - \tau_i(s)\right)\big) ds \right]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right)$$

$$+ \int_0^t \left[\sum_{i=1}^{N} H\left(s - \tau_i(s)\right) \left(1 - \tau_i^{/}(s)\right) g_i\left(x\left(s - \tau_i(s)\right)\right) \right]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right) ds$$

$$+ \int_0^t \Big[Q\big(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right), ..., x\left(s - \tau_N(s)\right)\big) \Big]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right) ds$$

$$- \int_0^t \Big[H(s) G\big(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right), ..., x\left(s - \tau_N(s)\right)\big) \Big]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right) ds$$

$$+ \int_0^t H(s) \left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right) \right]$$

$$\times \exp\left(- \left[\int_0^t H(v)dv + \int_s^0 H(v)dv \right] \right) ds.$$

$$= \left[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \int_0^t \left[\sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) du ds\right]$$

$$\times \ \exp\left(-\int_s^t H(v)dv\right)$$

$$- \int_0^t \left[\sum_{i=1}^{N} a_i(s) g_i\Big(x\left(s - \tau_i(s)\right)\Big) ds\right]$$

$$\times \exp\left(-\int_s^t H(v)dv\right)$$

$$+ \left[\int_0^t \sum_{i=1}^{N} H\big(s - \tau_i(s)\big)\left(1 - \tau_i^{/}(s)\right) g_i\Big(x\left(s - \tau_i(s)\right)\Big)\right]$$

$$\times \exp\left(-\int_s^t H(v)dv\right) ds$$

$$+ \int_0^t \Big[Q\Big(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right), ..., x\left(s - \tau_N(s)\right)\Big)\Big]$$

$$\times \ \exp\left(-\int_s^t H(v)dv\right) ds$$

$$- \int_0^t \Big[H(s) G\Big(s, x\left(s - \tau_1(s)\right), x\left(s - \tau_2(s)\right)$$

$$, ..., x\left(s - \tau_N(s)\right)\Big)\Big] \exp\left(-\int_s^t H(v)dv\right) ds$$

$$+\int_0^t H(s)\left[x(s)-\sum_{i=1}^{N} g_i\left(x(s)\right)\right]\exp\left(-\int_s^t H(v)dv\right)ds.$$

$$= \left[\psi(0)-G\left(0,\psi\left(-\tau_1(0)\right),\psi\left(-\tau_2(0)\right),...,\psi\left(-\tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+\int_0^t \exp\left(-\int_s^t H(v)dv\right)\sum_{i=1}^{N}\frac{d}{ds}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right)duds$$

$$+\int_0^t \sum_{i=1}^{N}\Big[-a_i(s)+H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big]$$

$$\times\ g_i\Big(x\left(s-\tau_i(s)\right)\Big)\exp\left(-\int_s^t H(v)dv\right)ds$$

$$+\int_0^t \Big[Q\big(s,x\left(s-\tau_1(s)\right),x\left(s-\tau_2(s)\right),...,x\left(s-\tau_N(s)\right)\big)$$

$$-H(s)G\big(s,x\left(s-\tau_1(s)\right),x\left(s-\tau_2(s)\right),...,x\left(s-\tau_N(s)\right)\big)\Big]$$

$$\times \exp\left(-\int_s^t H(v)dv\right)$$

$$+\int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\left[x(s)-\sum_{i=1}^{N} g_i\left(x(s)\right)\right]ds.$$

Thus,

$$x(t) = \left[\psi(0)-G\left(0,\psi\left(-\tau_1(0)\right),\psi\left(-\tau_2(0)\right),...,\psi\left(-\tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$
\begin{aligned}
&+ \Big[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\Big] \\
&+ \int_0^t \exp\left(-\int_s^t H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds \\
&+ \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big] \\
&\times\ g_i\Big(x\left(s-\tau_i(s)\right)\Big) \exp\left(-\int_s^t H(v)dv\right) ds \\
&+ \int_0^t \Big[Q\left(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\right) \\
&- H(s) G\left(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\right)\Big] \\
&\times\ \exp\left(-\int_s^t H(v)dv\right) \\
&+ \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \\
&\times\ \left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right] ds. \qquad (4.12)
\end{aligned}
$$

Integrating the third term on the right hand side of Equation (4.12) gives

$$
\begin{aligned}
&\int_0^t \exp\left(-\int_s^t H(v)dv\right) \sum_{i=1}^{N} \frac{d}{ds} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds \\
&= \exp\left(-\int_s^t H(v)dv\right) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds\Big|_0^t
\end{aligned}
$$

$$- \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds.$$

$$= \exp\left(-\int_t^t H(v)dv\right) \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds$$

$$- \exp\left(-\int_0^t H(v)dv\right) \sum_{i=1}^{N} \int_{0-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds$$

$$- \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds.$$

$$= \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds$$

$$- \exp\left(-\int_0^t H(v)dv\right) \sum_{i=1}^{N} \int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds$$

$$- \int_0^t H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds$$

$$\times \exp\left(-\int_s^t H(v)dv\right). \tag{4.13}$$

Substituting Equation (4.13) into Equation (4.12) gives

$$x(t) = \left[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right)\right]$$

$$\times \exp\left(-\int_0^t H(v)dv\right)$$

$$+ \left[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\right]$$

$$+ \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds$$

$$- \exp\left(-\int_0^t H(v)dv\right)\sum_{i=1}^{N}\int_{-\tau_i(0)}^{0} H(s)g_i\left(\psi(s)\right)ds$$

$$- \int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right)duds$$

$$+ \int_0^t \sum_{i=1}^{N}\Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big]$$

$$\times\, g_i\big(x\left(s-\tau_i(s)\right)\big)\exp\left(-\int_s^t H(v)dv\right)ds$$

$$+ \int_0^t \Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)$$

$$- H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right)$$

$$, ..., x\left(s-\tau_N(s)\right)\big)\Big]\exp\left(-\int_s^t H(v)dv\right)ds$$

$$+ \int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right]ds.$$

$$= \Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right)$$

$$-\sum_{i=1}^{N}\int_{-\tau_i(0)}^{0} H(s)g_i\left(\psi(s)\right)ds\Big]\exp\left(-\int_0^t H(v)dv\right)$$

$$+ \Big[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\Big]$$

$$+ \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t} H(s)g_i\left(x(s)\right)ds$$

$$
\begin{aligned}
&- \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds \\
&+ \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big] \\
&\times g_i\big(x\left(s-\tau_i(s)\right)\big) \exp\left(-\int_s^t H(v)dv\right) ds \\
&+ \int_0^t \Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big) \\
&- H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big) \Big] \\
&\times \exp\left(-\int_s^t H(v)dv\right) ds \\
&+ \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \left[x(s) - \sum_{i=1}^{N} g_i\left(x(s)\right)\right] ds. \\
= & \Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right) \\
&- \sum_{i=1}^{N} \int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds \Big] \exp\left(-\int_0^t H(v)dv\right) \\
&+ \Big[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right) \Big] \\
&+ \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(x(s)\right) ds \\
&- \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(x(u)\right) duds
\end{aligned}
$$

$$+\int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\Big]$$

$$\times\, g_i\big(x\left(s-\tau_i(s)\right)\big)\exp\left(-\int_s^t H(v)dv\right)ds$$

$$+\int_0^t \Big[Q\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)$$

$$-\,H(s)G\big(s, x\left(s-\tau_1(s)\right), x\left(s-\tau_2(s)\right), ..., x\left(s-\tau_N(s)\right)\big)\Big]$$

$$\times\, \exp\left(-\int_s^t H(v)dv\right)ds$$

$$+\int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\left[x(s)-\sum_{i=1}^{N} g_i\left(x(s)\right)\right]ds.$$

$$=\Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right)$$

$$-\sum_{i=1}^{N}\int_{-\tau_i(0)}^{0} H(s)g_i\left(\psi(s)\right)ds\Big]\exp\left(-\int_0^t H(v)dv\right)$$

$$+\Big[G\left(t, x\left(t-\tau_1(t)\right), x\left(t-\tau_2(t)\right), ..., x\left(t-\tau_N(t)\right)\right)\Big]$$

$$+\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t} H(s)g_i\left(x(s)\right)ds$$

$$-\int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)g_i\left(x(u)\right)duds$$

$$+\int_0^t \sum_{i=1}^{N}\Big[-a_i(s) + H\big(s-\tau_i(s)\big)\left(1-\tau_i^{/}(s)\right)\Big]$$

$$\times\ g_i\big(x\left(s-\tau_i(s)\right)\big)\exp\left(-\int_s^t H(v)dv\right)ds$$

$$+\int_0^t\Big[Q\big(s,x\left(s-\tau_1(s)\right),x\left(s-\tau_2(s)\right),...,x\left(s-\tau_N(s)\right)\big)$$

$$-H(s)G\big(s,x\left(s-\tau_1(s)\right),x\left(s-\tau_2(s)\right),...,x\left(s-\tau_N(s)\right)\big)\Big]$$

$$\times\ \exp\left(-\int_s^t H(v)dv\right)ds$$

$$+\int_0^t H(s)\left[x(s)-\sum_{i=1}^{N}g_i\left(x(s)\right)\right]\exp\left(-\int_s^t H(v)dv\right)ds.$$

This complete the proof.

Define

$$S_\psi=\Big\{\phi:[m(0),\infty)\to R,\ \phi(t)=\psi(t)\ for\ t\in[m(0),0],$$

$$and\ for\ t\geq 0,\ |\phi(t)|\leq\epsilon\Big\}$$

where

$$|\phi|_h=sup\left\{|\phi(t)|\exp\left(-\sum_{i=1}^{N}h_i(t)\right)\right\}$$

for $t\in[m(0),\infty)$. Then $(S_\psi,|.|)$ is a Banach space.

Let

$$h_i(t)=kl\sum_{i=1}^{N}\int_0^t\left[H(v)+w_i(v)\right]dv,$$

and

$$w_i(v) = L\Big|a_i(v) + H\left(v - \tau_i(v)\right)\left(1 - \tau_i^{/}(v)\right)\Big|, \; v \in [0, \infty) \qquad (4.14)$$

where $i = 1, 2, ..., N$.

Let $H : [m(0), \infty) \to R^+$ be a continuous function and define a mapping if $P : S_\psi \to S_\psi$ by

$$(P\phi)(t) = \psi(t), \; if \; t \in [m(0), 0]$$

and for $t > 0$,

$$\begin{aligned}
(P\phi)(t) \quad = \quad & \Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right) \\
& - \sum_{i=1}^{N} \int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds\Big] \exp\left(-\int_0^t H(v)dv\right) \\
& + \Big[G\left(t, \phi\left(t - \tau_1(t)\right), \phi\left(t - \tau_2(t)\right), ..., \phi\left(t - \tau_N(t)\right)\right)\Big] \\
& + \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(\phi(s)\right) ds \\
& - \int_0^t H(s) \sum_{i=1}^{N} \Big(\int_{s-\tau_i(s)}^{s} H(u) g_i\left(\phi(u)\right) du\Big) ds \\
& \times \exp\left(-\int_s^t H(v)dv\right) \\
& + \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\Big(s - \tau_i(s)\Big)\Big(1 - \tau_i^{/}(s)\Big)\Big] \\
& \times g_i\Big(\phi\left(s - \tau_i(s)\right)\Big) \exp\left(-\int_s^t H(v)dv\right) ds
\end{aligned}$$

$$+ \int_{s}^{t} \Big[Q\big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big)$$

$$- H(s)G, \big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big) \Big]$$

$$\times \exp\left(-\int_{s}^{t} H(v)dv\right) ds$$

$$+ \int_{0}^{t} \exp\left(-\int_{s}^{t} H(v)dv\right) H(s)$$

$$\times \left[\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\right] ds. \tag{4.15}$$

In Lemma 4.3 the conditions for the mapping P to be a contraction are given.

Lemma 4.3

Suppose that conditions (4.3) and (4.4) hold and there exists a constant $l > 0$ such that g_i is lipstchitz on $[-l, l]$. If L is a Lipstchitz constant for both $g_i(x)$ and $x - g_i(x)$ on $[-l, l]$ and $H(t) \geq 0$ for $t \geq m(0)$.

Assume further that

$$L\left[\frac{N}{k} + \left(\frac{4N}{kl} + \frac{1}{kl+1}\right)\right] \leq \alpha < 1. \tag{4.16}$$

Then $P : S_{\psi} \rightarrow S_{\psi}$ is a contraction and also continuous.

Proof.

Let $\phi, \eta, \in S_{\psi}$; Then

$$\left|P\phi(t) - P\eta(t)\right|_{h}$$

$$\leq \Big| G\left(t,\phi\left(t-\tau_1(t)\right),\phi\left(t-\tau_2(t)\right),...,\phi\left(t-\tau_N(t)\right)\right)$$

$$- G\left(t,\eta\left(t-\tau_1(t)\right),\eta\left(t-\tau_2(t)\right),...,\eta\left(t-\tau_N(t)\right)\right)\Big|$$

$$\times \ \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(t-\tau_i(t)\right) - \sum_{i=1}^{N} h_i\left(t-\tau_i(t)\right)\right)$$

$$+ \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t} H(S)\big|g_i(\phi(s)) - g_i(\eta(s))\big|$$

$$\times \ \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s) - \sum_{i=1}^{N} h_i(s)\right) ds$$

$$+ \int_0^t \exp\left(\int_s^t H(v)dv\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)\big|g_i(\phi(u)) - g_i(\eta(u))\big|$$

$$\times \ \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(u) - \sum_{i=1}^{N} h_i(u)\right) duds$$

$$+ \int_0^t \exp\left(-\int_s^t H(v)dv\right)\sum_{i=1}^{N}\left[a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\right]$$

$$\times \ \big|g_i\left(\phi\left(s-\tau_i(s)\right)\right) - g_i\left(\eta\left(s-\tau_i(s)\right)\right)\big|$$

$$\times \ \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(s-\tau_i(s)\right) - \sum_{i=1}^{N} h_i\left(s-\tau_i(s)\right)\right) ds$$

$$+ \int_0^t \Big[\Big| Q\left(s,\phi\left(s-\tau_1(s)\right),\phi\left(s-\tau_2(s)\right),...,\phi\left(s-\tau_N(s)\right)\right)$$

$$- Q\big(s,\eta\left(s-\tau_1(t)\right),\eta\left(s-\tau_2(s)\right),...,\eta\left(s-\tau_N(s)\right)\big)\Big|$$

$$+ H(s)\Big|G\big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big)$$

$$- G\big(s, \eta\left(s-\tau_1(t)\right), \eta\left(s-\tau_2(s)\right), ..., \eta\left(s-\tau_N(s)\right)\big)\Big|\Bigg] ds$$

$$\times \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(s-\tau_i(s)\right) - \sum_{i=1}^{N} h_i\left(s-\tau_i(s)\right)\right)$$

$$\times \exp\left(-\int_s^t H(v)dv\right)$$

$$+ \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s)\Big|\phi(s) - \sum_{i=1}^{N} g_i\left(\eta(s)\right)\Big|$$

$$\times \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s) - \sum_{i=1}^{N} h_i(s)\right) ds. \tag{4.17}$$

By applying lipschitz condition in Equation (4.17) it follows that

$$\big|P\phi(t) - P\eta(t)\big|_h$$

$$\leq \sum_{i=1}^{N} E_i\Big|\phi(t-\tau_i(t)) - \eta(t-\tau_i(t))\Big|$$

$$\times \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(t-\tau_i(t)\right) - \sum_{i=1}^{N} h_i\left(t-\tau_i(t)\right)\right)$$

$$+ \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(S)L\big|\phi(s) - \eta(s)\big|$$

$$
\begin{aligned}
&\times \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s) - \sum_{i=1}^{N} h_i(s)\right) ds \\
&+ \int_0^t \exp\left(\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u)L\big|\phi(u) - \eta(u)\big| \\
&\times \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(u) - \sum_{i=1}^{N} h_i(u)\right) duds \\
&+ \int_0^t \sum_{i=1}^{N} L\left[a_i(s) + H\left(s - \tau_i(s)\right)\left(1 - \tau_i^{/}(s)\right)\right] \\
&\times \left|\phi\left(s - \tau_i(s)\right) - \eta\left(s - \tau_i(s)\right)\right| \exp\left(-\int_s^t H(v)dv\right) \\
&\times \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(s - \tau_i(s)\right) - \sum_{i=1}^{N} h_i\left(s - \tau_i(s)\right)\right) ds \\
&+ \int_0^t \exp\left(-\int_s^t H(v)dv\right)\left[\sum_{i=1}^{N} M_i + H(s)\sum_{i=1}^{N} E_i\right] \\
&\times \sum_{i=1}^{N}\Big|\phi(s - \tau_i(s)) - \eta(s - \tau_i(s))\Big| \\
&\times \exp\left(\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(s - \tau_i(s)\right) - \sum_{i=1}^{N} h_i\left(s - \tau_i(s)\right)\right) ds \\
&+ \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s)L\Big|\phi(s) - \eta(s)\Big| \\
&\times \exp\left(-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s) - \sum_{i=1}^{N} h_i(s)\right) ds. \qquad (4.18)
\end{aligned}
$$

The terms on the right side in Equation (4.18) will be denoted by I_n, $n = 1, ..., 6$.

The expression

$$-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(t - \tau_i(t)\right)$$

in equation (4.18) is Simplified as follows.

$$\begin{aligned}
& -\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i\left(t - \tau_i(t)\right) \\
= \quad & -kl\sum_{i=1}^{N}\int_0^t \left[H(v) + w_i(v)\right] dv + kl\sum_{i=1}^{N}\int_0^{t-\tau_i(t)} \left[H(v) + w_i(v)\right] dv \\
= \quad & -kl\sum_{i=1}^{N}\int_0^t \left[H(v) + w_i(v)\right] dv - kl\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{0} \left[H(v) + w_i(v)\right] dv \\
= \quad & -kl\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t} \left[H(v) + w_i(v)\right] dv.
\end{aligned}$$

Also,

$$\begin{aligned}
& -\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s) \\
= \quad & -kl\sum_{i=1}^{N}\int_0^t \left[H(v) + w_i(v)\right] dv + kl\sum_{i=1}^{N}\int_0^{s} \left[H(v) + w_i(v)\right] dv \\
= \quad & -kl\sum_{i=1}^{N}\int_0^t \left[H(v) + w_i(v)\right] dv - kl\sum_{i=1}^{N}\int_{s}^{0} \left[H(v) + w_i(v)\right] dv \\
= \quad & -kl\sum_{i=1}^{N}\int_{s}^{t} \left[H(v) + w_i(v)\right] dv \\
\leq \quad & -kl\int_s^t H(v) dv.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(u) \\
&= -kl\sum_{i=1}^{N}\int_0^t [H(v)+w_i(v)]\,dv + kl\sum_{i=1}^{N}\int_0^u [H(v)+w_i(v)]\,dv \\
&= -kl\sum_{i=1}^{N}\int_0^t [H(v)+w_i(v)]\,dv - kl\sum_{i=1}^{N}\int_u^0 [H(v)+w_i(v)]\,dv \\
&= -kl\sum_{i=1}^{N}\int_u^t [H(v)+w_i(v)]\,dv \\
&\leq -kl\int_u^t H(v)dv
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&-\sum_{i=1}^{N} h_i(t) + \sum_{i=1}^{N} h_i(s-\tau_i(s)) \\
&= -kl\sum_{i=1}^{N}\int_0^t [H(v)+w_i(v)]\,dv + kl\sum_{i=1}^{N}\int_0^{s-\tau_i(s)} [H(v)+w_i(v)]\,dv \\
&= -kl\sum_{i=1}^{N}\int_0^t [H(v)+w_i(v)]\,dv - kl\sum_{i=1}^{N}\int_{s-\tau_i(s)}^0 [H(v)+w_i(v)]\,dv \\
&= -kl\sum_{i=1}^{N}\int_{s-\tau_i(s)}^t [H(v)+w_i(v)]\,dv \\
&\leq -kl\sum_{i=1}^{N}\int_s^t w_i(v)dv.
\end{aligned}
$$

Now, the terms in (4.18) are simplified one after the other.

Thus,

$$\begin{aligned} I_1 &= \exp\left(-kl\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}[H(v)+w_i(v)]\,dv\right) \\ &\quad \times \sum_{i=1}^{N}E_i\Big|\phi\left(s-\tau_i(s)\right)-\eta\left(s-\tau_i(s)\right)\Big| \\ &\quad \times \exp\left(-\sum_{i=1}^{N}h_i\left(t-\tau_i(t)\right)\right). \end{aligned} \tag{4.19}$$

Also,

$$\begin{aligned} I_2 &\leq \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}\exp\left(-kl\int_{s}^{t}H(v)dv\right)H(S)L\Big|\phi(s)-\eta(s)\Big| \\ &\quad \times \exp\left(-\sum_{i=1}^{N}h_i(s)\right)ds. \end{aligned} \tag{4.20}$$

Moreover,

$$\begin{aligned} I_3 &\leq \int_{0}^{t}\exp\left(-\int_{s}^{t}H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s}H(u)L\Big|\phi(u)-\eta(u)\Big| \\ &\quad \times \exp\left(-kl\int_{u}^{s}H(v)dv\right)\exp\left(-\sum_{i=1}^{N}h_i(u)\right)duds. \end{aligned} \tag{4.21}$$

Furthermore,

$$\begin{aligned} I_4 &\leq \sum_{i=1}^{N}\int_{0}^{t}\exp\left(-kl\sum_{i=1}^{N}\int_{s}^{t}w_i(v)dv\right)w_i(s) \\ &\quad \times \Big|\phi\left(s-\tau_i(s)\right)-\eta\left(s-\tau_i(s)\right)\Big| \end{aligned}$$

$$\times \ \exp\left(-\sum_{i=1}^{N} h_i\left(s-\tau_i(s)\right)\right) ds. \tag{4.22}$$

Similarly,

$$\begin{aligned} I_5 \ &\leq \ \int_0^t \exp\left(-\int_s^t H(v)dv\right)\Big[\sum_{i=1}^{N} M_i + H(s)\sum_{i=1}^{N} E_i\Big] \\ &\quad \times \sum_{i=1}^{N} |\phi(s-\tau_i(s)) - \eta(s-\tau_i(s))| \\ &\quad \times \ \exp\left(-kl\int_u^s H(v)dv\right)\exp\left(-\sum_{i=1}^{N}(s-\tau_i(s))\right) ds. \end{aligned} \tag{4.23}$$

Finally,

$$\begin{aligned} I_6 \ &\leq \ \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s)L\Big|\phi(s)-\eta(s)\Big| \\ &\quad \times \exp\left(-kl\int_s^t H(v)dv\right)\exp\left(-\sum_{i=1}^{N} h_i(s)\right) ds. \\ &\leq \ \int_0^t \exp\left(-\int_s^t H(v)dv - kl\int_s^t H(v)dv\right) H(s) \\ &\quad \times \ L\Big|\phi(s)-\eta(s)\Big|\exp\left(-\sum_{i=1}^{N} h_i(s)\right) ds. \\ &\leq \ \int_0^t \exp\left(-(1+kl)\int_s^t H(v)dv\right) H(s) \\ &\quad \times L\Big|\phi(s)-\eta(s)\Big|\exp\left(-\sum_{i=1}^{N} h_i(s)\right) ds. \end{aligned} \tag{4.24}$$

Substituting Inequalities (4.19), (4.20), (4.21), (4.22), (4.23), and (4.24) into (4.18) gives

$$\left|P\phi(t) - P\eta(t)\right|_h$$

$$\begin{aligned}
&\leq \exp\left(-kl\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}[H(v)+w_i(v)]\,dv\right) \\
&\quad\times \sum_{i=1}^{N}E_i\left|\phi(t-\tau_i(t))-\eta(t-\tau_i(t))\right|\exp\left(-\sum_{i=1}^{N}h_i\left(t-\tau_i(t)\right)\right) \\
&\quad+ \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}\exp\left(-kl\int_s^t H(v)dv\right)H(S)L\left|\phi(s)-\eta(s)\right| \\
&\quad\times \exp\left(-\sum_{i=1}^{N}h_i(s)\right)ds \\
&\quad+ \int_0^t \exp\left(\int_s^t H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s}H(u)L\left|\phi(u)-\eta(u)\right| \\
&\quad\times \exp\left(-kl\int_u^s H(v)dv\right)\exp\left(-\sum_{i=1}^{N}h_i(u)\right)duds \\
&\quad+ \sum_{i=1}^{N}\int_0^t w_i(s)\left|\phi\left(s-\tau_i(s)\right)-\eta\left(s-\tau_i(s)\right)\right| \\
&\quad\times \exp\left(-kl\sum_{i=1}^{N}\int_s^t w_i(v)dv\right)\exp\left(-\sum_{i=1}^{N}h_i\left(s-\tau_i(s)\right)\right)ds \\
&\quad+ \int_0^t \exp\left(-\int_s^t H(v)dv\right)\left[\sum_{i=1}^{N}M_i + H(s)\sum_{i=1}^{N}E_i\right] \\
&\quad\times \sum_{i=1}^{N}\left|\phi(s-\tau_i(s))-\eta(s-\tau_i(s))\right|
\end{aligned}$$

$$\times \ \exp\left(-kl\int_u^s H(v)dv\right)\exp\left(-\sum_{i=1}^N h_i\left(s-\tau_i(s)\right)\right)ds$$

$$+\int_0^t \exp\left(-(1+kl)\int_s^t H(v)dv\right)H(s)L\Big|\phi(s)-\eta(s)\Big|$$

$$\times \ \exp\left(-\sum_{i=1}^N h_i(s)\right)ds.$$

Consequently, by using (4.16), we obtain

$$\Big|(P\phi)(t)-(P\eta)(t)\Big|_h$$

$$\leq \left[\frac{N}{k}+\left(\frac{4N}{kl}+\frac{1}{kl+1}\right)\right]L|\phi-\eta|_h$$

$$< \alpha|\phi-\eta|_h.$$

Thus, showing that P is a contraction.

Given any $\epsilon > 0$, choose $\delta = \frac{\epsilon_1}{\rho}$, where

$$\rho = L\left[\frac{N}{k}+\left(\frac{4N}{kl}+\frac{1}{kl+1}\right)\right],$$

such that for $|\phi-\eta| \leq \delta$, then

$$\Big|(P\phi)(t)-(P\eta)(t)\Big|_h$$

$$\leq L\left[\frac{N}{k}+\left(\frac{4N}{kl}+\frac{1}{kl+1}\right)\right]\delta$$

$$= L\left[\frac{N}{k}+\left(\frac{4N}{kl}+\frac{1}{kl+1}\right)\right]\frac{\epsilon_1}{\rho}$$

$$= \epsilon_1.$$

Therefore showing that P is continuous.

This completes the proof.

In the next theorem the stability results for the zero solution of Equation (4.1) are stated.

Theorem 4.4

Suppose conditions (4.3) and (4.4) and the hypothesis of Lemma 4.3 hold. Suppose further that the following hold:

(A1) $g_i, (i = 1, 2, ..., N)$, are odd and strictly increasing on $[-\epsilon, \epsilon]$, $x - g_i(x)$ are non-decreasing on $[0, \epsilon]$.

(A2) There exists an $\alpha \in (0, 1)$ such that

$$\epsilon\sum_{i=1}^{N} E_i + \sum_{i=1}^{N} g_i(\epsilon)\int_{t-\tau_i(t)}^{t} H(s)ds + \sum_{i=1}^{N} g_i(\epsilon)$$

$$\times \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s)\int_{s-\tau_i(s)}^{s} |H(u)|duds$$

$$+ \sum_{i=1}^{N} g_i(\epsilon)\int_0^t \left|\left[-a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\right]\right|ds$$

$$\times \exp\left(-\int_s^t H(v)dv\right) + \epsilon\int_0^t \exp\left(-\int_s^t H(v)dv\right)$$

$$\times \sum_{i=1}^{N}\Big[M_i + E_i|H(s)|\Big]ds \le \alpha\sum_{i=1}^{N} g_i(\epsilon), \forall t \ge 0.$$

Then the solution of Equation (4.1) is stable.

Proof.

Given any $\epsilon > 0$, choose δ such that

$$\delta\left(1+\sum_{i=1}^{N} E_i\right)+\sum_{i=1}^{N} g_i(\delta)\int_{-\tau_i(0)}^{0} H(s)ds \leq (1-\alpha)\sum_{i=1}^{N} g_i(\epsilon) \qquad (4.25)$$

and $|\psi| \leq \delta$, then for an arbitrary $\phi \in S_\psi$, the terms on the right hand side of the Equation (4.15) will be denoted by $I_n, n = 1, ..., 7.$

Thus,

$$\begin{aligned}
\left|I_1\right| &= \Big|\psi(0) - G\big(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\big) \\
&\quad -\sum_{i=1}^{N}\int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds\Big| \\
&\leq \left|\psi(0)\right| + \Big|G\big(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\big)\Big| \\
&\quad + \Big|\sum_{i=1}^{N}\int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right)\Big| ds \\
&\leq \delta + \delta\sum_{i=1}^{N} E_i + \sum_{i=1}^{N}\int_{-\tau_i(0)}^{0}\left|H(s)\right| g_i(\delta) ds \\
&\leq \delta + \delta\sum_{i=1}^{N} E_i + \sum_{i=1}^{N} g_i(\delta)\int_{-\tau_i(0)}^{0}\left|H(s)\right| ds \\
&\leq \delta(1+\sum_{i=1}^{N} E_i) + \sum_{i=1}^{N} g_i(\delta)\int_{-\tau(0)}^{0}\left|H(s)\right| ds. \qquad (4.26)
\end{aligned}$$

$$\begin{aligned} \Big|I_2\Big| &= \Big|G,(t,\phi(t-\tau_1(t)),\phi(t-\tau_2(t)),...,\phi(t-\tau_N(t)))\Big| \\ &\leq \epsilon\sum_{i=1}^{N}E_i. \end{aligned} \tag{4.27}$$

$$\begin{aligned} \Big|I_3\Big| &= \Big|\sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t} H(s)g_i(\phi(s))\,ds\Big| \\ &\leq \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}\Big|H(s)\Big|\Big|g_i(\phi(s))\,ds\Big| \\ &\leq \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}\Big|H(s)\Big|\Big|g_i(\phi(s))\Big|ds \\ &\leq \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}|H(s)|g_i|(\phi(s))|ds \\ &\leq \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}H(s)g_i(\epsilon)ds \\ &\leq \sum_{i=1}^{N}g_i(\epsilon)\int_{t-\tau_i(t)}^{t}H(s)ds. \end{aligned} \tag{4.28}$$

$$\begin{aligned} \Big|I_4\Big| &= \Big|\int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\sum_{i=1}^{N}\left(\int_{s-\tau_i(s)}^{s}H(u)g_i(\phi(u))\,du\right)ds\Big| \\ &\leq \int_0^t \exp\left(-\int_0^t H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s}\Big|H(u)\Big|\Big|g_i(\phi(u))\Big|duds \\ &\leq \int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s}\Big|H(u)\Big|g_i\Big|\phi(u)\Big|duds \end{aligned}$$

$$\leq \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} \Big|H(u)\Big| g_i \Big|\phi(u)\Big| duds$$

$$\leq \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} \Big|H(u)\Big| g_i(\epsilon) duds$$

$$\leq \sum_{i=1}^{N} g_i(\epsilon) \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \int_{s-\tau_i(s)}^{s} \Big|H(u)\Big| duds. \quad (4.29)$$

$$\begin{aligned}
\Big|I_5\Big| &= \Big| \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\Big] \\
&\quad \times g_i\left(\phi\left(s-\tau_i(s)\right)\right) \exp\left(-\int_s^t H(v)dv\right) ds \Big| \\
&\leq \int_0^t \sum_{i=1}^{N} \Big| \Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big] \\
&\quad \times g_i\Big(\phi\left(s-\tau_i(s)\right)\Big) ds \Big| \exp\left(-\int_s^t H(v)dv\right) \\
&\leq \int_0^t \sum_{i=1}^{N} \Big| -a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\Big| \\
&\quad \times \Big| g_i\left(\phi\left(s-\tau_i(s)\right)\right)\Big| \exp\left(-\int_s^t H(v)dv\right) ds \\
&\leq \int_0^t \sum_{i=1}^{N} \Big| -a_i(s) + H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\Big| \\
&\quad \times g_i \Big| \left(\phi\left(s-\tau_i(s)\right)\right)\Big| \exp\left(-\int_s^t H(v)dv\right) ds
\end{aligned}$$

$$\leq \int_0^t \sum_{i=1}^{N} \Big| -a_i(s) + H\left(s - \tau_i(s)\right)\left(1 - \tau_i^{/}(s)\right)\Big|$$

$$\times \ \exp\left(-\int_s^t H(v)dv\right) g_i(\epsilon)ds$$

$$\leq \sum_{i=1}^{N} g_i(\epsilon) \int_0^t \exp\left(-\int_s^t H(v)dv\right)$$

$$\times \left| -a_i(s) + H\left(s - \tau_i(s)\right)\left(1 - \tau_i^{/}(s)\right)\right| ds. \qquad (4.30)$$

$$\left|I_6\right| = \left| \int_0^t \Big[Q\big(s, \phi\left(s - \tau_1(s)\right), \phi\left(s - \tau_2(s)\right), ..., \phi\left(s - \tau_N(s)\right)\big)\right.$$

$$- H(s)G\big(s, \phi\left(s - \tau_1(s)\right), \phi\left(s - \tau_2(s)\right), ..., \phi\left(s - \tau_N(s)\right)\big)\Big]$$

$$\left. \times \exp\left(-\int_s^t H(v)dv\right)\right| ds$$

$$\leq \int_0^t \Big| \Big[Q\big(s, \phi\left(s - \tau_1(s)\right), \phi\left(s - \tau_2(s)\right), ..., \phi\left(s - \tau_N(s)\right)\big)$$

$$- H(s)G\big(s, \phi\left(s - \tau_1(s)\right), \phi\left(s - \tau_2(s)\right), ..., \phi\left(s - \tau_N(s)\right)\big)\Big]\Big|$$

$$\times \ \exp\left(-\int_s^t H(v)dv\right) ds$$

$$\leq \int_0^t \Big[\Big| Q\big(s, \phi\left(s - \tau_1(s)\right), \phi\left(s - \tau_2(s)\right), ..., \phi\left(s - \tau_N(s)\right)\big)\Big|$$

$$
\begin{aligned}
& + \Big|H(s)\Big|\Big|G\big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big)\Big|\Bigg] \\
& \times \exp\left(-\int_s^t H(v)dv\right) ds \\
\leq & \int_0^t \exp\left(-\int_s^t H(v)dv\right) \left[\epsilon \sum_{i=1}^{N} M_i + \epsilon \sum_{i=1}^{N} E_i |H(s)|\right] ds \\
\leq & \int_0^t \exp\left(-\int_s^t H(v)dv\right) \epsilon \left[\sum_{i=1}^{N} M_i + \sum_{i=1}^{N} E_i |H(s)|\right] ds \\
\leq & \ \epsilon \int_0^t \exp\left(-\int_s^t H(v)dv\right) \left[\sum_{i=1}^{N} M_i + \sum_{i=1}^{N} E_i |H(s)|\right] ds \\
\leq & \ \epsilon \int_0^t \exp\left(-\int_s^t H(v)dv\right) \sum_{i=1}^{N} \Big[M_i + E_i |H(s)|\Big] ds. \qquad (4.31)
\end{aligned}
$$

$$
\begin{aligned}
\Big|I_7\Big| = & \ \Big|\int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s) \left[\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\right] ds\Big| \\
\leq & \int_0^t \exp\left(-\int_s^t H(v)dv\right) \Big|H(s)\Big| \Bigg|\left[\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\right]\Bigg| ds \\
\leq & \int_0^t \exp\left(-\int_s^t H(v)dv\right) \Big|H(s)\Big| \Big|\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\Big| ds \\
\leq & \int_0^t \exp\left(-\int_s^t H(v)dv\right) \Big|H(s)\Big| \Big|(\epsilon) - \sum_{i=1}^{N} g_i(\epsilon)\Big| ds \\
\leq & \left[\epsilon - \sum_{i=1}^{N} g_i(\epsilon)\right] \int_0^t \exp\left(-\int_s^t H(v)dv\right) |H(s)| ds. \qquad (4.32)
\end{aligned}
$$

Substituting Inequalities (4.26), (4.27), (4.28), (4.29), (4.31), (4.32), (4.33), and (4.34) into (4.15) gives

$$
\begin{aligned}
\left|P\phi(t)\right| &\leq \delta\left(1+\sum_{i=1}^{N} E_i\right)+\sum_{i=1}^{N} g_i(\delta)\int_{-\tau_i(0)}^{0}|H(s)|ds+\epsilon\sum_{i=1}^{N} E_i \\
&+\sum_{i=1}^{N} g_i(\epsilon)\int_{t-\tau_i(t)}^{t} H(s)ds+\sum_{i=1}^{N} g_i(\epsilon) \\
&\times\int_0^t \exp\left(-\int_s^t H(v)dv\right)H(s)\int_{s-\tau_i(s)}^{s}|H(u)|duds \\
&+\sum_{i=1}^{N} g_i(\epsilon)\int_0^t \exp\left(-\int_s^t H(v)dv\right) \\
&\times\left[-a_i(s)+H\left(s-\tau_i(s)\right)\left(1-\tau_i^{/}(s)\right)\right]ds \\
&+\epsilon\int_0^t \exp\left(-\int_s^t H(v)dv\right)\sum_{i=1}^{N}\Big[M_i+E_i|H(s)|\Big]ds \\
&+\left[\epsilon-\sum_{i=1}^{N} g_i(\epsilon)\right]\int_0^t \exp\left(-\int_s^t H(v)dv\right)|H(s)|ds. \\
&\leq \delta\left(1+\sum_{i=1}^{N} E_i\right)+\sum_{i=1}^{N} g_i(\delta)\int_{-\tau_i(0)}^{0}|H(s)|ds \\
&+\alpha\sum_{i=1}^{N} g_i(\epsilon)+\left(\epsilon-\sum_{i=1}^{N} g_i(\epsilon)\right) \\
&\leq (1-\alpha)\sum_{1=1}^{N} g_i(\epsilon)+\alpha\sum_{i=1}^{N} g_i(\epsilon)+\epsilon-\sum_{i=1}^{N} g_i(\epsilon) \\
&= \sum_{i=1}^{N} g_i(\epsilon)-\alpha\sum_{i=1}^{N} g_i(\epsilon)+\alpha\sum_{i=1}^{N} g_i(\epsilon)+\epsilon-\sum_{i=1}^{N} g_i(\epsilon)
\end{aligned}
$$

$$= \sum_{i=1}^{N} g_i(\epsilon) - \sum_{i=1}^{N} g_i(\epsilon) - \alpha \sum_{i=1}^{N} g_i(\epsilon) + \alpha \sum_{i=1}^{N} g_i(\epsilon) + \epsilon = \epsilon.$$

This implies that $|(P\phi)| \leq \epsilon$.

Therefore, showing that zero solution of Equation (4.1) is stable.

In the next theorem, results for the asymptotic stability of the zero solution of Equation (4.1) are stated.

Theorem 4.5

Assume that the hypothesis of Theorem (4.4) hold. Then the zero solution of Equation (4.1) is asymptotically stable if and only if

$$\int_0^t H(v)dv \to \infty \quad as \quad t \to \infty. \tag{4.33}$$

Proof.

First, assume that condition (4.33) holds. Then set

$$K := \sup_{t \geq 0}\{\exp\big(-\int_0^t H(v)dv\big)\}. \tag{4.34}$$

According to definition 4, the zero solution of a differential equation is asymptotically stable if it is stable and in addition for each $t_0 \geq 0$ there is an $r(t_0) > 0$ such that $||\psi|| < r(t_0)$ implies that $x(t) \to 0$ as $t \to \infty$. It has already been proved in Theorem 4.4 that the zero solution is stable.

Define

$$S_\psi^* = \Big\{\phi : [m(0), \infty) \to R,\ \phi(t) = \psi(t)\ for\ t \in [m(0), 0],$$

$$and\ for\ t \geq 0,\ |\phi(t)| \leq \epsilon,\ \phi(t) \to 0\ as\ t \to \infty\Big\}.$$

Also, define the mapping by $P : S^*_\psi \to S^*_\psi$ by

$$
\begin{aligned}
(P\phi)(t) \;=\; & \Big[\psi(0) - G\left(0, \psi\left(-\tau_1(0)\right), \psi\left(-\tau_2(0)\right), ..., \psi\left(-\tau_N(0)\right)\right) \\
& - \sum_{i=1}^{N} \int_{-\tau_i(0)}^{0} H(s) g_i\left(\psi(s)\right) ds\Big] \exp\left(-\int_0^t H(v)dv\right) \\
& + \Big[G\left(t, \phi\left(t-\tau_1(t)\right), \phi\left(t-\tau_2(t)\right), ..., \phi\left(t-\tau_N(t)\right)\right)\Big] \\
& + \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s) g_i\left(\phi(s)\right) ds \\
& - \int_0^t H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u) g_i\left(\phi(u)\right) du ds \\
& \times \exp\left(-\int_s^t H(v)dv\right) \\
& + \int_0^t \sum_{i=1}^{N} \Big[-a_i(s) + H\big(s - \tau_i(s)\big)\Big(1 - \tau_i^{/}(s)\Big)\Big] \\
& \times g_i\big(\phi\left(s - \tau_i(s)\right)\big) \exp\left(-\int_s^t H(v)dv\right) ds \\
& + \int_s^t \Big[Q\big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big) \\
& - H(s)G, \big(s, \phi\left(s-\tau_1(s)\right), \phi\left(s-\tau_2(s)\right), ..., \phi\left(s-\tau_N(s)\right)\big)\Big] \\
& \times \exp\left(-\int_s^t H(v)dv\right) ds \\
& + \int_0^t \exp\left(-\int_s^t H(v)dv\right) H(s)
\end{aligned}
$$

$$\times \left[\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\right] ds. \tag{4.35}$$

Consider $\big|(P\phi)(t)\big|$. But first note for any $\phi \in S_\psi^*$ that

$$\big|g_i(\phi(t))\big| \leq L\big|\phi(t)\big|.$$

Denote the seven terms on the right hand side of Equation (4.35) by $I_1, I_2, ..., I_7$ respectively. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (4.33). Also due to conditions (4.3), (4.4) and (4.5) and the facts that $\phi(t) \to 0$ and $t - \tau_i(t) \to \infty$ as $t \to \infty$, the second term in Equation (4.35) tends to zero as $t \to \infty$. What is left now is to show that each of the remaining terms in Equation (4.35) tends to zero as $t \to \infty$.

Let $\phi \in S_\psi^*$ be fixed. For a given $\epsilon > 0$ we choose $T_0 > 0$ large enough such that $t - \tau_i(t) \geq T_0$, implies $|\phi(s)| < \epsilon$, if $s \geq t - \tau_i(t)$. Therefore, the third term I_3 in Equation (4.35) satisfies

$$\begin{aligned} I_3 &\leq \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} |H(s)|g_i|\left(\phi(s)\right)|ds \\ &\leq \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s)L\big|\phi(s)\big|ds \\ &\leq L\epsilon \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} H(s)ds \\ &\leq L\alpha\epsilon. \end{aligned}$$

Thus $I_3 \to 0$ as $t \to \infty$.

Now consider I_4, for a given $\epsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $|\phi(s - \tau_i(s))| < \epsilon$. Thus, for $t \geq T_1$, the term I_4 in Equation (4.35) satisfies

$$
\begin{aligned}
I_4 &\leq \int_0^t \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)\Big|g_i\left(\phi(u)\right)\Big|duds\\
&\leq \int_0^{T_1} \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)L\Big|\left(\phi(u)\right)\Big|duds\\
&\quad+\int_{T_1}^{t} \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)L\Big|\left(\phi(u)\right)\Big|duds\\
&\leq \sup_{\delta\geq m(0)}\big|\phi(\delta)\big|L\int_0^{T_1} \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)duds\\
&\quad+L\epsilon\int_{T_1}^{t} \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)duds.
\end{aligned}
$$

By condition (4.33), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$
\begin{aligned}
&\sup_{\delta\geq m(0)}\big|\phi(\delta)\big|L\int_0^{T_1} \exp\left(-\int_s^t H(v)\right) H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)duds\\
&= \sup_{\delta\geq m(0)}\big|\phi(\delta)\big|L\left(\exp-\int_{T_2}^{t} H(v)dv\right)\int_0^{T_1} \exp\left(-\int_s^{T_2} H(v)\right)\\
&\quad\times H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)duds.
\end{aligned}
$$

Now, applying assumption $A2$ in Theorem (4.4) gives

$$
\begin{aligned}
&\leq \sup_{\delta\geq m(0)}\big|\phi(\delta)\big|L\left(\exp-\int_{T_2}^{t} H(v)dv\right)\int_0^{T_1} \exp\left(-\int_s^{T_2} H(v)\right)\\
&\quad\times H(s)\sum_{i=1}^{N}\int_{s-\tau_i(s)}^{s} H(u)duds
\end{aligned}
$$

$$+L\epsilon \int_{T_1}^{t} \exp\left(-\int_{s}^{t} H(v)\right) H(s) \sum_{i=1}^{N} \int_{s-\tau_i(s)}^{s} H(u)duds$$

$$\leq \quad L\epsilon + L\alpha\epsilon$$

$$\leq \quad 2L\epsilon.$$

Thus $I_4 \to 0$ as $t \to \infty$. Similarly by using conditions (4.3) and (4.4) and assumption $(A2)$ in Theorem (4.4), if $t \geq T_2$ then I_5, I_6 and I_7 in Equation (4.35) satisfy

$$I_5 \leq \int_{0}^{t} \exp\left(-\int_{s}^{t} H(v)dv\right) \sum_{i=1}^{N} \left[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\right]$$

$$\times g_i\Big|\big(\phi\left(s-\tau_i(s)\right)\big)\Big|ds$$

$$\leq \sup_{\delta \geq m(0)} |\phi(\delta)| \int_{0}^{T_1} L \sum_{i=1}^{N} \left[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\right]$$

$$\times \exp\left(-\int_{s}^{t} H(v)dv\right)$$

$$+ L\epsilon \int_{T_1}^{t} \sum_{i=1}^{N} \left[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\right]$$

$$\times \exp\left(-\int_{s}^{t} H(v)dv\right)$$

$$\leq \sup_{\delta \geq m(0)} |\phi(\delta)| \sum_{i=1}^{N} \left[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\right]$$

$$\times \exp\left(-\int_{T_2}^{t} H(v)dv\right)\int_0^{T_1} \exp\left(-\int_s^{T_2} H(v)dv\right)$$

$$+ L\epsilon \int_{T_1}^{t} \sum_{i=1}^{N}\Big[-a_i(s) + H\big(s-\tau_i(s)\big)\Big(1-\tau_i^{/}(s)\Big)\Big]$$

$$\times \exp\left(-\int_s^{t} H(v)dv\right)$$

$$\leq L\epsilon + L\alpha\epsilon$$

$$\leq 2L\epsilon,$$

$$I_6 \leq \int_0^t \Bigg[\Big|Q\big(s,\phi\left(s-\tau_1(s)\right),\phi\left(s-\tau_2(s)\right),...,\phi\left(s-\tau_N(s)\right)\big)\Big|$$

$$+ H(s)\Big|G,\big(s,\phi\left(s-\tau_1(s)\right),\phi\left(s-\tau_2(s)\right),...,\phi\left(s-\tau_N(s)\right)\big)\Big|\Bigg]$$

$$\times \exp\left(-\int_s^{t} H(v)dv\right)ds$$

$$\leq \int_0^t \exp\left(-\int_s^{t} H(v)dv\right)\sum_{i=1}^{N}\Big[M_i + E_i|H(s)|\Big]ds$$

$$\leq \sup_{\delta\geq(m(0))}\big|\phi(\delta)\big|\int_0^{T_1} \exp\left(-\int_s^{t} H(v)dv\right)\sum_{i=1}^{N}\Big[M_i + E_i|H(s)|\Big]ds$$

$$+\epsilon \int_{T_1}^{t} \exp\left(-\int_s^{t} H(v)dv\right)\sum_{i=1}^{N}\Big[M_i + E_i|H(s)|\Big]ds$$

$$\leq \sup_{\delta\geq m(0)}\big|\phi(\delta)\big|\exp\left(-\int_{T_2}^{t} H(v)dv\right)\int_0^{T_1} \exp\left(-\int_s^{T_2} H(v)dv\right)$$

$$\times \sum_{i=1}^{N}\Big[M_i + E_i H(s)\Big] ds$$

$$+\epsilon \int_{T_1}^{t} \exp\left(-\int_{s}^{t} H(v)dv\right) \sum_{i=1}^{N}\Big[M_i + E_i|H(s)|\Big] ds$$

$$\leq \epsilon + \alpha\epsilon$$

$$\leq 2\epsilon,$$

and

$$I_7 \leq \int_{0}^{t} \exp\left(-\int_{0}^{t} H(v)dv\right)\Big|\phi(s) - \sum_{i=1}^{N} g_i\left(\phi(s)\right)\Big| ds$$

$$\leq \sup_{\delta \geq m(0)} \big|\phi(\delta)\big| \int_{0}^{T_1} \exp\left(-\int_{s}^{t} H(v)dv\right) H(s)L\big|\phi(s)\big|$$

$$+L\epsilon \int_{T_1}^{t} \exp\left(-\int_{s}^{t} H(v)dv\right) H(s)ds$$

$$\leq L\epsilon + L\alpha\epsilon$$

$$\leq 2L\epsilon.$$

Thus I_5, I_6, and $I_7 \to 0$ as $t \to \infty$. In conclusion, $(P\phi)(t) \to 0$ as $t \to \infty$ as required.

Conversely, suppose condition (4.33) fails, then there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(v)dv = \beta$ for some $\beta \in R^+$. Also positive constant may be chosen to satisfy $-J \leq \int_0^{t_n} H(v)dv \leq J$ for all $n \geq 1$.

To simplify our expressions, define

$$\theta(s) = \sum_{i=1}^{N}\Bigg[g_i(\epsilon)\{|-a_i(s)+H(s-\tau_i(s))(1-\tau_i^{/}(s))|$$

$$+H(s)\int_{s-\tau_i(s)}^{s} H(u)du\}+M_i+E_iH(s)\Bigg]+LH(s)$$

for all $s \geq 0$. By condition $(A2)$ in Theorem 4.4 gives

$$\int_0^{t_n}\exp\left(-\int_s^{tn}H(v)dv\right)\theta(s)ds \leq \left(\alpha\sum_{i=1}^{N}g_i(\epsilon)+L\right). \tag{4.36}$$

Thus, Inequality (4.36) can be written in the form

$$\int_0^{t_n}\exp\left(-\int_0^{tn}H(v)dv+\int_0^{s}H(v)dv\right)\theta(s)ds$$

$$\leq \left(\alpha\sum_{i=1}^{N}g_i(\epsilon)+L\right).$$

This implies that

$$\int_0^{t_n}\exp\left(-\int_0^{tn}H(v)dv\right)\exp\left(\int_0^{s}H(v)dv\right)\theta(s)ds$$

$$\leq \left(\alpha\sum_{i=1}^{N}g_i(\epsilon)+L\right). \tag{4.37}$$

Dividing through Equation (4.37) by $\exp\left(-\int_0^{tn}H(v)dv\right)$ gives

$$\int_0^{t_n}\exp\left(\int_0^{s}H(v)dv\right)\theta(s)ds$$

$$\leq \left(\alpha \sum_{i=1}^{N} g_i(\epsilon) + L\right) \exp\left(\int_0^{tn} H(v)dv\right)$$

$$\leq \left(\alpha \sum_{i=1}^{N} g_i(\epsilon) + L\right) \exp(J).$$

Thus the sequence $\left\{ \int_0^{t_n} \exp\left(\int_0^s H(v)dv\right) \theta(s)ds \right\}$ is bounded so there exists a convergent sub-sequence and for brevity of notation, assume that

$$\lim_{n\to\infty} \int_0^{t_n} \exp\left(\int_0^s H(v)dv\right) \theta(s)ds = \gamma,$$

for some $\gamma \in R^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} \exp\left(\int_0^s H(v)dv\right) \theta(s)ds < \frac{\delta_0}{4K},$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies

$$\left\{ \delta_0 \left(1 + \sum_{i=1}^{N} E_i\right) + \sum_{i=1}^{N} g_i(\delta_0) \int_{t_m - \tau_i(t_m)}^{t_m} H(s)ds \right\} K \exp(J)$$

$$\leq (1-\alpha) \sum_{i=1}^{N} g_i(\epsilon).$$

Now consider the solution $x(t) = x(t, t_m, \psi)$ of Equation (4.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$.

Choose ψ so that

$$\Big\{ \psi(t_m) - G(t_m, \psi(t_m - \tau_1(t_m)), \psi(t_m - \tau_2(t_m)), ..., \psi(t_m - \tau_N(t_m)))$$

$$- \sum_{i=1}^{N} \int_{t_m - \tau_i(t_m)}^{t_m} H(s) g_i(\psi(s))ds \Big\} \geq \frac{1}{2}\delta_0.$$

It follows from Equation (4.35) with $x(t) = (Px)(t)$ that for $n \geq m$

$$\Big|\psi(t_n) - G(t_n, \psi(t_n - \tau_1(t_n)), \psi(t_n - \tau_2(t_n)), ..., \psi(t_n - \tau_N(t_n)))$$

$$-\sum_{i=1}^{N} \int_{t_n-\tau_i(t_n)}^{t_n} H(s) g_i(\psi(s)) ds\Big|$$

$$\geq \frac{1}{2}\delta_0 \exp\left(-\int_{t_m}^{t_n} H(v)dv\right) - \int_{t_m}^{t_n} \exp\left(-\int_{s}^{t_n} H(v)dv\right)\theta(s)ds$$

$$= \frac{1}{2}\delta_0 \exp\left(-\int_{t_m}^{t_n} H(v)dv\right) - \exp\left(-\int_{0}^{t_n} H(v)dv\right)$$

$$\times \int_{t_m}^{t_n} \exp\left(\int_{0}^{s} H(v)dv\right)\theta(s)ds$$

$$= \exp\left(-\int_{t_m}^{t_n} H(v)dv\right)\left[\frac{1}{2}\delta_0 - \exp\left(-\int_{0}^{t_m} H(v)dv\right)\right.$$

$$\left.\times \int_{t_m}^{t_n} \exp\left(\int_{0}^{s} H(v)dv\right)\theta(s)ds\right]$$

$$\geq \exp\left(-\int_{t_m}^{t_n} H(v)dv\right)\left(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} \exp\left(\int_{0}^{s} H(v)dv\right)\theta(s)ds\right)$$

$$\geq \frac{1}{4}\delta_0 \exp\left(-\int_{t_m}^{t_n} H(v)dv\right) \geq \frac{1}{4}\delta_0 \exp(-2J) > 0 \qquad (4.38)$$

On the other hand, if the zero solution of Equation (4.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n \to \tau_i(t_n)$ as $n \to \infty$ and assumption $A2$ in Theorem 4.4 holds, it follows that

$$\psi(t_n) - G(t_n, \psi(t_n - \tau_(t_n)), \psi(t_n - \tau_(t_n)), ..., \psi(t_n - \tau_N(t_n)))$$

$$-\sum_{i=1}^{N}\int_{t_n-\tau_i(t_n)}^{t_n} H(s)g_i(\psi(s))ds \to 0 \; as \; n \to \infty$$

which contradicts Equation (4.38). Hence condition (4.33) is necessary for the asymptotic stability of the zero solution of Equation (4.1).

The proof is complete.

Chapter Summary

This chapter dealt with sufficient conditions for which the zero solution of nonlinear neutral differential equations is asymptotically stable by the contraction mapping principle.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter gives a brief summary and conclusions of the study. The summary briefly gives a general overview of the research problem, objectives, methods, and results of the study.

Summary

As indicated in the objectives of the research, the research is on the stability properties of solutions of a certain nonlinear neutral delay differential equation with multiple varying delays. The main tool that was used to investigate the stability properties of this nonlinear neutral differential equation is the Fixed point theory. The nonlinear differential equation was transformed into an equivalent integral equation. The integral equation was then used to define a mapping that was used to study the stability behaviour of the nonlinear neutral differential equation with multiple time varying delays. Since the mappings were contractions, the contraction mapping principle was used. An asymptotic stability of the zero solution of a nonlinear neutral delay differential equation was proved.

Conclusion

Sufficient conditions for the asymptotic stability of the zero solution of the nonlinear delay differential equation with multiple time varying delays have been established.

Recommendations

This problem can be fruitfully studied by using a variable delay or a finite delay.

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