UNIVERSITY OF CAPE COAST

STOCHASTIC OPTIMAL CONTROL OF SYSTEMS DRIVEN BY STOCHASTIC DIFFERENTIAL EQUATIONS OF MEAN-FIELD TYPE WITH IRREGULAR DRIFT COEFFICIENTS

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BY

SORELLE MURIELLE TOUKAM TCHOUMEGNE

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature: Date: 27/11/2023

Name: Sorelle Murielle Toukam Tchoumegne

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature: Date: 27/11/2023

Name: Prof. Olivier Menoukeu Pamen

Co-Supervisor's Signature: Date: 27/11/2023

Name: Prof. Natalia Mensah

ABSTRACT

The main objective of this work is to maximize a performance functional subjected to a controlled stochastic differential equation of mean-field type using the stochastic maximum principle approach. The controlled mean-field stochastic differential equation has a non smooth drift and is driven by a one dimensional Brownian motion. We started by first showing that, considering a corresponding sequence of mean-field stochastic differential equations with a smooth drift coefficient, the corresponding sequence of solutions will converge to the solution of the mean-field stochastic differential equation. We study the representation of the stochastic (Sobolev) differential flow, via a time-space local time integration argument. Lastly, we look at a control problem where the state process follows the dynamics of a mean-field stochastic differential equation. Since the drift coefficient is non smooth, we characterize the optimal control through an approximate performance functional which is derived using the Ekeland's variational principle. Afterwards, we pass to the limit and prove convergence of the stochastic maximum principle.

KEY WORDS

First Variation Process (in the Sobolev sense)

Irregular Drift Coefficient

Mean-Field Stochastic Differential Equation

Stochastic Maximum Principle

Time-Space Local Time

Weak Differentiability

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DEDICATION

To my family.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

SDE	Stochastic Differential Equation
MF	Mean Field
RV	Random Variable
MFSDE	Mean-Field Stochastic Differential Equation
IID	Independent and Identically Distributed
MKV	McKean Vlasov
BM	Brownian Motion
ODE	Ordinary Differential Equation
W.R.T	With Respect To

CHAPTER ONE

INTRODUCTION

This work is mainly dedicated to the study of an optimal control problem subjected to a mean field stochastic differential equation with irregular coefficients using the maximum principle. Early results in the literature have shown solutions to optimal control problems subjected to stochastic differential equations with smooth coefficients. However, since in our setting, the coefficients of the mean field stochastic differential equations are not smooth, we cannot use the same approach to arrive at the solution. We aim at approaching this particular setting using a time-space local time integration approach, hence the novelty in this work.

1.1 Background to the Study

The mean-field theory finds its origin in the field of statistical mechanics, where the interest lies in studying the evolution of an interacting system of particles. The term mean-field comes from the fact that we expect some form of averaging in the marginal distribution of the particles as the number of particles approaches infinity. A SDE of mean-field type is a stochastic differential equation in which we allow the coefficients of the stochastic differential equation to depend on some functional of the distribution of the states.

As an illustration, let us consider the following mean-field stochastic differential equation defined as follows:

$$\begin{cases} dX_t = b(t, X_t, \mathbb{E}(\Phi(X_t)), u_t)dt + \sigma(t, X_t, \mathbb{E}(\Psi(X_t)), u_t)dB_t \\ X(0) = X_0, \end{cases} \tag{1.1}$$

for some functions b, σ, Φ and Ψ and a Brownian motion B_t. In the above equation, we clearly see that the coefficients b and σ depend on the expectation of the state which makes it a SDE of mean-field type. That mean-field

SDE represents the mean-square limit of the following:

$$dX_t^{i,n} = b\Big(t, X_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n} \Phi(X_t^{i,n}), u_t\Big)dt + \sigma\Big(t, X_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n} \Psi(X_t^{i,n}), u_t\Big)dB_t^i$$

as $n \to \infty$. The prior case is one example which has been studied in Andersson & Djehiche (2011). The authors in Andersson & Djehiche (2011) succeeded in characterizing an optimal control for a control problem where the state process follows the dynamics of the MFSDE written in (1.1), assuming that the four functions b, σ, Φ and Ψ are all differentiable in the space variable. However, the pioneering work in mean-field theory has been done by Lasry and Lions in Lasry & Lions (2007a) in which the authors consider a system of interacting players and the objective is to look out for equilibria as the number of players tends to infinity. The authors did it by studying the optimal behavior of one player, after fixing the strategy of all other players, however, by considering that if one player slightly modify his strategy, it will not affect the overall outcome of the game.

In this thesis, we are also interested in the dynamics of a stochastic differential equation of mean-field type. However, instead of having the expectation $\mathbb{E}$ as our measure variable, it will be the law of the process X_t itself, denoted by $\mathbb{P}_{X_t}$. In our settings, we will impose some conditions on the drift coefficient and later on provide some analysis with regards to characterizing the optimal control considering the assumptions provided.

1.2 Research Objective

In this thesis, our objective is to characterize an optimal control for a system driven by a SDE of mean-field type having an irregular drift coefficient b. More precisely, the drift is of at most linear growth. Our goal is to maximize the following performance functional J:

$$J(\alpha) := \mathbb{E}\bigg[\int_0^T f(s, X_s^x, \mathbb{P}_{X_s^x}, \alpha_s)ds + g(X_T^x, \mathbb{P}_{X_T^x})\bigg],$$

subjected to:

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)dt + dB_t, \qquad X_0^x = x, \quad t \in [0, T] \tag{1.2}$$

where B_t is a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\alpha = \{\alpha_t, t \in [0, T]\}$ is a suitable control process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and X_t^x denotes the state of the system controlled by α_t, and f and g some given functions. Further details and definitions will be provided along the thesis.

1.3 Significance of the Study

One straightforward example of optimal control problem is a production planning problem which has been described in Yong & Zhou (1999), page 52. This is established in the context of wanting to minimize the cost of having the inventory at a certain level, at a given production rate. Consequently, it is important to define the rate at which we want to produce a material, also because the factory has a control over the rate it uses. The two other important processes are the demand rate, the rate at which the material is sold and the inventory level of the material. To formulate the control problem in this scenario, the factory wants to work with the optimal production rate in order to spend the minimum cost for the production of material. One logical constraint in this scenario will be to specify the storage size for the inventory. Solving this optimal control problem means finding the optimal production rate taking into consideration the constraint. The study developed in this thesis is more theoretical and abstract, we mostly look at the mathematical properties of the rate itself that we refer to as the drift coefficient and we make some analysis assuming that the drift coefficient fulfills some conditions.

1.4 Delimitation

There are several approaches to solving stochastic optimal problems, which are but not limited to :

- Dynamic programming: establishes a link between the effect of a strategy at a specific point in time and the initial condition of the control problem on one hand and the effect of other strategies that are chosen due to that initial condition on the other hand,
- Stochastic maximum principle: through which we can characterize an optimal control by providing necessary conditions for optimality.

The main aim of our work is to solve an optimal control problem via stochastic maximum principle assuming that the drift coefficient of the controlled state process is non smooth.

1.5 Limitation

Existence and uniqueness have been shown in the case where the mean-field stochastic differential equation has a drift that is not random. The case where the drift is random remains an open problem. However in our case, we can assume that the drift b is of the form $b(t, X_t, \mathbb{P}_{X_t}, \alpha(t, X_t))$ where $\alpha_t = \alpha(t, X_t)$ is a bounded and measurable function, and therefore, we have existence of a solution of the stochastic differential equation.

1.6 Definition of Terms

In this section, we provide definitions of some concepts that will be used throughout this thesis. We also give some useful results that are of a great relevance in our research objective. We will also mention the references and sources we will mostly rely on for our definitions.

1.6.1 Concepts in Mathematical Analysis

This part is dedicated to defining useful concepts of mathematical analysis mainly real and functional analysis, which will be used in several proofs we are going to develop throughout the thesis.

Theorem 1.6.1 (Mean Value Theorem)**.** *If f is a continuous function on a closed interval $[a, b]$, there is at least one number c, $a \leq c \leq b$, so that:*

$$f(b) - f(b) = f'(c)(b - a),$$

on the other hand, we have, by the fundamental theorem of calculus followed by a change of variables,

$$f(x + h) - f(x) = \int_x^{x+h} f'(u)du = \left(\int_0^1 f'(x + th)dt \right) \cdot h,$$

Kouba (2003).

Lemma 1.6.1 (Fatou's Lemma)**.** Considering the measure space $(Y, \mathcal{D}, \nu)$, let $A \in \mathcal{D}$. If $\{g_n\}_{n\geq 1}$ is sequence of positive measurable function on A, hence,

$$\int_A \liminf g_n d\nu \leq \liminf \int_A g_n d\nu,$$

Heinonen (2005a).

Definition 1.6.1.1 (The Gateaux Differential)**.** Considering the following three elements, a function $g : I \longrightarrow J$, a value $\lambda \neq 0$ and a vector $z \in I$, the Gateaux differential $d_\lambda g$ in the direction λ expressed as:

$$d_\lambda g(z + \varepsilon\lambda)_{|\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{g(z + \varepsilon\lambda) - g(z)}{\varepsilon},$$

Long (2009).

1.7 Organisation of the Study

In this thesis, we will develop in the second chapter an overview of well-known results in the literature in stochastic maximum principle, mean-field stochastic differential equations and optimal control of systems where, the state process is a stochastic differential equation with an irregular drift coefficient. We will also explain how the mean-field theory came to light and how it is assimilated to the theory of stochastic analysis. The third chapter is dedicated to study of the representation of the stochastic differential flow using time-space local time which is important because the drift coefficient in this thesis is not differentiable. The fourth chapter provides a thought study on the characterization of a control optimizing a given performance functional for a control problem, where the state is given by a mean-field stochastic differential equation. In that chapter, we prove necessary condition for optimality through the stochastic maximum principle. We will close the thesis in the last chapter with a summary of our study, conclusions and recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Stochastic control theory takes into account uncertainties that are present in a given state process. One task of stochastic control theory is to define the path that will take the state variable subjected to noise when the control involved leads to a minimal cost spent. A control problem affected by randomness in its environment is a stochastic control problem. Stochastic control problems are usually solved or at least studied using two well-known approaches that are common in the literature. The first one is the dynamic programming principle. Richard Bellman in 1952 initiated the dynamic programming principle in Bellman (1952), which was performed at two levels. The first level establishes a link between the effect of a strategy at a specific point in time and the initial condition of the control problem. The second level however establishes the effect of other strategies that are chosen due to that initial condition.

On the other hand, the second well-known approach to solving a stochastic control problem is the stochastic maximum principle, which is the method we will use to study the control problem at our hand. Indeed, the stochastic maximum principle makes it possible to characterize an optimal control by providing necessary and with further convexity assumptions, sufficient conditions for optimality. The author in Bismut (1976) made a breakthrough contribution which paved the way to the field of stochastic control theory with the use of stochastic maximum principle to provide necessary condition for optimality. In Bismut (1976), the stochastic maximum principle was applied to a control problem with a quadratic cost-function. Nevertheless, with the content of the current literature, stochastic maximum principle can be applied to control problems where the cost-function or the performance functional need not to be quadratic.

2.2 Stochastic Maximum Principle

The maximum principle in optimal control was initially formulated in Boltyanskiy et al. (1962) where the authors develop an approach to find the optimal path for a state process with a given constraint. The state variable was deterministic and their approach consisted in removing the constraint on the optimization problem using the Hamiltonian. Once the control problem is affected by noise in its dynamics, one way of assessing the study of the control problem is by using the stochastic maximum principle. For a start, the author in U. Haussmann (1981) applied the stochastic maximum principle to solve stochastic control problems including those which were already solved using the Bellman dynamic programming, however, U. Haussmann (1981) succeeded in obtaining explicit optimal feedback control using the stochastic maximum principle.

Early results in the literature reveal the application of the stochastic maximum principle to control problems where the control is chosen to belong to a convex space. Also, for the controlled state variable, the control parameter was only put in the drift coefficient of the stochastic differential equation, not in the diffusion. It is the case for example in Bismut (1976), Bismut (1978), U. G. Haussmann (1986), Kushner (1972a). An adjoint process of first order was enough to study the stochastic control problem in early studies. The very first author who studied a control problem where in the controlled state variable, the control parameter was put in both the drift coefficient and the diffusion coefficient was Peng in Peng (1990). Also in the aforementioned work, Peng does not need the control space to be convex. The author introduced an adjoint equation of second order to characterize the optimal control for this type of control problem. The specific application of the stochastic maximum principle initially formulated by Peng is developed in the book Yong & Zhou (1999).

2.3 Irregularity in the Control Problem

There are results in the literature of stochastic control theory where authors study the stochastic optimal control of systems consisting of a stochastic differential equations having nonsmooth coefficients. One approach used to tackling this kind of control problems is by using a variational approach which consists in deriving the maximum principle through a pertubation of the optimal control. The first work which went in that direction was done by Kushner in Kushner (1972b). In Kushner (1972b), the author derive a maximum principle using the first order convex approximation of a set of controls which is defined by using the variational result of Neustadt. The drift and diffusion coefficients in Kushner (1972b) were differentiable. On the contrary, the author in Bensoussan (1982) derive the stochastic maximum principle for a control problem, where the coefficients of the state variable do not need to be differentiable everywhere. More precisely, both drift and diffusion coefficients are Lipschitz and admits the linear growth property. The author study the control problem using a variational approach which consist in convergence of the approximate control problem with pertubation to the initial control problem without pertubation.

Also, going in that same direction, the author in Brahim (1988) was able to give necessary conditions for optimality of a system driven by a stochastic differential equation, where the drift does not need to be differentiable everywhere. The author in Brahim (1988) shows stable convergence of the maximum principle by approximating the initial control problem into a sequence of control problems which are smooth using the approximation developed by Frankowska in Frankowska (1984). In addition, we also have the study in S. Bahlali & Mezerdi (2005) which extended the result in Peng (1990) to singular control problems where in their case as well, the control set does not need to be convex . Later on, the authors in K. Bahlali et al. (2007) used a similar approach as in Brahim (1988) but with a diffusion coefficient which is degenerate. More authors have contributed in this direction among which we can cite K. Bahlali et al. (1996) and S. Bahlali & Chala (2005).

One important contribution where the study was done for a control problem in which the drift coefficient of the state variable was neither differentiable nor Lipschitz, has been tackled in Menoukeu-Pamen & Tangpi (2021a), where a variational approach is also used, but with the difference that the authors express the adjoint equation using time-space local time.

In the aforementioned literature, the state process under study was not dependent on a measure variable.

2.4 Mean-Field Stochastic Differential Equation

The idea of having the dynamics depending on the probability law was seen in the pioneering work of Lasry & Lions (2007b), in which the authors succeed in deriving a mean-field control problem consisting of nonlinear differential equations.

In addition, stochastic maximum principle has been used in the literature to characterize optimal controls of systems driven by stochastic differential equations which have a dependence on the law of the state process. It is for example the case in Carmona & Delarue (2013) where the authors showed existence and uniqueness of solutions of a system of two SDE, one is the forward controlled state dynamics and the second is the adjoint equation. However, let us point out that in their case, the drift b is differentiable in the state variable and linear in the state and the measure variables, therefore, deriving the adjoint equation can be done in a classical way. In this thesis, We consider a case where the drift is irregular, i.e. we allow our drift to be measurable and to admit the linear growth property. An attempt of characterizing an optimal control for a system having an irregular drift was done in Menoukeu-Pamen & Tangpi (2021a) where the authors' idea was to explicitly write the adjoint process, but in terms of the flow of the state process. We need to remember that they did the work without the dependence on the measure variable. We will draw our inspiration from their approach, however, this time, we will consider the dependence on the law of the state process.

2.5 Chapter Summary

The objective of this part was to present well-known results in the literature of stochastic control theory. We have seen that it initially started by considering a control problem where the controls were taken in a convex space, and the state dynamics had smooth drift and diffusion coefficients. It gradually moved to the use of variational approaches to derive the stochastic maximum principle in cases where the coefficients of the state variable are not necessarily differentiable everywhere. But still, in those cases, there was no dependence of the state dynamics on its law. However, we presented cases where it is possible to apply the stochastic maximum principle with the state driven by a mean-field stochastic differential equation. At last, we presented a contribution where the state dynamics was a mean-field stochastic differential having an irregular drift coefficient.

CHAPTER THREE

RESEARCH METHODS

3.1 Introduction

In this chapter, we consider the following mean-field stochastic differential equation:

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)dt + dB_t, \qquad X_0^x = x, \quad t \in [0, T], \tag{3.1}$$

where the drift coefficient b fulfills some assumptions which will be stated at the end of this overview. In this part, we assume b to be measurable and of at most linear growth. $\{B_t\}_{t\in[0,T]}$ is the Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ and is one dimensional. The process $\{\alpha_t\}_{t\geq 0}$ belongs to the space defined as follows:

$$\mathcal{D} = \Big\{\alpha : [0, T] \times \Omega \to \mathbb{R}, \text{ is progressive such that (3.1) has a unique strong solution}\Big\}. \tag{3.2}$$

The mean-field stochastic differential equation (3.1) admits a strong solution depending on how the process $\{\alpha_t\}_{t\geq 0}$ is defined.

3.2 Existence and Properties of a Strong Solution of the MFSDE under Study

This part is focused on describing the setup used to lay down the properties of a stron solution of the MFSDE aforementioned. For example, we have existence of strong solution if the process $\{\alpha_t\}_{t\geq 0}$ is a Markovian control. Since a strong solution exists, we will show that the approximate sequence of strong solutions denoted by $\{X^{n,x}\}_{n\geq 0}$ will converge to the strong solution itself. We prove so by supposing that the following assumptions holds:

- the drift b can be decomposed in this form:

$$b(t, z, \mu, a) = b_1(t, z, \mu) + b_2(t, z, a), \tag{3.3}$$

- $b_1 : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is defined such that:

  - there exists a constant $C > 0$ such that for all $t \in [0, T]$, $z \in \mathbb{R}$ and $\mu \in \mathcal{P}_1(\mathbb{R})$,

$$|b_1(t, z, \mu)| \leq C(1 + |z| + \mathcal{K}(\mu, \delta_0)), \tag{3.4}$$

  where b_1 has the following particular form of the linear growth condition:

$$b_1(t, z, \mu) = \hat{b}_1(t, z, \mu) + \tilde{b}_1(t, z, \mu), \tag{3.5}$$

  - $\hat{b}_1$ is bounded and measurable,
  - $\tilde{b}_1$ is of at most linear growth and differentiable in z with bounded derivative,
  - b_1 is continuous in the third variable i.e. for all $\mu \in \mathcal{P}_1(\mathbb{R})$ and all $\epsilon > 0$, $\exists \delta > 0$ such that,

$$(\forall \nu \in \mathcal{P}_1(\mathbb{R}) : \mathcal{K}(\mu, \nu) < \delta) \Rightarrow |b_1(t, z, \mu) - b_1(t, z, \nu)| < \epsilon, \quad t \in [0, T], \quad z \in \mathbb{R}, \tag{3.6}$$

- we will also assume that b_1 is Lipschitz continuous in the measure variable uniformly in the other variables which means that we can find a constant $C > 0$ such that:

$$|b_1(t, z, \mu) - b_1(t, z, \nu)| \leq C\mathcal{K}(\mu, \nu). \tag{3.7}$$

- We assume $b_2 : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ to be adapted such that and fulfilling the following conditions:

– $\left|\frac{\partial}{\partial z} b_2(t,z,\omega)\right| + |b_2(t,z,\omega)| \leq L(\omega)$ for all $(t,z,\omega) \in [0,T]\times\mathbb{R}\times\Omega$ with $L^e := \mathbb{E}[e^{\hat{C}|L(\omega)|^2}] < \infty$, where we have $\hat{C} = 48T$, where the value of $\hat{C}$ is given according to Lemma 3.1 in Menoukeu-Pamen & Tangpi (2019).

– we assume Malliavin differentiability of $b_2(t,z,\cdot)$ for every $(t,z) \in [0,T]\times\mathbb{R}$ and we can find a process $\hat{L}(t,\omega)$ such that the Malliavin derivative of b_2 denoted by $D_t b_2(s,z,\omega)$ satisfies $|D_t b_2(s,z,\omega)| \leq \hat{L}(s,t,\omega)$ $\mathbb{P}\otimes dt$ *a.s.* for all $(t,z)\in[0,T]\times\mathbb{R}$,

– $L^p := \sup_{0\leq s\leq T} \mathbb{E}\left[(\int_0^T |\hat{L}(s,t,\omega)|^2 dt)^4\right] < \infty$ and we can find constants $C, \beta > 0$, such that $\mathbb{E}[|D_{t'} b_2(s,z,\omega) - D_t b_2(s,z,\omega)|^4] \leq C|t'-t|^\beta$,

- the drift b is of at most linear growth i.e. there exists a random variable $C(\omega) > 0$ and a constant $C_1 > 0$ such that for all $t \in [0,T], z\in\mathbb{R}$, and $\mu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t,z,\mu,\omega)| \leq C_1(C(\omega) + |z| + \mathcal{K}(\mu,\delta_0)), \tag{3.8}$$

where we have,

$$L^e := \mathbb{E}[e^{\tilde{C}|C(\omega)|^2}] < \infty, \tag{3.9}$$

with $\tilde{C} = 48T$, where the value of $\tilde{C}$ is given according to Lemma 3.1 in Menoukeu-Pamen & Tangpi (2019). Also, the Kantorovich metric $\mathcal{K}$ is defined as :

$$\mathcal{K}(\mu,\nu) := \sup_{h\in \mathrm{Lip1}(\mathbb{R})} \left|\int_{\mathbb{R}} h(y)(\mu-\nu)(dy)\right|, \quad \mu,\nu\in\mathcal{P}_1(\mathbb{R}),$$

and,

$$\mathcal{P}_1(\mathbb{R}) = \left\{\mu \middle| \mu \text{ probability measure on } (\mathbb{R},\mathcal{B}(\mathbb{R})) \text{ with } \int_{\mathbb{R}} |y| d\mu(y) < \infty\right\}.$$

we must also take note of the following notation:

$$\mathcal{E}\left(\int_0^T b(t, X_t, \mu_t, \alpha_t)dB_t\right) = e^{\int_0^T b(t,X_t,\mu_t,\alpha_t)dB_t - \frac{1}{2}\int_0^T b(t,X_t,\mu_t,\alpha_t)^2 dt} \tag{3.10}$$

3.3 Compactness of the approximating sequence $X_t^{n,x}$ in L^2

Considering the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$, assume that b satisfies the decomposition:

$$b(t, z, \mu, \alpha) = \hat{b}_1(t, z, \mu) + \tilde{b}_1(t, z, \mu) + b_2(t, z, \alpha),$$

such that,

- $\hat{b}_1 : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is bounded and adapted,
- $\tilde{b}_1 : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is differentiable in the space variable with bounded derivative,
- b_2 is bounded measurable and continuously differentiable with bounded first derivative.

For a given sequence,

$$b_n(t, z, \mu, \alpha) = b_{1,n}(t, z, \mu) + b_2(t, z, \alpha), \tag{3.11}$$

such that,

$b_{1,n} : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}, n \geq 1$ admits the decomposition,

$$b_{1,n} = \hat{b}_{1,n} + \tilde{b}_1, \tag{3.12}$$

where $\hat{b}_{1,n}$ are smooth coefficients converging almost everywhere to $\hat{b}_1$, we show that the sequence of corresponding strong solutions $(X_t^{n,x})_{n\geq 1}$ of the

MFSDEs:

$$dX_t^{n,x} = b_n(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}}, \alpha_t)dt + dB_t, \quad 0 \le t \le T, \quad X_0^{n,x} = x \in \mathbb{R}, \quad n \ge 1 \tag{3.13}$$

has the property of relative compactness in $L^2(\mathbb{P}, \mathbb{R})$ for each $0 \le t \le T$. Before stating our result, let us define what a Malliavin derivative is. In this direction, we first give a definition of a symmetric real function.

Definition 3.3.0.1 (Symmetric real function)**.** A real function $g : [0, T]^n \to \mathbb{R}$ is called symmetric if:

$$g(t_{\eta_1}, \cdots, g(t_{\eta_n}) = g(t_1, \cdots, t_n),$$

for all permutations $\eta = (\eta_1, \cdots, \eta_n)$ of $(1, \cdots, n)$, Di Nunno et al. (2009).

Definition 3.3.0.2 (Malliavin derivative)**.** Let $F \in L^2(\mathbb{P})$ be $\mathcal{F}_T-$measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2([0, T]^n), n = 1, 2, \cdots$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f_n(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n},$$

with $\tilde{L}^2([0, T]^n) \subset L^2([0, T]^n)$ being the space of symmetric square integrable Borel real functions on $[0, T]^n$.

(1) *We say that* $F \in \mathbb{D}_{1,2}$ *if:*

$$||F||^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2([0,T]^n)} < \infty.$$

(2) If $F \in \mathbb{D}_{1,2}$ *we define the Malliavin derivative* D_tF *of* F *at time* t *as the expansion:*

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)), \qquad t \in [0, T],$$

where $I_{n-1}(f_n(\cdot,t))$ *is the* $(n-1)$*-fold iterated integral of* $f_n(t_1,t_2,\cdots,t_{n-1},t)$ *with respect to the first to* $n-1$ *variables* $t_1,t_2,\cdots,t_{n-1}$ *and* $t_n=t$ *left as parameter. Di Nunno et al. (2009).*

Our compactness argument is based on the following result:

Lemma 3.3.1. For $T>0$ small enough, we can find a constant C_{T,L^p} depending on T and L^p such that the strong solution $X_t^{n,x}$ of the stochastic differential equation (3.13) fulfills the following property:

$$\mathbb{E}[|D_{t'}X_s^{n,x}-D_tX_s^{n,x}|^2]\leq C_{T,L^p}|t'-t|^{\bar{\alpha}},\quad \text{for } 0\leq t'\leq t\leq s\leq T \text{ and some } \bar{\alpha}>0.$$

The following also holds:

$$\mathbb{E}[|D_tX_s^{n,x}|^2]\leq C_{T,L^p},\quad \text{for a constant } C_{T,L^p}>0.$$

The proof of Lemma 3.3.1 relies on the following results which are provided with proofs:

Lemma 3.3.2. Let $b:[0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})\times\Omega\to\mathbb{R}$ be a function of at most linear growth, i.e., for some random variable $C(\omega)>0$, and a constant $C_1>0$,

$$|b(t,y,\mu,\omega)|\leq C_1(C(\omega)+|y|+\mathcal{K}(\mu,\delta_0)),$$

all $x\in\mathbb{R}$ and $t\in[0,T]$ with T sufficiently small, then for any compact subset $K\subset\mathbb{R}$,

$$\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)^2\right]<\infty, \tag{3.14}$$

$$\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)^4\right]<\infty, \tag{3.15}$$

the second Lemma is the following:

Lemma 3.3.3. Let $f : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$, be a bounded measurable function, then for every $t \in [0, T], \lambda \in \mathbb{R}$ and compact subset $K \in \mathbb{R}$, we have:

$$\sup_{x \in K} \mathbb{E}\left[e^{\lambda \int_0^t \int_{\mathbb{R}} f(s,y,\mu) L^{B^x}(ds,dy)}\right] < \infty, \tag{3.16}$$

where $L^{B^x}(ds, dy)$ denotes the integration with respect to local time of the Brownian motion $B_t^x = B_t + x$ in both time and space.

Proof of Lemma 3.3.2. Indeed, after splitting the Doléans-Dade exponential, and applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) dB_u\right)^2\right] \\
&= \mathbb{E}\left[e^{\int_0^T 2b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u - \int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right] \\
&= \mathbb{E}\left[e^{\int_0^T 2b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u - 4\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du + 3\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right] \\
&\le \mathbb{E}\left[e^{\int_0^T 4b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u - 8\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right]^{\frac{1}{2}} \mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right]^{\frac{1}{2}} \\
&= \mathbb{E}\left[\mathcal{E}\left(\int_0^T 4b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) dB_u\right)\right]^{\frac{1}{2}} \mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right]^{\frac{1}{2}} \\
&= \mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right]^{\frac{1}{2}},
\end{aligned}$$

the last line holds true because:

$$\mathbb{E}\left[\mathcal{E}\left(\int_0^T 4b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) dB_u\right)\right] = 1,$$

we have,

$$
\begin{aligned}
\mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0) &= \sup_{h \in \mathrm{Lip1}} \left| \int_{\mathbb{R}} h(y)(\mathbb{P}_{X_t^x} - \delta_0)(dy) \right| \\
&= \sup_{h \in \mathrm{Lip1}} \left| \int_{\mathbb{R}} h(y)\mathbb{P}_{X_t^x}(dy) - \int_{\mathbb{R}} h(y)\delta_0(dy) \right| \\
&= \sup_{h \in \mathrm{Lip1}} \left| \int_{\mathbb{R}} h(y)\mathbb{P}_{X_t^x}(dy) - \int_{\mathbb{R}} h(0)\mathbb{P}_{X_t^x}(dy) \right| \\
&= \sup_{h \in \mathrm{Lip1}} \left| \int_{\mathbb{R}} h(y)\mathbb{P}_{X_t^x}(dy) - \int_{\mathbb{R}} h(0)\mathbb{P}_{X_t^x}(dy) \right| \\
&= \sup_{h \in \mathrm{Lip1}} \left| \int_{\mathbb{R}} (h(y) - h(0))\mathbb{P}_{X_t^x}(dy) \right| \\
&\leq \sup_{h \in \mathrm{Lip1}} \int_{\mathbb{R}} |h(y) - h(0)|\mathbb{P}_{X_t^x}(dy) \\
&\leq \int_{\mathbb{R}} |y|\mathbb{P}_{X_t^x}(dy) = \mathbb{E}[|X_t^x|],
\end{aligned}
$$

now, let us evaluate $\mathbb{E}[|X_t^x|]$,

$$
\begin{aligned}
\mathbb{E}[|X_t^x|] &= \mathbb{E}\left[\left| x + \int_0^t b(u, X_u^x, \mathbb{P}_{X_u^x}, \alpha_u)du + B_t \right| \right] \\
&\leq |x| + \mathbb{E}\left[\int_0^t |b(u, X_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|du + |B_t| \right] \\
&\leq |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega) + |X_u^x| + \mathcal{K}(X_u^x, \delta_0)\big)du + \mathbb{E}[|B_t|] \\
&\leq |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega) + |X_u^x| + \mathbb{E}[|X_u^x|]\big)du + \mathbb{E}[|B_t|] \\
&= |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega)] + 2\mathbb{E}[|X_u^x|]\big)du + \mathbb{E}[|B_t|] \\
&= |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega)] + 2\mathbb{E}[|X_u^x|]\big)du + \sqrt{\frac{2t}{\pi}} \\
&\leq |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega)] + 2\mathbb{E}[|X_u^x|]\big)du + \sqrt{\frac{2T}{\pi}} \\
&= |x| + C_1\mathbb{E}[C(\omega)]t + 2\int_0^t \mathbb{E}[|X_u^x|]du + \sqrt{\frac{2T}{\pi}}
\end{aligned}
$$

$$\leq |x| + C_1\mathbb{E}[C(\omega)]T + 2\int_0^t \mathbb{E}[|X_u^x|]du + \sqrt{\frac{2T}{\pi}}$$
$$\leq |x| + C_1\mathbb{E}[e^{|C(\omega)|}]T + 2\int_0^t \mathbb{E}[|X_u^x|]du + \sqrt{\frac{2T}{\pi}}$$
$$\leq |x| + C_1\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}T + 2\int_0^t \mathbb{E}[|X_u^x|]du + \sqrt{\frac{2T}{\pi}}$$

after applying the Grönwall's inequality, we get:

$$\mathbb{E}[|X_t^x|] \leq e^{2C_1T}\left(|x| + \sqrt{\frac{2T}{\pi}} + C_1T\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}\right),$$

we will use this estimate of $\mathbb{E}[|X_t^x|]$ to prove the estimate (3.14) of the lemma. So,

$$\begin{aligned}
&|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \\
&\leq C_1(C(\omega) + |B_t^x| + \mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0)) \\
&\leq C_1(C(\omega) + |B_t^x| + \mathbb{E}[|X_t^x|]) \\
&= C_1(C(\omega) + |B_t| + |x| + \mathbb{E}[|X_t^x|]) \\
&\leq C_1\left\{C(\omega) + |x| + |B_t| + e^{2C_1T}\left(|x| + \sqrt{\frac{2T}{\pi}} + C_1T\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}\right)\right\} \\
&= C_1\left\{C(\omega) + |x|\big(1 + e^{2C_1T}\big) + |B_t| + e^{2C_1T}\sqrt{\frac{2T}{\pi}} + C_1Te^{2C_1T}\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}\right\},
\end{aligned}$$

let $\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}$ be bounded by a certain constant we denote C_2, then we have,

$$\begin{aligned}
&|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \\
&\leq C_1\left\{C(\omega) + |x|\big(1 + e^{2C_1T}\big) + |B_t| + e^{2C_1T}\sqrt{\frac{2T}{\pi}} + C_1Te^{2C_1T}\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}}\right\} \\
&\leq C_1\left\{C(\omega) + |x|\big(1 + e^{2C_1T}\big) + |B_t| + e^{2C_1T}\sqrt{\frac{2T}{\pi}} + C_1Te^{2C_1T}C_2\right\} \\
&= C_1\left\{C(\omega) + |x|\big(1 + e^{2C_1T}\big) + |B_t| + e^{2C_1T}\sqrt{\frac{2}{\pi}}T^{\frac{1}{2}} + C_1Te^{2C_1T}C_2\right\}
\end{aligned}$$

we now add the following extra term:

$$C_1 T e^{2C_1 T},$$

and we get,

$$\begin{aligned}
&|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \\
&\leq C_1 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + C_1 T e^{2C_1 T} C_2 + C_1 T e^{2C_1 T} \Big\} \\
&\leq C_1 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + (C_1 C_2 + C_1) T e^{2C_1 T} \Big\} \\
&\leq C_1 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + (C_1 C_2 + C_1) T e^{2C_1 T} \Big\} \\
&\quad + C_1 C_2 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + (C_1 C_2 + C_1) T e^{2C_1 T} \Big\}
\end{aligned}$$

the last line before the last comes from the fact that we just added the following extra term:

$$C_1 C_2 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + (C_1 C_2 + C_1) T e^{2C_1 T} \Big\},$$

we now get,

$$\begin{aligned}
&|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \\
&\leq (C_1 + C_1 C_2) \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + (C_1 C_2 + C_1) T e^{2C_1 T} \Big\} \\
&\leq (C_1 + C_1 C_2) \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T + 2C_1 C_2 T}\big) + |B_t| + e^{2C_1 T + 2C_1 C_2 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} \\
&\quad + (C_1 C_2 + C_1) T e^{2C_1 T + 2C_1 C_2 T} \Big\},
\end{aligned}$$

let us now update the value of C_1 by $C_1(C_2 + 1)$, we get,

$$|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \leq C_1 \Big\{ C(\omega) + |x|\big(1 + e^{2C_1 T}\big) + |B_t| + e^{2C_1 T}\sqrt{\frac{2}{\pi}} T^{\frac{1}{2}} + C_1 T e^{2C_1 T} \Big\},$$

let us now assume that $T \leq 1$, therefore,

$$\begin{aligned} |b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| &\leq C_1\left\{C(\omega) + |x|\left(1 + e^{2C_1T}\right) + |B_t| + e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right\} \\ &\leq C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)\left(C(\omega) + |x| + |B_t| + 1\right) \\ &= C_{4,T}\left(C(\omega) + |x| + |B_t| + 1\right), \end{aligned}$$

with $C_{4,T} = C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)$. The last inequality holds true because:

$$\begin{cases} 1 - \dfrac{1}{C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)} \geq 0 \\ 1 - \dfrac{1 + e^{2C_1T}}{C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)} \geq 0, \end{cases}$$

and it is true for any value of C_1T. Our estimate for $|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)|$ will be,

$$|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \leq C_{4,T}(C(\omega) + |x| + |B_t| + 1), \tag{3.17}$$

where $C_{4,T} = C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)$, with this estimate of $|b(t, B_t^x, \mathbb{P}_{X_t^x}, \alpha_t)|$, we can now find an estimate of $\mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right]$, we will use the approach seen in Menoukeu-Pamen & Tangpi (2019) at page 9 and what follows will be an adaptation of their proof to our settings,

$$\begin{aligned} \mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2 du}\right] &= \mathbb{E}\left[e^{\int_0^T 6C_{4,T}^2(C(\omega)+|x|+|B_u|+1)^2 du}\right] \\ &\leq \mathbb{E}\left[e^{\int_0^T 18C_{4,T}^2(C(\omega)^2+(1+|x|)^2+|B_u|^2) du}\right] \end{aligned}$$

where the last expression comes from using the inequality:

$$(d_1 + d_2 + \cdots + d_n)^2 \leq n(d_1^2 + d_2^2 + \cdots + d_n^2),$$

so,

$$\begin{aligned}\mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\right] &\leq e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{\int_0^T 18C_{4,T}^2C(\omega)^2du}e^{\int_0^T 18C_{4,T}^2|B_u|^2du}\right] \\ &= e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{18C_{4,T}^2C(\omega)^2T}e^{\int_0^T 18C_{4,T}^2|B_u|^2du}\right] \\ &\leq e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{18C_{4,T}^2C(\omega)^2T}e^{18C_{4,T}^2T(\sup_{t\in[0,T]}|B_t|^2)}\right],\end{aligned}$$

next, we apply the Cauchy-Schwarz inequality to get:

$$\mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\right] \leq e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{36C_{4,T}^2|C(\omega)|^2T}\right]^{\frac{1}{2}} \times \mathbb{E}\left[e^{36C_{4,T}^2T(\sup_{t\in[0,T]}|B_t|^2)}\right]^{\frac{1}{2}},$$

let us first estimate $\mathbb{E}\left[e^{36C_{4,T}^2T(\sup_{t\in[0,T]}|B_t|^2)}\right]$, we will apply the Taylor expansion of the exponential function e^x. Therefore, we have:

$$\begin{aligned}\mathbb{E}\left[e^{36C_{4,T}^2T(\sup_{t\in[0,T]}|B_t|^2)}\right] &= \mathbb{E}\left[1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2(\sup_{t\in[0,T]}|B_t|^2)T)^q}{q!}\right], \\ &= \mathbb{E}\left[1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2T)^q}{q!}(\sup_{t\in[0,T]}|B_t|^2)^q\right], \\ &= \left(1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2T)^q}{q!}\mathbb{E}[(\sup_{t\in[0,T]}|B_t|^2)^q]\right), \\ &\leq \left(1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2T)^q}{q!}\mathbb{E}[\sup_{t\in[0,T]}|B_t|^{2q}]\right),\end{aligned}$$

before going forward, we need to estimate $\mathbb{E}[\sup_{t\in[0,T]}|B_t|^{2q}]$ using the Doob's maximal inequality,

$$\begin{aligned}\mathbb{E}[\sup_{t\in[0,T]}|B_t|^{2q}] &\leq \left(\frac{2q}{2q-1}\right)^{2q}\mathbb{E}[|B_T|^{2q}] \\ &= \left(\frac{2q}{2q-1}\right)^{2q}\times\frac{T^q(2q)!}{2^q q!},\end{aligned}$$

we substitute back the expression of $\mathbb{E}[\sup_{t\in[0,T]}|B_t|^{2q}]$ into the inequality with $\mathbb{E}\left[e^{\int_0^T 36C_{4,T}^2|B_u|^2du}\right]$ to have,

$$\begin{aligned}
\mathbb{E}\left[e^{\int_0^T 36C_{4,T}^2|B_u|^2du}\right] &\leq \left(1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2T)^q}{q!}\mathbb{E}[\sup_{t\in[0,T]}|B_t|^{2q}]\right)\\
&\leq \left(1+\sum_{q=1}^{\infty}\frac{(36C_{4,T}^2T)^q}{q!}\left(\frac{2q}{2q-1}\right)^{2q}\times\frac{T^q(2q)!}{2^q q!}\right)\\
&= \left(1+\sum_{q=1}^{\infty}a_q\right),
\end{aligned}$$

where, $a_q = \dfrac{(36C_{4,T}^2T)^q}{q!}\left(\dfrac{2q}{2q-1}\right)^{2q}\times\dfrac{T^q(2q)!}{2^q q!}$,

we will now check the convergence of a_q using the ratio test,

<u>ratio test for a_q:</u>

$$\begin{aligned}
M = \lim_{q\to\infty}\left|\frac{a_{q+1}}{a_q}\right| &= \lim_{q\to\infty}\left|\frac{\frac{(36C_{4,T}^2T)^{q+1}}{(q+1)!}\left(\frac{2q+2}{2q+1}\right)^{2q+2}\times\frac{T^{q+1}(2q+2)!}{2^{q+1}(q+1)!}}{\frac{(36C_{4,T}^2T)^q}{q!}\left(\frac{2q}{2q-1}\right)^{2q}\times\frac{T^q(2q)!}{2^q q!}}\right|\\
&= \lim_{q\to\infty}\left|\frac{\frac{36C_{4,T}^2T}{q+1}\times\left(\frac{2q+2}{2q+1}\right)^{2q+2}\times\frac{T(2q+2)(2q+1)}{2(q+1)}}{\left(\frac{2q}{2q-1}\right)^{2q}}\right|\\
&= \lim_{q\to\infty}\left|\frac{\frac{36C_{4,T}^2T^2(2q+2)(2q+1)}{2(q+1)^2}\times\left(\frac{2q+2}{2q+1}\right)^{2q+2}}{\left(\frac{2q}{2q-1}\right)^{2q}}\right|\\
&= \lim_{q\to\infty}\left|\frac{\frac{36C_{4,T}^2T^2\times 4q^2}{2q^2}\times\frac{(2q)^{2q+2}}{(2q)^{2q+2}}}{\frac{(2q)^{2q}}{(2q)^{2q}}}\right| = 72C_{4,T}^2T^2,
\end{aligned}$$

the ratio test states that for the series a_q to converge, we need to have $M \leq 1$, which means $72C_{4,T}^2 T^2 \leq 1$,

$$
\begin{aligned}
72C_{4,T}^2 T^2 \leq 1 \Rightarrow 72\left(C_1\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)\right)^2 T^2 \leq 1 \\
\Rightarrow 72\left(C_1T\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right)\right)^2 \leq 1 \\
\Rightarrow -\frac{1}{\sqrt{72}} \leq C_1T\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right) \leq \frac{1}{\sqrt{72}} \\
\Rightarrow -\frac{1}{6\sqrt{2}} \leq C_1T\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right) \leq \frac{1}{6\sqrt{2}},
\end{aligned}
$$

since the left part of the previous inequality is always true, we can remove it and deal only with the right part,

$$
\begin{aligned}
& C_1T\left(e^{2C_1T}\sqrt{\frac{2}{\pi}} + C_1Te^{2C_1T}\right) \leq \frac{1}{6\sqrt{2}}, \\
& \Rightarrow p\left(e^{2p}\sqrt{\frac{2}{\pi}} + pe^{2p}\right) - \frac{1}{6\sqrt{2}} \leq 0, \text{ where } p = C_1T, \\
& \Rightarrow p \leq 0.105605,
\end{aligned}
$$

this means,

$$
\begin{aligned}
p &\leq 0.105605, \\
\Rightarrow C_1T &\leq 0.105605, \\
\Rightarrow T &\leq \frac{0.105605}{C_1},
\end{aligned}
$$

we have,

$$
T \leq \frac{0.105605}{C_1}, \tag{3.18}
$$

therefore, we can conclude that the series a_q will converge if the condition (3.18) is satisfied.

Let us scroll back to what we wrote previously, indeed we had:

$$\mathbb{E}\left[e^{\int_0^T 36C_{4,T}^2|B_u|^2du}\right] \leq \left(1+\sum_{q=1}^{\infty} a_q\right),$$

since we have shown that the series a_q converges provided that the condition on T is fulfilled, we can say:

$$\mathbb{E}\left[e^{\int_0^T 36C_{4,T}^2|B_u|^2du}\right] < \infty, \tag{3.19}$$

we also had,

$$\mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\right] \leq e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{36C_{4,T}^2|C(\omega)|^2T}\right]^{\frac{1}{2}} \times \mathbb{E}\left[e^{36C_{4,T}^2T(\sup_{t\in[0,T])}|B_t|^2}\right]^{\frac{1}{2}}$$
$$\leq \hat{C}e^{18C_{4,T}^2(1+|x|)^2T}\mathbb{E}\left[e^{36C_{4,T}^2|C(\omega)|^2T}\right]^{\frac{1}{2}},$$

the term $\mathbb{E}\left[e^{18C_{4,T}^2|C(\omega)|^2T}\right]$ looks like our L^e defined in (3.9), therefore is finite. We can write next,

$$\mathbb{E}\left[e^{\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\right] \leq \hat{C}e^{18C_T^2(1+|x|)^2T}, \text{ with } \hat{C} \text{ changing to a new } \hat{C},$$

since $\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u,x+B_u,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)^2\right] \leq \mathbb{E}\left[e^{6\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\right]^{\frac{1}{2}}$ (you can see it shown at the beginning of the proof of Lemma 3.3.2), we can therefore conclude that for any compact subset $K \subset \mathbb{R}$ and for T which is sufficiently small, the following holds:

$$\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)^2\right] < \infty.$$

The next part consists in proving that:

$$\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)^4\right] < \infty, \tag{3.20}$$

the proof follows the same approach as in the case in the case of

$\sup_{x\in K}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\Big)^2\Big]<\infty$. In this direction, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\Big)^4\Big]\\
&=\mathbb{E}\Big[e^{4\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u-2\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]\\
&=\mathbb{E}\Big[e^{4\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u-16\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du+14\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]\\
&\le\mathbb{E}\Big[e^{8\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u-32\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]^{\frac{1}{2}}\mathbb{E}\Big[e^{28\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]^{\frac{1}{2}}\\
&=\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T 8b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\Big)\Big]^{\frac{1}{2}}\mathbb{E}\Big[e^{28\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]^{\frac{1}{2}}\\
&=\mathbb{E}\Big[e^{28\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]^{\frac{1}{2}},
\end{aligned}$$

so using the same approach as in the proof of the estimate (3.14), one can arrive at the step where showing that $\mathbb{E}\Big[e^{28\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Big]$ is finite means finding the range where the following applies:

$$p\Big(e^{2p}\sqrt{\frac{2}{\pi}}+pe^{2p}\Big)-\frac{1}{4\sqrt{21}}\le 0,\ \text{ where } p=C_1T$$

which is true when $p=C_1T\le 0.0569487$, which means we should have:

$$T\le\frac{0.0569487}{C_1}. \tag{3.21}$$

Therefore, we can conclude that $\sup_{x\in K}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\Big)^4\Big]$ is finite if the condition (3.21) is satisfied which ends our proof. □

Let us state next a strong theorem which has been derived in Eisenbaum (2000) which will play a key role in the proof of Lemma 3.3.3.

Theorem 3.3.1. *Let f be a measurable function from $[0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})$ into $\mathbb{R}$. For a given measure μ, let us define $f^\mu:\mathbb{R}\times[0,T]\to\mathbb{R}$ such that*

$f^\mu(s,x) = f(s,x,\mu)$. We define the norm $||\cdot||$ by:

$$||f^\mu|| = 2\left(\int_0^1\int_{\mathbb{R}}(f^\mu)^2 e^{-\frac{x^2}{2r}}\frac{dxdr}{\sqrt{2\pi r}}\right)^{\frac{1}{2}} + \int_0^1\int_{\mathbb{R}}|xf^\mu(r,x)|e^{-\frac{x^2}{2r}}\frac{dxdr}{\sqrt{2\pi r}},$$

consider the set $\mathcal{H}$ of functions f^μ such that $||f^\mu|| < \infty$. Then,

$$\int_0^t\int_{\mathbb{R}} f^\mu(r,y)L^{B^x}(dr,dy) = \int_0^t f^\mu(r,B_r^x)dB_r + \int_{T-t}^T f^\mu(T-r,\hat{B}_r^x)dW_r - \int_{T-r}^T f^\mu(T-r,\hat{B}_r^x)\frac{\hat{B}_r}{T-r}dr, \tag{3.22}$$

where, $\hat{B}_t := B_{T-t}, 0 \le t \le T$ is the time-reversed Brownian motion, and $W_t := \hat{B}_t - B_T + \int_0^t \frac{\hat{B}_r}{T-r}dr$ is a Brownian motion with respect to the filtration of $\hat{B}$, Eisenbaum (2000).

Next, we recall the Tanaka's formula:

Definition 3.3.0.3 (Tanaka's formula)**.** In stochastic calculus, the Tanaka's formula states that:

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)dB_s + L_t, \tag{3.23}$$

where B_t is a Brownian motion defined under a suitable filtered probability space, sgn denotes the signum function:

$$\operatorname{sgn}(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

Proof of Lemma 3.3.3. Using the decomposition (3.22), we have,

$$\begin{aligned} &\mathbb{E}\left[e^{\lambda\int_0^t\int_{\mathbb{R}} f^\mu(s,y)L^{B^x}(ds,dy)}\right] \\ &= \mathbb{E}\left[e^{\lambda\int_0^t f^\mu(s,B_s^x)dB_s + \lambda\int_{T-t}^T f^\mu(T-s,\hat{B}_s^x)dW_s - \lambda\int_{T-t}^T f^\mu(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right], \\ &= \mathbb{E}[e^{\lambda\int_0^t f^\mu(s,B_s^x)dB_s}e^{\lambda\int_{T-t}^T f^\mu(T-s,\hat{B}_s^x)dW_s}e^{-\lambda\int_{T-t}^T f^\mu(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}], \end{aligned}$$

after using Cauchy-Schwarz inequality two times, we get:

$$
\begin{aligned}
&\mathbb{E}\left[e^{\lambda\int_0^t\int_{\mathbb{R}} f^{\mu}(s,y)L^{B^x}(ds,dy)}\right]\\
&\leq \mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}}\mathbb{E}\left[e^{2\lambda\int_{T-t}^T f^{\mu}(T-s,\hat{B}_s^x)dW_s}e^{-2\lambda\int_{T-t}^T f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]^{\frac{1}{2}}\\
&\leq \mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}}\mathbb{E}\left[e^{4\lambda\int_{T-t}^T f^{\mu}(T-s,\hat{B}_s^x)dW_s}\right]^{\frac{1}{4}}\mathbb{E}\left[e^{-4\lambda\int_{T-t}^T f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]^{\frac{1}{4}},\\
&= I\times II\times III,
\end{aligned}
$$

where $W_t = B_{T-t} - B_T + \int_0^t \frac{B_{T-s}}{T-s}ds$, is a Brownian motion. Let us next show that I and II are finite. First, we use the Taylor series expansion of the exponential function:

$$
\begin{aligned}
\mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\sum_{q=0}^{\infty}\frac{(2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s)^q}{q!}\right]^{\frac{1}{2}}\\
&= \left(\sum_{q=0}^{\infty}\frac{\mathbb{E}\left[(2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s)^q\right]}{q!}\right)^{\frac{1}{2}}\\
&= \left(\sum_{q=0}^{\infty}\frac{(2\lambda)^q\mathbb{E}\left[(\int_0^t f^{\mu}(s,B_s^x)dB_s)^q\right]}{q!}\right)^{\frac{1}{2}},
\end{aligned}
$$

next, we use the Burkholder-Davis-Gundy on the expectation term with the best possible estimates and we get :

$$
\begin{aligned}
\mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}} &= \left(\sum_{q=0}^{\infty}\frac{(2\lambda)^q\mathbb{E}\left[(\int_0^t f^{\mu}(s,B_s^x)dB_s)^q\right]}{q!}\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{q=0}^{\infty}\frac{(2\lambda)^q 2^q q^{\frac{q}{2}}\mathbb{E}\left[(\int_0^t (f^{\mu})^2(s,B_s^x)ds)^{\frac{q}{2}}\right]}{q!}\right)^{\frac{1}{2}},
\end{aligned}
$$

since f^{μ} is a bounded function, we can write:

$$
\begin{aligned}
\mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}} &\leq \left(\sum_{q=0}^{\infty}\frac{(2\lambda)^q C^q q^{\frac{q}{2}} C^{\frac{q}{2}}(\int_0^t ds)^{\frac{q}{2}}}{q!}\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{q=1}^{\infty}\frac{C^q q^{\frac{q}{2}}}{q!}\right)^{\frac{1}{2}}\\
&= \left(\sum_{q=0}^{\infty}a_q\right)^{\frac{1}{2}}.
\end{aligned}
$$

The ratio test implies:

$$
\begin{aligned}
M = \frac{a_{q+1}}{a_q} &= \frac{\dfrac{C^{q+1}(q+1)^{\frac{q+1}{2}}}{(q+1)!}}{\dfrac{C^q q^{\frac{q}{2}}}{q!}} \\
&= \frac{\dfrac{C^{q+1}(q+1)^{\frac{q}{2}}(q+1)^{\frac{1}{2}}}{(q+1)!}}{\dfrac{C^q q^{\frac{q}{2}}}{q!}} \\
&= \frac{\dfrac{C^{q+1}(q+1)^{\frac{q}{2}}(q+1)^{\frac{1}{2}}}{(q+1)q!}}{\dfrac{C^q q^{\frac{q}{2}}}{q!}} \\
&= \frac{\dfrac{CC^{q}(q+1)^{\frac{q}{2}}(q+1)^{\frac{1}{2}}}{(q+1)q!}}{\dfrac{C^q q^{\frac{q}{2}}}{q!}} \\
&= \frac{CC^{q}(q+1)^{\frac{q}{2}}(q+1)^{\frac{1}{2}}}{(q+1)q!} \times \frac{q!}{C^q q^{\frac{q}{2}}} \\
&= C\frac{(q+1)^{q/2}}{(q+1)^{\frac{1}{2}} q^{\frac{q}{2}}} \\
&= C\left(\frac{q+1}{q}\right)^{\frac{q}{2}} \frac{1}{(q+1)^{\frac{1}{2}}} \\
&= C\left[\left(1+\frac{1}{q}\right)^{q}\right]^{\frac{1}{2}} \frac{1}{(q+1)^{\frac{1}{2}}}.
\end{aligned}
$$

Thus, $\lim_{q\to\infty} C\left[\left(1+\frac{1}{q}\right)^{q}\right]^{\frac{1}{2}} \frac{1}{(q+1)^{\frac{1}{2}}} = \lim_{q\to\infty} Ce^{\frac{1}{2}} \frac{1}{(q+1)^{\frac{1}{2}}} = 0 < 1$. Since, the series $\{a_q\}_{q\geq 0}$ converges, $\mathbb{E}\left[e^{2\lambda\int_0^t f^{\mu}(s,B_s^x)dB_s}\right]^{\frac{1}{2}}$ is finite. The same reasoning applies to $\mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]^{\frac{1}{4}}$. Therefore, both I and II are finite. Let us now show that the term $III = \mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]^{\frac{1}{4}}$ is also finite.

For the term III, we have using the Taylor series expansion of the exponential function:

$$\begin{aligned}
&\mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]\\
&=\mathbb{E}\Big[\sum_{q=0}^{\infty}\frac{\big(-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds\big)^q}{q!}\Big]\\
&=\mathbb{E}\Big[\sum_{q=0}^{\infty}\frac{\big(-4\lambda\int_{T-t}^{T} 2f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{\sqrt{T-s}}\frac{1}{2\sqrt{T-s}}ds\big)^q}{q!}\Big]\\
&=\sum_{q=0}^{\infty}\frac{\mathbb{E}\Big[\big(-4\lambda\int_{T-t}^{T} 2f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{\sqrt{T-s}}\frac{1}{2\sqrt{T-s}}ds\big)^q\Big]}{q!}\\
&=\sum_{q=0}^{\infty}\frac{(-4\lambda)^q\mathbb{E}\Big[\big(\int_{T-t}^{T} 2f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{\sqrt{T-s}}\frac{1}{2\sqrt{T-s}}ds\big)^q\Big]}{q!},
\end{aligned}$$

next, we apply the Jensen's inequality with the measure $d\nu_s = \frac{1}{2\sqrt{T-s}}ds$ and we get:

$$\begin{aligned}
&\mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]\\
&=\sum_{q=0}^{\infty}\frac{(-4\lambda)^q\mathbb{E}\Big[\big(\int_{T-t}^{T} 2f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{\sqrt{T-s}}\frac{1}{2\sqrt{T-s}}ds\big)^q\Big]}{q!}\\
&=\sum_{q=0}^{\infty}\frac{(-4\lambda)^q\mathbb{E}\Big[\big(\int_{T-t}^{T} 2f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{\sqrt{T-s}}d\nu_s\big)^q\Big]}{q!}\\
&\le\sum_{q=0}^{\infty}\frac{(-4\lambda)^qC^q\mathbb{E}\Big[\int_{T-t}^{T} \big(f^{\mu}(T-s,\hat{B}_s^x)\big)^q\big(\frac{\hat{B}_s}{\sqrt{T-s}}\big)^q d\nu_s\Big]}{q!},
\end{aligned}$$

since f^{μ} is a bounded function, we update the value of C to a new C to get:

$$\begin{aligned}
&\mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]\\
&\le\sum_{q=0}^{\infty}\frac{(-4\lambda)^qC^q\mathbb{E}\Big[\int_{T-t}^{T} \big(f^{\mu}(T-s,\hat{B}_s^x)\big)^q\big(\frac{\hat{B}_s}{\sqrt{T-s}}\big)^q d\nu_s\Big]}{q!}\\
&\le\sum_{q=0}^{\infty}\frac{(-4\lambda)^qC^q\mathbb{E}\Big[\int_{T-t}^{T} \big(\frac{\hat{B}_s}{\sqrt{T-s}}\big)^q d\nu_s\Big]}{q!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=0}^{\infty} \frac{(-4\lambda)^q C^q \int_{T-t}^{T} \mathbb{E}\big[\big|\frac{\hat{B}_s}{\sqrt{T-s}}\big|^q\big] d\nu_s}{q!} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \int_{T-t}^{T} \frac{1}{(T-s)^{\frac{q}{2}}} \mathbb{E}\big[|\hat{B}_s|^q\big] d\nu_s}{q!} \\
&\leq \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \int_{T-t}^{T} \frac{1}{(T-s)^{\frac{q}{2}}} \mathbb{E}\big[|\hat{B}_s|^{2q}\big]^{\frac{1}{2}} d\nu_s}{q!} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \int_{T-t}^{T} \frac{1}{(T-s)^{\frac{q}{2}}} \mathbb{E}\big[|B_{T-s}|^{2q}\big]^{\frac{1}{2}} d\nu_s}{q!} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \big(\int_{T-t}^{T} \frac{1}{(T-s)^{\frac{q}{2}}} \big[\frac{(2q)!(T-s)^q}{2^q q!}\big]^{\frac{1}{2}} d\nu_s \big)}{q!} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \big(\int_{T-t}^{T} \frac{1}{(T-s)^{\frac{q}{2}}} \frac{[(2q)!]^{\frac{1}{2}} (T-s)^{\frac{q}{2}}}{2^{\frac{q}{2}} (q!)^{\frac{1}{2}}} d\nu_s \big)}{q!} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \frac{[(2q)!]^{\frac{1}{2}}}{2^{\frac{q}{2}} (q!)^{\frac{1}{2}}} \int_{T-t}^{T} d\nu_s}{q!} \\
&\leq \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \frac{[(2q)!]^{\frac{1}{2}}}{2^{\frac{q}{2}} (q!)^{\frac{1}{2}}}}{q!} \\
&\leq \sum_{q=0}^{\infty} (4\lambda)^q C^q \frac{[(2q)!]^{\frac{1}{2}}}{q! 2^{\frac{q}{2}} (q!)^{\frac{1}{2}}} \\
&= \sum_{q=0}^{\infty} \frac{(4\lambda)^q C^q \big[(2q)!\big]^{\frac{1}{2}}}{q!(2^q q!)^{\frac{1}{2}}} \\
&= \sum_{q=0}^{\infty} a_q,
\end{aligned}
$$

where $a_q = \dfrac{(4\lambda)^q C^q \big[(2q)!\big]^{\frac{1}{2}}}{q!(2^q q!)^{\frac{1}{2}}}$, the ratio test of a_q gives:

$$
M = \lim_{q \to \infty} \frac{a_{q+1}}{a_q}
$$

$$M = \lim_{q\to\infty} \frac{\dfrac{(4\lambda)^{q+1}C^{q+1}\big((2(q+1))!\big)^{\frac{1}{2}}}{(q+1)!(2^{q+1}(q+1)!)^{\frac{1}{2}}}}{\dfrac{(4\lambda)^{q}C^{q}\big[(2q)!\big]^{\frac{1}{2}}}{q!(2^{q}q!)^{\frac{1}{2}}}}$$

$$= \lim_{q\to\infty} \frac{\dfrac{(4\lambda)^{q}(4\lambda)C^{q}C\big((2q+2)(2q+1)(2q)!\big)^{\frac{1}{2}}}{(q+1)q!(2\times 2^{q}(q+1)q!)^{\frac{1}{2}}}}{\dfrac{(4\lambda)^{q}C^{q}\big[(2q)!\big]^{\frac{1}{2}}}{q!(2^{q}q!)^{\frac{1}{2}}}}$$

$$= \lim_{q\to\infty} \frac{\dfrac{(4\lambda)^{q}C^{q}\big[(2q)!\big]^{\frac{1}{2}}}{q!(2^{q}q!)^{\frac{1}{2}}} \times \dfrac{(4\lambda)\times C\big((2q+2)(2q+1)\big)^{\frac{1}{2}}}{(q+1)\times(2(q+1))^{\frac{1}{2}}}}{\dfrac{(4\lambda)^{q}C^{q}\big[(2q)!\big]^{\frac{1}{2}}}{q!(2^{q}q!)^{\frac{1}{2}}}}$$

$$= \lim_{q\to\infty} \frac{(4\lambda)\times C\big((2q+2)(2q+1)\big)^{\frac{1}{2}}}{(q+1)\times(2(q+1))^{\frac{1}{2}}}$$

$$= \lim_{q\to\infty} \frac{(4\lambda)\times C(2q+2)^{\frac{1}{2}}(2q+1)^{\frac{1}{2}}}{(q+1)\times(2q+2)^{\frac{1}{2}}}$$

$$= \lim_{q\to\infty} \frac{(4\lambda)\times C(2q+1)^{\frac{1}{2}}}{q+1}$$

$$= \lim_{q\to\infty} \frac{(4\lambda)\times C(2q+1)^{\frac{1}{2}}}{(q+1)^{\frac{1}{2}}(q+1)^{\frac{1}{2}}}$$

$$= \lim_{q\to\infty} \frac{(4\lambda)\times C}{(q+1)^{\frac{1}{2}}} \times \left(\frac{2q+1}{q+1}\right)^{\frac{1}{2}} = 0 < 1.$$

Since the ratio is less than 1, the series $\{a_q\}_{q\geq 0}$ converges, therefore the term $\mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]$ is finite. Since it is finite, the term $III = \mathbb{E}\left[e^{-4\lambda\int_{T-t}^{T} f^{\mu}(T-s,\hat{B}_s^x)\frac{\hat{B}_s}{T-s}ds}\right]^{\frac{1}{4}}$ is also finite. Hence, the three terms I, II and III are finite. We can then conclude that,

$$\sup_{x\in K}\mathbb{E}\left[e^{\lambda\int_0^t\int_{\mathbb{R}} f(s,y,\mu)L^{B^x}(ds,dy)}\right] < \infty.$$

This is where the proof ends. □

Proof of Lemma 3.3.1. First, we derive the explicit representation for the following expression $D_{t'}X_s^{n,x} - D_tX_s^{n,x}$. We know that:

$$dX_t^{n,x} = b_n(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}}, \alpha_t)dt + dB_t,$$
$$\Rightarrow X_t^{n,x} = x + B_t + \int_0^t b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u)du,$$

we also have:

$$\begin{aligned} X_s^{n,x} &= x + B_s + \int_0^s b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u)du \\ &= x + B_s + \int_0^t b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u)du + \int_t^s b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u)du \\ &= x + B_s + \int_0^t b_n(u, X_u^{n,x}, \mathbb{P}_{X_u n,x}, \alpha_u)du + \int_t^s (b_{1,n}(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}) + b_2(u, X_u^{n,x}, \alpha_u))du, \end{aligned}$$

now, let us take the Malliavin derivative of $X_s^{n,x}$, we get:

$$\begin{aligned} &D_tX_s^{n,x} \\ &= D_t\left[x + B_s + \int_0^t b_n(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr + \int_t^s (b_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}) + b_2(r, X_r^{n,x}, \alpha_r))dr\right] \\ &= D_tx + D_tB_s + D_t\left[\int_0^t b_n(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_t)dr\right] + D_t\left[\int_t^s b_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right] \\ &\quad + D_t\left[\int_t^s b_2(r, X_r^{n,x}, \alpha_r)dr\right] \\ &= \mathbb{1}_{\{t\le s\}} + \int_t^s D_t[b_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})]dr + \int_t^s D_t[b_2(r, X_r^{n,x}, \alpha_r)]dr \\ &= 1 + \int_t^s b'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})D_tX_r^{n,x}dr + \int_t^s [b'_2(r, X_r^{n,x}, \alpha_r)D_tX_r^{n,x} + D_tb_2(r, X_r^{n,x}, \alpha_r)]dr \\ &= 1 + \int_t^s b'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})D_tX_r^{n,x}dr + \int_t^s b'_2(r, X_r^{n,x}, \alpha_r)D_tX_r^{n,x}dr + \int_t^s D_tb_2(r, X_r^{n,x}, \alpha_r)dr \\ &= 1 + \int_t^s (b'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}) + b'_2(r, X_r^{n,x}, \alpha_r))D_tX_r^{n,x}dr + \int_t^s D_tb_2(r, X_r^{n,x}, \alpha_r)dr \\ &= 1 + \int_t^s b'_n(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)D_tX_r^{n,x}dr + \int_t^s D_tb_2(r, X_r^{n,x}, \alpha_r)dr, \end{aligned}$$

where b'_2 and $b'_{1,n}$ are the derivatives of b_2 and $b_{1,n}$ with respect to the second variable which appears after we apply the chain rule. We get:

$$D_tX_s^{n,x} = 1 + \int_t^s b'_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u)D_tX_u^{n,x}du + \int_t^s D_tb_2(u, X_u^{n,x}, \alpha_u)du. \tag{3.24}$$

We obtain a linear ODE of order 1 in $D_tX_s^{n,x}$, with variable coefficient.

Therefore, one can show that the solution to the above equation can be explicitly written as follows :

$$D_t X_s^{n,x} = e^{\int_t^s b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} \left(1 + \int_t^s e^{-\int_t^u b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} D_t b_2(u, X_u^{n,x}, \alpha_u)du\right).$$

Now, let us first derive the explicit representation for the following expression $D_{t'} X_s^{n,x} - D_t X_s^{n,x}$:

$$\begin{aligned} &D_{t'} X_s^{n,x} - D_t X_s^{n,x} \\ &= e^{\int_{t'}^s b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} \left(1 + \int_{t'}^s e^{-\int_{t'}^u b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} D_{t'} b_2(u, X_u^{n,x}, \alpha_u)du\right) \\ &\quad - e^{\int_t^s b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} \left(1 + \int_t^s e^{-\int_t^u b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr} D_t b_2(u, X_u^{n}, \alpha_u)du\right) \end{aligned}$$

$$
\begin{aligned}
&= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + e^{\int_{t'}^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\int_{t'}^s e^{-\int_{t'}^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad - e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\int_t^s e^{-\int_t^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_t b_2(u,X_u^{n,x},\alpha_u)du\\
&= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + \int_{t'}^s e^{\int_{t'}^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} e^{-\int_{t'}^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad - \int_t^s e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} e^{-\int_t^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_t b_2(u,X_u^{n,x},\alpha_u)du\\
&= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + \int_{t'}^s e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad - \int_t^s e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_t b_2(u,X_u^{n,x},\alpha_u)du,\\
&= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + \int_{t'}^t e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\omega)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad + \int_t^s e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad - \int_t^s e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_t b_2(u,X_u^{n,x},\alpha_u)du\\
&= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + \int_{t'}^t e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad + \int_t^s e^{-\int_s^u b'_n(r,X_r^{n,x},\alpha_r)dr} (D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u))du,
\end{aligned}
$$

therefore, the explicit representation of $D_{t'}X_s^{n,x} - D_tX_s^{n,x}$ is:

$$
\begin{aligned}
D_{t'}X_s^{n,x} - D_tX_s^{n,x} &= e^{\int_t^s b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\\
&\quad + \int_{t'}^t e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\\
&\quad + \int_t^s e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} (D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u))du.
\end{aligned}
$$

Now, let:

$$D_{t'}X_s^{n,x} - D_tX_s^{n,x} = I_1 + I_2 + I_3,$$

where,
$$I_1 = e^{\int_t^s b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right),$$
$$I_2 = \int_{t'}^t e^{-\int_s^u b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} D_{t'}b_2(u, X_u^{n,x}, \alpha_u)du,$$
$$I_3 = \int_t^s e^{-\int_s^u b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}(D_{t'}b_2(u, X_u^{n,x}, \alpha_u) - D_tb_2(u, X_u^{n,x}, \alpha_u))du,$$

our goal is to find the following compactness criteria:

$$\mathbb{E}[|D_{t'}X_s^{n,x} - D_tX_s^{n,x}|^2] \leq C|t'-t|^m, \quad \text{for } 0 \leq t' \leq t \leq s \leq T \text{ and } m \in \left(\frac{1}{2}, \frac{\beta}{2}\right).$$

Hence, we have:

$$\begin{aligned}\mathbb{E}[|D_{t'}X_s^{n,x} - D_tX_s^{n,x}|^2] &= \mathbb{E}[|I_1 + I_2 + I_3|^2] \\ &\leq 3\mathbb{E}[|I_1|^2 + |I_2|^2 + |I_3|^2] \\ &= 3\mathbb{E}[|I_1|^2] + 3\mathbb{E}[|I_2|^2] + 3\mathbb{E}[|I_3|^2],\end{aligned}$$

we used the following inequality to get the prior expression:

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

First step: we compute $\mathbb{E}[|I_1|^2]$:

$$\begin{aligned}&\mathbb{E}[|I_1|^2] \\ &= \mathbb{E}\left[\left|e^{\int_t^s b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)\right|^2\right] \\ &= \mathbb{E}\left[e^{\int_t^s 2b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\left(e^{\int_{t'}^t b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)^2\right] \\ &= \mathbb{E}\left[e^{\int_t^s 2\left(b_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})+b_2(r,X_r^{n,x},\alpha_r)\right)dr}\left(e^{\int_{t'}^t b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)^2\right] \\ &= \mathbb{E}\left[e^{\int_t^s 2b_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}e^{\int_t^s 2b_2(r,X_r^{n,x},\alpha_r)dr}\left(e^{\int_{t'}^t b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} - 1\right)^2\right],\end{aligned}$$

we get after applying the Cauchy-Schwarz inequality:

$$\begin{aligned}&\mathbb{E}[|I_1|^2]\\&=\mathbb{E}\left[e^{\int_t^s 2b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}e^{\int_t^s 2b_2(r,X_r^{n,x},\alpha_r)dr}\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^2\right]\\&\leq\mathbb{E}\left[e^{\int_t^s 4b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}e^{\int_t^s 4b_2(r,X_r^{n,x},\alpha_r)dr}\right]^{\frac{1}{2}}\mathbb{E}\left[\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^4\right]^{\frac{1}{2}},\end{aligned}$$

after applying once again the Cauchy-Schwarz inequality on the first term, we get:

$$\begin{aligned}&\mathbb{E}[|I_1|^2]\\&\leq\mathbb{E}\left[e^{\int_t^s 4b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}e^{\int_t^s 4b_2(r,X_r^{n,x},\alpha_r)dr}\right]^{\frac{1}{2}}\mathbb{E}\left[\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^4\right]^{\frac{1}{2}}\\&\leq\mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}}\mathbb{E}\left[e^{\int_t^s 8b'_2(r,X_r^{n,x},\alpha_r)dr}\right]^{\frac{1}{4}}\mathbb{E}\left[\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^4\right]^{\frac{1}{2}}\end{aligned}$$

since the space derivative of b_2 is bounded by a random variable with finite exponential moment, we can write:

$$\mathbb{E}[|I_1|^2]\leq C\mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}}\mathbb{E}\left[\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^4\right]^{\frac{1}{2}},$$

we next apply the mean value theorem to $e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1$, the mean value theorem states that,

$$\begin{aligned}g(y+h)-g(y)&=\int_y^{y+h}g'(u)du,\\&=\left(\int_0^1 g'(y+\theta h)d\theta\right)\cdot h,\end{aligned}$$

if we consider the function $g(y)=e^y$ with $y=0$ and $h=\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr$,

$$\begin{aligned}e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1&=e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-e^0,\\&=\left(\int_0^1 e^{\theta\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}d\theta\right)\cdot\left(\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr\right)\end{aligned}$$

after substituting that right hand side into the expression of $\mathbb{E}[|I_1|^2]$, we get,

$$\begin{aligned}
&\mathbb{E}[|I_1|^2] \\
&\lesssim C\mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}}\mathbb{E}\left[\left(e^{\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}-1\right)^4\right]^{\frac{1}{2}}, \\
&=\mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}}\mathbb{E}\left[\left(\int_0^1 e^{\theta\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}d\theta\right)^4\cdot\left(\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr\right)^4\right]^{\frac{1}{2}}
\end{aligned}$$

we apply once again the Cauchy-Schwarz inequality to obtain:

$$\begin{aligned}
&\mathbb{E}[|I_1|^2] \\
&\lesssim \mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}}\mathbb{E}\left[\left(\int_0^1 e^{\theta\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}d\theta\right)^8\right]^{\frac{1}{4}} \\
&\quad\mathbb{E}\left[\left(\int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr\right)^8\right]^{\frac{1}{4}} \\
&= J_1^n\cdot J_2^n\cdot J_3^n,
\end{aligned}$$

now, we have $\mathbb{E}[|I_1|^2]\leq CJ_1^n\cdot J_2^n\cdot J_3^n$, let us evaluate J_1^n,

$$\begin{aligned}
J_1^n &= \mathbb{E}\left[e^{\int_t^s 8b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}} \\
&= \mathbb{E}\left[e^{\int_t^s\int_{\mathbb{R}} 8\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_t^s 8\tilde{b}'_1(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}},
\end{aligned}$$

afterwards, we apply the Girsanov transform with a change of trajectory and we get:

$$\begin{aligned}
J_1^n &\leq \mathbb{E}\left[e^{\int_t^s\int_{\mathbb{R}} 8\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_t^s 8\tilde{b}'_1(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\right]^{\frac{1}{4}} \\
&= \mathbb{E}\left[e^{\int_t^s\int_{\mathbb{R}} 8\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)+\int_t^s 8\tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr}\mathcal{E}\left(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\right)\right]^{\frac{1}{4}},
\end{aligned}$$

we separate the previous expression using Cauchy-Schwarz inequality to get:

$$
\begin{aligned}
&J_1^n \\
&\leq \mathbb{E}\left[e^{\int_t^s \int_{\mathbb{R}} 8\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) + \int_t^s 8\tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr} \mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dB_r \right) \right]^{\frac{1}{4}} \\
&= \mathbb{E}\left[e^{\int_t^s \int_{\mathbb{R}} 16\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) + \int_t^s 16\tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr} \right]^{\frac{1}{8}} \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dB_r \right)^2 \right]^{\frac{1}{8}}
\end{aligned}
$$

another Cauchy-Schwarz inequality yields:

$$
\begin{aligned}
&J_1^n \\
&\leq \mathbb{E}\left[e^{\int_t^s \int_{\mathbb{R}} 32\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)} \right]^{\frac{1}{16}} \mathbb{E}\left[e^{\int_t^s 32\tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr} \right]^{\frac{1}{16}} \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dB_r \right)^2 \right]^{\frac{1}{8}} \\
&< \infty,
\end{aligned}
$$

J_1^n is finite because the first term is finite due to Lemma 3.3.3 since $\hat{b}_1$ is bounded, the second term as well is finite because of the assumption on $\tilde{b}_1$ which says that it has bounded space derivative, and the last term is finite due to Lemma 3.3.2. We continue with J_2^n:

$$
\begin{aligned}
J_2^n &= \mathbb{E}\left[\left(\int_0^1 e^{\theta \int_{t'}^t b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr} d\theta \right)^8 \right]^{\frac{1}{4}} \\
&= \mathbb{E}\left[\left(\int_0^1 e^{\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr + \theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr} d\theta \right)^8 \right]^{\frac{1}{4}}
\end{aligned}
$$

next, we apply Cauchy-Schwarz inequality two times and we get:

$$
\begin{aligned}
J_2^n &= \mathbb{E}\left[\left(\int_0^1 e^{\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr + \theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr} d\theta \right)^8 \right]^{\frac{1}{4}} \\
&\leq \mathbb{E}\left[\left(\int_0^1 e^{2\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr} d\theta \right)^4 \left(\int_0^1 e^{2\theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr} d\theta \right)^4 \right]^{\frac{1}{4}} \\
&\leq \mathbb{E}\left[\left(\int_0^1 e^{2\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr} d\theta \right)^8 \right]^{\frac{1}{8}} \mathbb{E}\left[\left(\int_0^1 e^{2\theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr} d\theta \right)^8 \right]^{\frac{1}{8}}
\end{aligned}
$$

next, we apply the Minkowski inequality to get:

$$J_2^n \leq \mathbb{E}\Big[\Big(\int_0^1 e^{2\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr} d\theta\Big)^8\Big]^{\frac{1}{8}} \mathbb{E}\Big[\Big(\int_0^1 e^{2\theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr} d\theta\Big)^8\Big]^{\frac{1}{8}}$$
$$\leq \Big(\int_0^1 \mathbb{E}\big[e^{16\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]^{\frac{1}{8}} d\theta\Big)\Big(\int_0^1 \mathbb{E}\Big[e^{16\theta \int_{t'}^t b'_2(r,X_r^{n,x},\alpha_r)dr}\Big]^{\frac{1}{8}} d\theta\Big),$$

we now use the fact that b'_2 is bounded by a random variable with exponential moment and we write:

$$J_2^n \leq C\Big(\int_0^1 \mathbb{E}\big[e^{16\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]^{\frac{1}{8}} d\theta\Big).$$

Let us now evaluate $\mathbb{E}\big[e^{16\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]$, we have applying the Girsanov transform with a change of trajectory:

$$\mathbb{E}\big[e^{16\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]$$
$$= \mathbb{E}\Big[e^{16\theta \int_{t'}^t b'_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr} \mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)\Big]$$
$$\leq \mathbb{E}\Big[e^{32\theta \int_{t'}^t b'_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr}\Big]^{\frac{1}{2}} \mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)^2\Big]^{\frac{1}{2}},$$

we then apply the Cauchy-Schwarz inequality two times and we get:

$$\mathbb{E}\big[e^{16\theta \int_{t'}^t b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]$$
$$\leq \mathbb{E}\Big[e^{32\theta \int_{t'}^t \hat{b}'_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr + 32\theta \int_{t'}^t \tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr}\Big]^{\frac{1}{2}} \mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)^2\Big]^{\frac{1}{2}}$$
$$\leq \mathbb{E}\Big[e^{64\theta \int_{t'}^t \int_{\mathbb{R}} \hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\Big]^{\frac{1}{4}} \mathbb{E}\Big[e^{64\theta \int_{t'}^t \tilde{b}'_1(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dr}\Big]^{\frac{1}{4}}$$
$$\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)^2\Big]^{\frac{1}{2}}$$
$$< \infty.$$

Therefore, using the same reasoning as in the proof of J_1^n, we can assert that J_2^n is finite. We continue with J_3^n:

$$\begin{aligned}
J_3^n &= \mathbb{E}\Bigg[\bigg(\int_{t'}^{t} b_n'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dr\bigg)^8\Bigg]^{\frac{1}{4}} \\
&= \mathbb{E}\Bigg[\bigg(\int_{t'}^{t} b_{1,n}'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr + \int_{t'}^{t} b_2'(r, X_r^{n,x}, \alpha_r)dr\bigg)^8\Bigg]^{\frac{1}{4}}, \\
&\leq \mathbb{E}\left[2^7\left(\left|\int_{t'}^{t} b_{1,n}'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8 + \left|\int_{t'}^{t} b_2'(r, X_r^{n,x}, \alpha_r)dr\right|^8\right)\right]^{\frac{1}{4}}, \\
&\leq 2^7\mathbb{E}\left[\left|\int_{t'}^{t} b_{1,n}'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{4}} + 2^7\mathbb{E}\left[\left|\int_{t'}^{t} b_2'(r, X_r^{n,x}, \alpha_r)dr\right|^8\right]^{\frac{1}{4}},
\end{aligned}$$

let us now evaluate $\mathbb{E}\left[\left|\int_{t'}^{t} b_2'(r, X_r^{n,x}, \alpha_r)dr\right|^8\right]^{\frac{1}{8}}$, we have using the Minkowski inequality:

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{t'}^{t} b_2'(r, X_r^{n,x}, \alpha_r)dr\right|^8\right]^{\frac{1}{8}} &\leq \int_{t'}^{t} \mathbb{E}[|b_2'(r, X_r^{n,x}, \alpha_r)|^8]^{\frac{1}{8}}dr \\
&\leq \int_{t'}^{t} \mathbb{E}[4!e^{|b_2'(r, X_r^{n,x}, \alpha_r)|^2}]^{\frac{1}{8}}dr \\
&\leq C|t - t'|
\end{aligned}$$

let us now evaluate $\mathbb{E}\left[\left|\int_{t'}^{t} b_{1,n}'(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}}$. In the following, we will apply the Girsanov transform for a change of trajectory and use the

Minkowski inequality and the Cauchy-Schwarz inequality to separate terms,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{t'}^{t} b'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} \\
&= \mathbb{E}\left[\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr + \int_{t'}^{t} \tilde{b}'_1(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} \\
&\leq \mathbb{E}\left[2^7\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8 + 2^7\left|\int_{t'}^{t} \tilde{b}'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} \\
&\leq C\mathbb{E}\left[\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} + C\mathbb{E}\left[\left|\int_{t'}^{t} \tilde{b}'_1(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} \\
&\leq C\mathbb{E}\left[\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} + C||\tilde{b}'_1||_\infty|t-t'| \\
&\leq C\mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dB_r\right)\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{8}} + C||\tilde{b}'_1||_\infty|t-t'| \\
&\leq C\mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r)dB_r\right)^2\right]^{\frac{1}{16}}\mathbb{E}\left[\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}})dr\right|^{16}\right]^{\frac{1}{16}} \\
&\quad + C||\tilde{b}'_1||_\infty|t-t'| \\
&\leq C\mathbb{E}\left[\left|\int_{t'}^{t} \hat{b}'_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}})dr\right|^{16}\right]^{\frac{1}{16}} + C||\tilde{b}'_1||_\infty|t-t'| \\
&= C\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}} \hat{b}'_{1,n}(r, z, \mathbb{P}_{X_r^{n,x}})L^{B^x}(dr, dz)\right|^{16}\right]^{\frac{1}{16}} + C||\tilde{b}'_1||_\infty|t-t'|,
\end{aligned}
$$

let us continue with $\mathbb{E}\Big[\Big|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\Big|^{16}\Big]^{\frac{1}{16}}$, using the decomposition in (3.22), the expression becomes:

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{16}}\\
&=\mathbb{E}\Bigg[\Bigg|\int_{t'}^{t}\hat{b}_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dB_r+\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})dW_r\\
&\quad-\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\frac{\hat{B}_r}{T-r}dr\Bigg|^{16}\Bigg]^{\frac{1}{16}}\\
&\leq C\mathbb{E}\Bigg[\Bigg|\int_{t'}^{t}\hat{b}_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dB_r\Bigg|^{16}+\Bigg|\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})dW_r\Bigg|^{16}\\
&\quad+\Bigg|\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\frac{\hat{B}_r}{T-r}dr\Bigg|^{16}\Bigg]^{\frac{1}{16}},\\
&\leq C\mathbb{E}\Bigg[\Bigg|\sup_{0\leq t\leq T}\int_{t'}^{t}\hat{b}_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})dB_r\Bigg|^{16}+\Bigg|\sup_{0\leq t\leq T}\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})dW_r\Bigg|^{16}\\
&\quad+\Bigg|\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\frac{\hat{B}_r}{T-r}dr\Bigg|^{16}\Bigg]^{\frac{1}{16}},
\end{aligned}
$$

after applying the Burkholder-Davis-Gundy inequality on the martingale terms, we get,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{16}}\\
&\leq C\mathbb{E}\Bigg[\left(\int_{t'}^{t}\left|\hat{b}_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})\right|^2dr\right)^8+\left(\int_{T-t}^{T-t'}\left|\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\right|^2dr\right)^8\\
&\quad+\Bigg|\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\frac{\hat{B}_r}{T-r}dr\Bigg|^{16}\Bigg]^{\frac{1}{16}},\\
&\leq C\mathbb{E}\left[\left(\int_{t'}^{t}\left|\hat{b}_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}})\right|^2dr\right)^8\right]^{\frac{1}{16}}+C\mathbb{E}\left[\left(\int_{T-t}^{T-t'}\left|\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\right|^2dr\right)^8\right]^{\frac{1}{16}}\\
&\quad+C\mathbb{E}\left[\left|\int_{T-t}^{T-t'}\hat{b}_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}})\frac{\hat{B}_r}{T-r}dr\right|^{16}\right]^{\frac{1}{16}},
\end{aligned}
$$

since $\hat{b}_{1,n}$ is uniformly bounded, we have,

$$
\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{16}}\leq C|t-t'|^{\frac{1}{2}}||\hat{b}_{1,n}||_\infty+C||\hat{b}_{1,n}||_\infty\mathbb{E}\left[\left|\int_{T-t}^{T-t'}\frac{B_{T-r}}{T-r}dr\right|^{16}\right]^{\frac{1}{16}},
$$

let us evaluate separately $\mathbb{E}\left[\left|\int_{T-t}^{T-t'} \frac{B_{T-r}}{T-r}dr\right|^{16}\right]^{\frac{1}{16}}$,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{T-t}^{T-t'} \frac{B_{T-r}}{T-r}dr\right|^{16}\right]^{\frac{1}{16}} &= \left(\int_{\Omega}\left|\int_{T-t}^{T-t'} \frac{B_{T-r}}{T-r}dr\right|^{16} d\mathbb{P}\right)^{\frac{1}{16}}, \\
&= \left(\int_{\Omega}\left|\int_{T-t}^{T-t'} \frac{1}{\sqrt{T-r}}\frac{B_{T-r}}{\sqrt{T-r}}dr\right|^{16} d\mathbb{P}\right)^{\frac{1}{16}}, \\
&= \left(\int_{-\infty}^{\infty}\left|\int_{T-t}^{T-t'} \frac{1}{\sqrt{T-r}}zdr\right|^{16} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right)^{\frac{1}{16}}, \\
&= \left(\left|\int_{T-t}^{T-t'} \frac{1}{\sqrt{T-r}}dr\right|^{16} \int_{-\infty}^{\infty}|z|^{16}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right)^{\frac{1}{16}}, \\
&\leq C\left|\int_{T-t}^{T-t'} \frac{1}{\sqrt{T-r}}dr\right|, \\
&= C|t'^{\frac{1}{2}} - t^{\frac{1}{2}}|, \\
&\leq C|t'-t|^{\frac{1}{2}},
\end{aligned}$$

the last inequality coming from the following version of the triangular inequality:

$$|v^{\frac{1}{p}} - w^{\frac{1}{p}}| \leq 2|v-w|^{\frac{1}{p}}, \text{ for any real number } v, w > 0 \text{ and for any } p > 1.$$

Therefore, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}} \hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{16}} \\
&\leq C|t-t'|^{\frac{1}{2}}||\hat{b}_{1,n}||_\infty + C||\hat{b}_{1,n}||_\infty \mathbb{E}\left[\left|\int_{T-t}^{T-t'} \frac{B_{T-r}}{T-r}dr\right|\right]^{\frac{1}{16}}
\end{aligned}$$

$$\begin{aligned}
&\leq C|t-t'|^{\frac{1}{2}}||\hat{b}_{1,n}||_\infty + C|t-t'|^{\frac{1}{2}}||\hat{b}_{1,n}||_\infty, \\
&\leq C|t-t'|^{\frac{1}{2}}||\hat{b}_{1,n}||_\infty,
\end{aligned}$$

thus, we can find a constant C such that:

$$\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{16}}\leq C|t-t'|^{\frac{1}{2}},$$

since we had,

$$\begin{aligned}
J_3^n &\leq 2^7\mathbb{E}\left[\left|\int_{t'}^{t} b'_{1,n}(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr\right|^8\right]^{\frac{1}{4}}+2^7\mathbb{E}\left[\left|\int_{t'}^{t} b'_2(r,X_r^{n,x},\alpha_r)dr\right|^8\right]^{\frac{1}{4}},\\
&\leq C\mathbb{E}\left[\left|\int_{t'}^{t}\int_{\mathbb{R}}\hat{b}'_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)\right|^{16}\right]^{\frac{1}{8}}+C||\tilde{b}'_1||^2_\infty|t-t'|^2+C|t-t'|^2,\\
&\leq C|t-t'|||\hat{b}_{1,n}||^2_\infty+C||\tilde{b}'_1||^2_\infty|t-t'|^2+C|t-t'|^2\\
&\leq C|t-t'|,
\end{aligned}$$

after obtaining all these estimates, let us return back to $\mathbb{E}[|I_1|^2]$, we had:

$$\begin{aligned}
\mathbb{E}[|I_1|^2] &\leq CJ_1^n\cdot J_2^n\cdot J_3^n,\\
&\leq C|t-t'|,
\end{aligned}$$

Second step: we compute $\mathbb{E}[|I_2|^2]$:

$$\mathbb{E}[|I_2|^2]=\mathbb{E}\left[\left(\int_{t'}^{t}e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\right)^2\right],$$

after applying the Cauchy-Schwarz inequality several times, we get,

$$\begin{aligned}
&\mathbb{E}[|I_2|^2]\\
&=\mathbb{E}\left[\left(\int_{t'}^{t}e^{-\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}D_{t'}b_2(u,X_u^{n,x},\alpha_u)du\right)^2\right]\\
&\leq\mathbb{E}\left[\left(\int_{t'}^{t}(D_{t'}b_2(u,X_u^{n,x},\alpha_u))^2du\right)\cdot\left(\int_{t'}^{t}e^{-2\int_s^u b'_n(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\right)\right]
\end{aligned}$$

$$\leq \mathbb{E}\Big[\Big(\int_{t'}^{t}(D_{t'}b_2(u,X_u^{n,x},\alpha_u))^2du\Big)^2\Big]^{\frac{1}{2}}\cdot\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-4\int_s^u b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\Big)\Big]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\Big[\Big(\int_{t'}^{t}|\tilde{M}_2(u,t',\omega)|^2du\Big)^4\Big]^{\frac{1}{4}}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-4\int_s^u b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\Big)\Big]^{\frac{1}{2}}$$

$$\leq C_{L^p}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-8\int_s^u b_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\Big]^{\frac{1}{4}}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-8\int_s^u b_2'(r,X_r^{n,x},\alpha_r)dr}du\Big)\Big]^{\frac{1}{4}}$$

$$\leq C_{L^p}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u \hat{b}_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\Big]^{\frac{1}{8}}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u \tilde{b}_1'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\Big]^{\frac{1}{8}}$$

$$\times\Big(\int_{t'}^{t}\mathbb{E}\big[e^{-8\int_s^u b_2'(r,X_r^{n,x},\alpha_r)dr}\big]du\Big)^{\frac{1}{4}}$$

we now apply the Girsanov transform to get:

$$\mathbb{E}[|I_2|^2]$$

$$\leq C_{L^p}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u \hat{b}_{1,n}'(r,B_r^{x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)\Big]^{\frac{1}{8}}$$

$$\times\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u \tilde{b}_1'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\Big]^{\frac{1}{8}}\times\Big(\int_{t'}^{t}\mathbb{E}\big[e^{-8\int_s^u b_2'(r,X_r^{n,x},\alpha_r)dr}\big]du\Big)^{\frac{1}{4}}$$

$$\leq C_{L^p}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u \hat{b}_{1,n}'(r,B_r^{x},\mathbb{P}_{X_r^{n,x}})dr}du\Big)\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)\Big]^{\frac{1}{8}}$$

$$\times|t-t'|^{\frac{1}{8}}\times|t-t'|^{\frac{1}{4}}$$

$$\leq C_{L^p}\mathbb{E}\Big[\Big(\int_{t'}^{t}e^{-16\int_s^u\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}du\Big)^2\Big]^{\frac{1}{16}}$$

$$\times\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)^2\Big]^{\frac{1}{16}}|t-t'|^{\frac{1}{8}}\times|t-t'|^{\frac{1}{4}}$$

$$\leq C_{L^p}\Big(\int_{t'}^{t}\mathbb{E}\big[e^{-32\int_s^u\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\big]^{\frac{1}{2}}du\Big)^{\frac{1}{8}}|t-t'|^{\frac{1}{8}}\times|t-t'|^{\frac{1}{4}},$$

$$\leq C_{L^p}|t-t'|^{\frac{1}{2}},$$

by Lemma 3.3.2, we have $\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\Big)^2\Big]$ which is finite, and also, we used the following assumption:

$$L^p := \sup_{0\leq t'\leq T}\mathbb{E}\Big[\Big(\int_0^T|\hat{L}(u,t',\omega)|^2du\Big)^4\Big]<\infty,$$

third step: we compute $\mathbb{E}[|I_3|^2]$:

$$\mathbb{E}[|I_3|^2] = \mathbb{E}\left[\left(\int_t^s e^{-\int_s^u b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}(D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u))du\right)^2\right],$$

after applying Cauchy-Schwarz inequality, we get,

$$\mathbb{E}[|I_3|^2] \leq \mathbb{E}\left[\left(\int_t^s e^{-\int_s^u 2b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\right)\left(\int_t^s |D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u)|^2 du\right)\right],$$

we next apply the Cauchy-Schwarz inequality and the Minkowski inequality to get,

$$\begin{aligned}
&\mathbb{E}[|I_3|^2] \\
&\leq \mathbb{E}\left[\left(\int_t^s e^{-\int_s^u 2b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\right)\left(\int_t^s |D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u)|^2 du\right)\right] \\
&\leq \mathbb{E}\left[\left(\int_t^s e^{-\int_s^u 2b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}du\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_t^s |D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u)|^2 du\right)^2\right]^{\frac{1}{2}} \\
&\leq \left(\int_t^s \mathbb{E}\big[e^{-\int_s^u 4b_n'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\big]^{\frac{1}{2}}du\right)\mathbb{E}\left[\left(\int_t^s |D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u)|^2 du\right)^2\right]^{\frac{1}{2}} \\
&\leq \left(\int_t^s \mathbb{E}\big[e^{-\int_s^u 16\hat{b}_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]^{\frac{1}{8}}\mathbb{E}\big[e^{-\int_s^u 16\tilde{b}_1'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]^{\frac{1}{8}}\mathbb{E}\big[e^{-\int_s^u 8b_2'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}},\alpha_r)dr}\big]^{\frac{1}{4}}du\right) \\
&\quad\times\left(\int_t^s \mathbb{E}\big[|D_{t'}b_2(u,X_u^{n,x},\alpha_u) - D_t b_2(u,X_u^{n,x},\alpha_u)|^4\big]^{\frac{1}{2}}du\right) \\
&\leq C\left(\int_t^s \mathbb{E}\big[e^{-\int_s^u 16\hat{b}_{1,n}'(r,X_r^{n,x},\mathbb{P}_{X_r^{n,x}})dr}\big]^{\frac{1}{8}}du\right)\times|s-t||t-t'|^{\frac{\beta}{2}} \\
&\leq C\left(\int_t^s \mathbb{E}\big[e^{-\int_s^u\int_{\mathbb{R}} 32\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\big]^{\frac{1}{16}}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r)dB_r\right)^2\right]^{\frac{1}{16}}du\right) \\
&\quad\times|s-t||t-t'|^{\frac{\beta}{2}} \\
&\leq C|s-t|^2|t-t'|^{\frac{\beta}{2}}.
\end{aligned}$$

Therefore,

$$\mathbb{E}[|I_3|^2] \leq C|s-t|^2|t-t'|^{\frac{\beta}{2}}.$$

After putting everything together, we get the compactness argument as follows,

$$\begin{aligned}
\mathbb{E}[|D_{t'}X_s^{n,x} - D_t X_s^{n,x}|^2] &\leq 3\mathbb{E}[|I_1|^2] + 3\mathbb{E}[|I_2|^2] + 3\mathbb{E}[|I_3|^2], \\
&\leq 3|t-t'| + 3C_{L^p}|t-t'|^{\frac{1}{2}} + 3CT^2|t-t'|^{\frac{\beta}{2}}, \\
&\leq C_{T,L^p}|t-t'|^m,
\end{aligned}$$

where we can find a constant C_{T,L^p} depending on T and L^p, with $m = \min\left(\frac{1}{2}, \frac{\beta}{2}\right)$. Consequently, the first part of Lemma 3.3.1 is proved. For the second part, let us notice that taking $t' > s$ yields $D_{t'}X_s^{n,x} = 0$, which means:

$$\sup_{0\le t\le T} \mathbb{E}[|D_t X_s^{n,x}|^2] \le C_{T,L^p},$$

and this is the end of the proof. □

3.4 Weak convergence of $X_t^{n,x}$ to $\mathbb{E}[X_t^x|\mathcal{F}_t]$ in L^2

This step consists in proving that the aforementioned sequence $(X_t^{n,x})_{n\ge 0}$ weakly converges to $\mathbb{E}[X_t^x|\mathcal{F}_t]$ in the space L^2 for each $0 \le t \le T$. Therefore, we state the following lemma:

Lemma 3.4.1. Assume $L^e < \infty$ and Ω is considered to be the canonical space. We consider a sequence $b_{1,n} : [0,T] \times \mathbb{R} \to \mathbb{R}$, $n \ge 1$ as expressed in (3.12), and we denote by $(X_t^{n,x})_{n\ge 1}$ the corresponding strong solutions to the MFSDE (3.13). Hence, for each $0 \le t \le T$ with T sufficiently small, for each function $\kappa : \mathbb{R} \to \mathbb{R}$ of polynomial growth, the sequence $(\kappa(X_t^{n,x}))_{n\ge 1}$ is uniformly bounded in L^2 and weakly converges to $\mathbb{E}[\kappa(X_t^x)|\mathcal{F}_t]$ in this space.

Proof of Lemma 3.4.1. As done in Lemma 2.5 in Menoukeu-Pamen & Tangpi (2019), let us first show that $(\kappa(X_t^{n,x}))_{n\ge 1}$ is uniformly bounded in L^2.

$$\sup_{n\ge 1} \mathbb{E}[|\kappa(X_t^{n,x})|^2] = \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) dB_r\right) |\kappa(B_t^x)|^2\right],$$

after applying Cauchy-Schwarz inequality, we get,

$$\begin{aligned}\sup_{n\ge 1} \mathbb{E}[|\kappa(X_t^n)|^2] &\le \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) dB_r\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[|\kappa(B_t^x)|^4\right]^{\frac{1}{2}},\\ &\le \left(\sup_{x\in K} \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) dB_r\right)^2\right]\right)^{\frac{1}{2}} \mathbb{E}\left[|\kappa(B_t^x)|^4\right]^{\frac{1}{2}},\end{aligned}$$

after applying Lemma 3.3.2, we have,

$$\sup_{n\geq 1}\mathbb{E}[|\kappa(X_t^{n,x})|^2]\leq C\mathbb{E}\left[|\kappa(B_t^x)|^4\right]^{\frac{1}{2}},$$

let us now evaluate $\mathbb{E}\left[|\kappa(B_t^x)|^4\right]$,

$$\mathbb{E}\left[|\kappa(B_t^x)|^4\right]=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}|\kappa(x+z)|^4e^{-\frac{|z|^2}{2t}}dz,$$

since κ is of polynomial growth and also using the following inequality,

$$(v+w)^q\leq 2^{q-1}(v^q+w^q),$$

we have,

$$\begin{aligned}\mathbb{E}\left[|\kappa(B_t^x)|^4\right]&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}|\kappa(x+z)|^4e^{-\frac{|z|^2}{2t}}dz,\\&\leq\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}C(1+|x+z|^4)e^{-\frac{|z|^2}{2t}}dz,\\&\leq\frac{C}{\sqrt{2\pi t}}\int_{\mathbb{R}}(1+|x|^4+|z|^4)e^{-\frac{|z|^2}{2t}}dz,\\&=\frac{C}{\sqrt{2\pi t}}\int_{\mathbb{R}}(1+|z|^4)e^{-\frac{|z|^2}{2t}}dz+\frac{C}{\sqrt{2\pi t}}\int_{\mathbb{R}}|x|^4e^{-\frac{|z|^2}{2t}}dz,\end{aligned}$$

given the following bound:

$$(1+|z|^p)e^{-\frac{|z|^2}{2s}}<C_pe^{-\frac{|z|^2}{2^{p+1}s}},$$

we get,

$$\begin{aligned}\mathbb{E}\left[|\kappa(B_t^x)|^4\right]&<\frac{C}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-\frac{|z|^2}{2^5t}}dz+\frac{C}{\sqrt{2\pi t}}\int_{\mathbb{R}}|x|^4e^{-\frac{|z|^2}{2t}}dz,\\&<\infty,\end{aligned}$$

therefore,

$$\begin{aligned}\sup_{n\geq 1}\mathbb{E}[|\kappa(X_t^{n,x})|^2]&\leq C\mathbb{E}\left[|\kappa(x+B_t)|^4\right]^{\frac{1}{2}},\\&<\infty,\end{aligned}$$

after establishing the boundedness of $(\kappa(X_t^{n,x}))_{n\geq 1}$, let us now show that the sequence $(\kappa(X_t^{n,x}))_{n\geq 1}$ converges weakly to $\mathbb{E}[\kappa(X_t^x)|\mathcal{F}_t]$, in L^2. The space,

$$\left\{\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right) : \varphi \in C_b^1([0,T],\mathbb{R})\right\} \tag{3.25}$$

spans $L^2(\Omega,\mathbb{P})$. $\dot{\varphi}$ is the derivative of φ with respect to time, $C_b^1([0,T],\mathbb{R})$ is the space of continuous bounded functions that are differentiable on $[0,T]$ and with values in $\mathbb{R}$. Consequently, we show the weak convergence of $(\kappa(X_t^{n,x}))_{\geq 1}$ to $\mathbb{E}[X_t^x|\mathcal{F}_t]$ by proving convergence in expectation of $\left(\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)\right)_{n\geq 1}$ to $\mathbb{E}\left[\kappa(X_t^x)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)$. Since Ω is a Wiener space, the Cameron-Martin theorem states that for every κ measurable,

$$\mathbb{E}\left[\kappa(X_t^x)\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)\right] = \int_\Omega \kappa(X_t^x(\omega+\varphi))d\mathbb{P}(\omega), \tag{3.26}$$

let $\varphi \in C_b^1([0,T],\mathbb{R})$, the process $\tilde{X}^{n,x}$ defined by $\tilde{X}^{n,x}(\omega) := X^{n,x}(\omega+\varphi)$ is solution to the stochastic differential equation,

$$d\tilde{X}_t^{n,x} = (b_{1,n}(t,\tilde{X}_t^{n,x},\mathbb{P}_{X_t^{n,x}}) + \tilde{b}_2(t,\tilde{X}_t^{n,x},\alpha_t) + \dot{\varphi}_t)dt + dB_t, \tag{3.27}$$

for every $n\geq 1$, where $\tilde{b}_2(t,z,\alpha(\omega)) = b_2(t,z,\alpha(\omega+\varphi))$. To see where the state dynamics for $\tilde{X}_t^{n,x}$ comes from, let $\Gamma \in L^2(\Omega,\mathbb{P})$, make use 3.26 and the fact that $X^{n,x}$ is solution of the stochastic differential equation 3.13 to obtain:

$$\begin{aligned}
&\mathbb{E}[\tilde{X}_t^{n,x}\Gamma(\omega)]\\
&= \mathbb{E}[X_t^{n,x}(\omega+\varphi)\Gamma(\omega)] = \mathbb{E}\left[X_t^{n,x}(\omega)\Gamma(\omega-\varphi)\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)\right]\\
&= \mathbb{E}\left[\left(x + \int_0^t (b_{1,n}(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x}}) + b_2(u,X_u^{n,x},\alpha_u))du + B_t(\omega)\right)\Gamma(\omega-\varphi)\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)\right]\\
&= \mathbb{E}\left[\left(x + \int_0^t (b_{1,n}(u,X_u^{n,x}(\omega+\varphi),\mathbb{P}_{X_u^{n,x}}) + b_2(u,X_u^{n,x}(\omega+\varphi),\alpha_u(\omega+\varphi))du + B_t(\omega+\varphi)\right)\Gamma(\omega)\right]\\
&= \mathbb{E}\left[\left(x + \int_0^t (b_{1,n}(u,X_u^{n,x}(\omega+\varphi),\mathbb{P}_{X_u^{n,x}}) + b_2(u,X_u^{n,x}(\omega+\varphi),\alpha_u(\omega+\varphi))du + B_t(\omega)+\varphi\right)\Gamma(\omega)\right]\\
&= \mathbb{E}\left[\left(x + \int_0^t (b_{1,n}(u,X_u^{n,x}(\omega+\varphi),\mathbb{P}_{X_u^{n,x}}) + b_2(u,X_u^{n,x}(\omega+\varphi),\alpha_u(\omega+\varphi)+\dot{\varphi}_u)du + B_t(\omega)\right)\Gamma(\omega)\right],
\end{aligned}$$

where the last equality holds true to the fact that $B_t(\omega+\varphi) = B_t(\omega) + \varphi$, since B is by definition the canonical process. Consequently, the last equality shows that $\tilde{X}^{n,x}(\omega) := X^{n,x}(\omega+\varphi)$ satisfies the dynamics 3.27 $\mathbb{P}-a.s.$. Now, let us get back to showing that $\left(\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)\right)_{n\geq 1}$ converges to $\mathbb{E}\left[\kappa(X_t)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_u dB_u\right)$ in expectation. We denote,

$$\begin{cases} \tilde{b}_n(t,\tilde{X}_t^{n,x},\mathbb{P}_{X_t^{n,x}},\alpha_t) = b_n(t,\tilde{X}_t^{n,x},\mathbb{P}_{X_t^{n,x}},\alpha_t(\omega+\varphi)) \\ = b_{1,n}(t,\tilde{X}_t^{n,x},\mathbb{P}_{X_t^{n,x}}) + b_2(t,\tilde{X}_t^{n,x},\alpha_t(\omega+\varphi)) \\ = b_{1,n}(t,\tilde{X}_t^{n,x},\mathbb{P}_{X_t^{n,x}}) + \tilde{b}_2(t,\tilde{X}_t^{n,x},\alpha_t), \\ \tilde{b}(t,\tilde{X}_t,\mathbb{P}_{X_t^x},\alpha_t) = b(t,\tilde{X}_t^x,\mathbb{P}_{X_t^x},\alpha_t(\omega+\varphi)) \\ = b_1(t,\tilde{X}_t^x,\mathbb{P}_{X_t^x}) + b_2(t,\tilde{X}_t^x,\alpha_t(\omega+\varphi)) \\ = b_1(t,\tilde{X}_t^x,\mu_t) + \tilde{b}_2(t,\tilde{X}_t^x,\alpha_t), \end{cases} \tag{3.28}$$

we have,

$$\begin{aligned}
&\mathbb{E}\left[\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right) - \mathbb{E}\left[\kappa(X_t^x)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right] \\
&= \mathbb{E}\left[\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right] - \mathbb{E}\left[\kappa(X_t^x)\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right] \\
&= \mathbb{E}\left[(\kappa(X_t^{n,x}) - \kappa(X_t^x))\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right] \\
&= \mathbb{E}\left[\kappa(\tilde{X}_t^{n,x}) - \kappa(\tilde{X}_t^x)\right] \\
&= \mathbb{E}\left[\kappa(B_t^x)\mathcal{E}\left(\int_0^T (\tilde{b}_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r) + \dot{\varphi}_r)dB_r\right) - \kappa(B_t^x)\mathcal{E}\left(\int_0^T (\tilde{b}(r,B_r^x,\mathbb{P}_{X_r^x},\alpha_r) + \dot{\varphi}_r)dB_r\right)\right] \\
&= \mathbb{E}\left[\kappa(B_t^x)\left(\mathcal{E}\left(\int_0^T (\tilde{b}_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r) + \dot{\varphi}_r)dB_r\right) - \mathcal{E}\left(\int_0^T (\tilde{b}(r,B_r^x,\mathbb{P}_{X_r^x},\alpha_r) + \dot{\varphi}_r)dB_r\right)\right)\right],
\end{aligned}$$

applying the inequality $|e^v - e^w| \leq |e^v + e^w||v-w|$, we get,

$$\begin{aligned}
&\mathbb{E}\left[\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right) - \mathbb{E}\left[\kappa(X_t^x)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right] \\
&\leq \mathbb{E}\left[|\kappa(B_t^x)|\left|\mathcal{E}\left(\int_0^T (\tilde{b}_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\alpha_r) + \dot{\varphi}_r)dB_r\right) - \mathcal{E}\left(\int_0^T (\tilde{b}(r,B_r^x,\mathbb{P}_{X_r^x},\alpha_r) + \dot{\varphi}_r)dB_r\right)\right|\right]
\end{aligned}$$

$$\leq \mathbb{E}\Big[|\kappa(B_t^x)|\Big|\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right) + \mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)\Big|$$
$$\times \Big|\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r - \frac{1}{2}\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 dr$$
$$- \int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r + \frac{1}{2}\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2 dr\Big|\Big],$$

below we separate the two Doléans-Dade exponentials from the remaining terms using the Cauchy-Schwarz inequality and we apply Lemma 3.3.2:

$$\mathbb{E}\left[\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right) - \mathbb{E}\left[\kappa(X_t^x)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right]$$
$$\leq \mathbb{E}\Big[|\kappa(B_t^x)|^2\Big|\int_0^T \tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)dB_r$$
$$- \frac{1}{2}\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2 dr\Big|^2\Big]^{\frac{1}{2}}$$
$$\times \mathbb{E}\Big[\Big|\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right) + \mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)\Big|^2\Big]^{\frac{1}{2}}$$
$$\leq \mathbb{E}[|\kappa(B_t^x)|^4]^{\frac{1}{4}}$$
$$\times \mathbb{E}\Big[\Big|\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r))dB_r - \frac{1}{2}\int_0^T ((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2$$
$$- (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2)dr\Big|^4\Big]^{\frac{1}{4}} \times \mathbb{E}\Big[\Big|\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right)$$
$$+ \mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)\Big|^2\Big]^{\frac{1}{2}}$$
$$= J_1 \times J_2^n \times J_3^n,$$

we have shown before that $\mathbb{E}[|\kappa(B_r^x)|^4]$ is finite, therefore $J_1 < \infty$. Next, we continue with J_3^n,

$$J_3^n$$
$$= \mathbb{E}\left[\Big|\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right) + \mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)\Big|^2\right]^{\frac{1}{2}}$$
$$\leq \left(2\mathbb{E}\left[\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right)^2\right] + 2\mathbb{E}\left[\mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)^2\right]\right)^{\frac{1}{2}}$$
$$\leq C\mathbb{E}\left[\mathcal{E}\left(\int_0^T (\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)dB_r\right)^2\right]^{\frac{1}{2}} + C\mathbb{E}\left[\mathcal{E}\left(\int_0^T (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)dB_r\right)^2\right]^{\frac{1}{2}}$$
$$< \infty.$$

The last line holds true due to Lemma 3.3.2. Indeed, J_3^n is bounded uniformly in n. Let us continue with J_2^n, we have:

$$\begin{aligned}
&J_2^n \\
&= \mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r \\
&\quad - \frac{1}{2}\int_0^T \big((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2\big) dr\Big|^4\Big]^{\frac{1}{4}} \\
&\leq C\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4 \\
&\quad + \Big|\int_0^T \big((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2\big) dr\Big|^4\Big]^{\frac{1}{4}} \\
&\leq C\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}} \\
&\quad + C\mathbb{E}\Big[\Big|\int_0^T \big((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2\big) dr\Big|^4\Big]^{\frac{1}{4}},
\end{aligned}$$

let us now evaluate $\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}}$, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}} \\
&\leq \mathbb{E}\Big[\Big|\sup_{0\leq t\leq T}\int_0^t \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}},
\end{aligned}$$

after applying Burkholder-Davis-Gundy inequality, we get,

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}} \\
&\leq \mathbb{E}\Big[\Big|\sup_{0\leq t\leq T}\int_0^t \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}}, \\
&\leq C\mathbb{E}\Big[\Big(\int_0^T |\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^2 dr\Big)^2\Big]^{\frac{1}{4}},
\end{aligned}$$

after applying the Minkowski inequality to the expression above, we get,

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^T \big(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big) dB_r\Big|^4\Big]^{\frac{1}{4}} \\
&\leq \left(\int_0^T \left(\int_{\alpha_r} |\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4 d\mathbb{P}\right)^{\frac{1}{2}} dr\right)^{\frac{1}{2}} \\
&= \left(\int_0^T \mathbb{E}\left[|\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4\right]^{\frac{1}{2}} dr\right)^{\frac{1}{2}},
\end{aligned}$$

we continue as follows,

$$\begin{aligned}
&\mathbb{E}\left[|\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4\right]^{\frac{1}{4}} \\
&= \mathbb{E}\left[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) + \tilde{b}_2(r, B_r^x, \alpha_r) - b_1(r, B_r^x, \mathbb{P}_{X_r^x}) - \tilde{b}_2(r, B_r^x, \alpha_r)|^4\right]^{\frac{1}{4}} \\
&\leq C\mathbb{E}\big[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x})|^4 \\
&\quad + |b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) - b_1(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4\big]^{\frac{1}{4}} \\
&\leq C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + \mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^4]^{\frac{1}{4}} \\
&= C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + \left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^4 \frac{1}{\sqrt{2\pi r}} e^{-\frac{(z-x)^2}{2r}} dz\right)^{\frac{1}{4}} \\
&= C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + \left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mu_r)|^4 \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} e^{-\frac{(z-2x)^2}{4r}} e^{\frac{x^2}{2r}} dz\right)^{\frac{1}{4}} \\
&\leq C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + e^{\frac{x^2}{8r}} \left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^4 \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} dz\right)^{\frac{1}{4}},
\end{aligned}$$

the last inequality follows from the following inequality,

$$\begin{aligned}
e^{-\frac{(z-x)^2}{2r}} &= e^{-\frac{z^2}{4r}} e^{-\frac{(z-2x)^2}{4r}} e^{\frac{x^2}{2r}}, \\
&\leq e^{-\frac{z^2}{4r}} e^{\frac{x^2}{2r}}.
\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[|\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4\right]^{\frac{1}{4}} \\
&\leq C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + e^{\frac{x^2}{8r}} \left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^4 \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} dz\right)^{\frac{1}{4}},
\end{aligned}$$

the second term will converge by dominated convergence as n grows large.

We are left with showing that $\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x})$ will converge to 0 as n grows large. By definition, we have:

$$\begin{aligned}\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) &\leq \mathbb{E}[|X_r^{n,x} - X_r^x|] \\ &= \mathbb{E}\Big[|B_r^x|\Big|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\Big|\Big] \\ &\leq \mathbb{E}\big[|B_r^x|^2\big]^{\frac{1}{2}}\mathbb{E}\Big[\Big|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\Big|^2\Big]^{\frac{1}{2}},\end{aligned} \tag{3.29}$$

since $\mathbb{E}\big[|B_r^x|^2\big]^{\frac{1}{2}}$ is finite, convergence follows by application of Lemma 3.5.5. We continue by evaluating the second term of J_2^n which is,

$\mathbb{E}\Big[\Big|\int_0^T ((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2)dr\Big|^4\Big]^{\frac{1}{4}}$, we also apply the Minkowski inequality to get,

$$\begin{aligned}&\mathbb{E}\Big[\Big|\int_0^T ((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2)dr\Big|^4\Big]^{\frac{1}{4}} \\ &\leq \int_0^T \mathbb{E}\big[|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2|^4\big]^{\frac{1}{4}}dr \\ &= \int_0^T \mathbb{E}\big[|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)^2|^4\big]^{\frac{1}{4}}dr \\ &\leq \int_0^T \mathbb{E}\big[|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r) - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)|^8\big]^{\frac{1}{8}} \\ &\quad \times \mathbb{E}\big[|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r) + (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}\alpha_r) + \dot{\varphi}_r)|^8\big]^{\frac{1}{8}}dr,\end{aligned}$$

where the last inequality holds by using the following identity:

$$v^2 - w^2 = (v - w)(v + w),$$

since the estimate (3.34) in Lemma 3.5.2 states that:

$$\sup_{x \in K} \mathbb{E}\left[\sup_{t \in [0,T]} |b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)|^p\right] < \infty,$$

we have:

$$\mathbb{E}\big[\big|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r) + (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}\alpha_r) + \dot{\varphi}_r)\big|^8\big] < \infty,$$

we can continue as follows:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^T ((\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r)^2 - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \omega) + \dot{\varphi}_r)^2)dr\Big|^4\Big]^{\frac{1}{4}} \\
&\leq C\int_0^T \mathbb{E}\big[\big|(\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) + \dot{\varphi}_r) - (\tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) + \dot{\varphi}_r)\big|^8\big]^{\frac{1}{8}} dr \\
&\leq C\int_0^T \mathbb{E}\big[\big|\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)\big|^8\big]^{\frac{1}{8}} dr \\
&\leq C\int_0^T \big(\mathbb{E}\big[\big|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x}) + \tilde{b}_2(r, B_r^x, \alpha_r) - \tilde{b}_2(r, B_r^x, \alpha_r)\big|^8\big]^{\frac{1}{8}}\big) dr \\
&= C\int_0^T \mathbb{E}\big[\big|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})\big|^8\big]^{\frac{1}{8}} dr \\
&= C\int_0^T \mathbb{E}\big[\big|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) + b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})\big|^8\big]^{\frac{1}{8}} dr \\
&\leq C\int_0^T \big(\mathbb{E}\big[\big|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x})\big|^8\big]^{\frac{1}{8}} + \mathbb{E}\big[\big|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) \\
&\quad - b_1(r, B_r^x, \mathbb{P}_{X_r^x})\big|^8\big]^{\frac{1}{8}}\big) dr \\
&\leq C\int_0^T \mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + e^{\frac{x^2}{16r}}\left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^8 \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} dz\right)^{\frac{1}{8}} dr
\end{aligned}$$

combining the results for J_2^n, we get,

$$\begin{aligned}
J_2^n \leq C\Big\{\int_0^T \Big[\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + e^{\frac{x^2}{8r}}\left(\int_{\mathbb{R}} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^4 \frac{1}{\sqrt{2\pi r}} e^{-\frac{x^2}{4r}} dz\right)^{\frac{1}{4}}\Big]^2 dr\Big\}^{\frac{1}{2}} \\
+ C\int_0^T \mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x}) + e^{\frac{x^2}{16r}}\left(\int_{\mathbb{R}} |\tilde{b}_{1,n}(r, z, \mathbb{P}_{X_r^x}) - \tilde{b}_1(r, z, \mathbb{P}_{X_r^x})|^8 \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} dz\right)^{\frac{1}{8}} dr,
\end{aligned}$$

as $n \to \infty$, we have $J_2^n \to 0$ by dominated convergence and by convergence of $\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x})$ towards 0 as n tends to ∞, which has shown when proving the convergence of $\mathbb{E}\left[|\tilde{b}_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) - \tilde{b}(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r)|^4\right]^{\frac{1}{4}}$. As $n \to \infty$, we have J_1 finite, J_3^n finite and $J_2^n \to 0$.

Thus, $\mathbb{E}\left[\kappa(X_t^{n,x})\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right) - \mathbb{E}\left[\kappa(X_t^x)|\mathcal{F}_t\right]\mathcal{E}\left(\int_0^T \dot{\varphi}_r dB_r\right)\right]$ tends to 0 as $n \to \infty$. Proving therefore the weak convergence of $\kappa(X_t^{n,x})$ to $\mathbb{E}[\kappa(X_t^x)|\mathcal{F}_t]$.

This is where we end our proof. □

Before going to the final part, let us state and prove the following proposition:

Proposition 3.4.0.1. For any t such that $0 \leq t \leq T$ with T sufficiently small, and $z \in \mathbb{R}$, we have strong convergence of the sequence $(X_t^{n,x})_{n\geq 1}$ of strong solutions of the stochastic differential equation (3.13) to $\mathbb{E}[X_t^x|\mathcal{F}_t] = X_t^x$ in $L^2(\Omega, \mathbb{P}; \mathbb{R})$.

Proof of Proposition 3.4.0.1. The starting point of our argument is to notice that by the compactness argument in Lemma 3.3.1, for each t such that $0 \leq t \leq T$, we can find a subsequence $(X_t^{n_k,x})_{k\geq 1}$ that converges strongly to $\mathbb{E}[X_t^x|\mathcal{F}_t]$ in $L^2(\Omega, \mathbb{P})$. We can notice from Lemma 3.4.1 that we get by considering $\kappa(z) = z, z \in \mathbb{R}$ that $(X_t^{n,x})_{n\geq 1}$ weakly converges in $L^2(\Omega, \mathbb{P})$ to $\mathbb{E}[X_t^x|\mathcal{F}_t]$. Consequently, by the uniqueness of the limit, we can find a subsequence $(n_k)_{k\geq 1}$ such that we have $(X_t^{n_k,x})_{k\geq 1}$ which strongly converges to $\mathbb{E}[X_t^x|\mathcal{F}_t]$ in $L^2(\Omega, \mathbb{P})$. Thus, the strong convergence holds not only for the subsequence, but for the entire sequence by the uniqueness of the limit. □

3.5 Representation of the Stochastic Differential Flow by Time-Space Local Time

It has been shown in the literature that, under some conditions fulfilled by the coefficients of a stochastic differential equation, there exist a flow process, derivative of the solutions of the stochastic differential equation, however it is a derivative in the sense of distribution. Bouleau & Hirsch (1988) proved the existence of the stochastic differential flow when the drift and the diffusion of the SDE are Lipschitz and have the linear growth property. Menoukeu-Pamen & Tangpi (2019) investigated the representation of the flow for the solutions of an SDE having a random drift coefficient. The existence and representation of the flow this time for a mean-field stochastic differential equation has been studied in Bauer et al. (2018), where the authors consider their drift coefficient to be at most linear growth and continuous in the measure variable. Our goal is to investigate the representation of the flow for solutions of a MFSDE having a random drift coefficient. Following this direction, let us state the following theorems:

Theorem 3.5.1. *Assume that the drift b can be decomposed as in (3.3) and b_1 is uniformly Lipschitz continuous in the measure variable (3.7). The first variation process (in the Sobolev sense) of the strong unique solution $(X_t^x)_{0\leq t\leq T}$ of the MFSDE (3.32) admits $dt\otimes d\mathbb{P}$ almost surely the representation,*

$$\mathcal{G}^{\alpha}_{0,t} := e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r} \tag{3.30}$$
$$+\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;X_s^x)\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}]\mathrm{d}s.$$

for every $x\in\mathbb{R}$, $0\leq s\leq t\leq T$, where $-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(dr,dz)$ represents the integration w.r.t the time-space local time of X.

Theorem 3.5.2. *Assume that the drift b can be decomposed as in (3.3) and b_1 is Lipschitz continuous uniformly in the measure variable (3.7). We consider $(X_t^x)_{0\leq t\leq T}$ as the strong unique solution of the MFSDE (3.32). Thus, we can find a constant $C>0$ sastisfying the following:*

$$\mathbb{E}[|X_t^x - X_{t'}^z|^2]\leq C(|t-t'|+|x-z|^2),$$

$\forall t,t'\in[0,T]$ and $x,z\in K$ with K a compact subset of $\mathbb{R}$.

Before developing the proof of Theorem 3.5.1, let us state and prove the following lemmas and propositions.

Lemma 3.5.1. Assume that the drift b can be decomposed as in (3.3) and b_1 can also be decomposed as in (3.5). Consider $(X_t^x)_{t\in[0,T]}$, the unique strong solution of the MFSDE (3.32). In addition, we take $\{b_n\}_{n\geq 0}$ as the approximating sequence of b as expressed in (3.45). Also, $(X_t^{n,x})_{n\geq 0}$ is the corresponding strong solution of the MFSDE (3.46). Consequently,

$$\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]<\infty, \tag{3.31}$$

for every compact subset K in $\mathbb{R}$ and $\beta\in\mathbb{R}$.

Proof of Lemma 3.5.1. We have,

$$\begin{aligned}&\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]\\&=\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}e^{\int_s^t\tilde{b}_1'(r,z,\mathbb{P}_{X_r^{n,x}})dr}\right],\end{aligned}$$

from the assumptions made in (3.5), we know that $\tilde{b}_1'$ is a bounded quantity. We are left with,

$$\begin{aligned}&\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]\\&\leq\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}C\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}\hat{b}_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right],\end{aligned}$$

since $\hat{b}_1$ is bounded and measurable, Lemma 3.3.3 applies uniformly in n. Hence,

$$\sup_{n\geq 0}\sup_{0\leq t\leq T}\sup_{x\in K}\mathbb{E}\left[e^{-\beta\int_s^t\int_{\mathbb{R}}b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]<\infty,$$

which proves the lemma. □

Lemma 3.5.2. Let us consider a measurable function $b:[0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})\times\Omega\to\mathbb{R}$ fulfilling the property of linear growth (3.8). In addition, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P},B,X)$ be a weak solution of the MFSDE:

$$dX_t^x=b(t,X_t^x,\mathbb{P}_{X_t^x},\alpha_t)dt+dB_t,\qquad X_0^x=x,\quad t\in[0,T]. \tag{3.32}$$

Thus, we can find a constant C satisfying the following:

$$|b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)| \le C \left(1 + C(\omega) + |x| + \sup_{u\in[0,T]} |B_u|\right), \tag{3.33}$$

therefore, for any compact set $K \subset \mathbb{R}$, and $1 \le p < \infty$, we can find an ϵ where the following estimates hold:

$$\sup_{x\in K} \mathbb{E}\left[\sup_{t\in[0,T]} |b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)|^p\right] < \infty, \tag{3.34}$$

$$\sup_{x\in K} \sup_{t\in[0,T]} \mathbb{E}\left[|X_t^x|^p\right] < \infty, \tag{3.35}$$

Proof of Lemma 3.5.2. we have seen from the proof of Lemma 3.3.2 that:

$$\mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0) \le \mathbb{E}[|X_t^x|],$$

now, let us evaluate $\mathbb{E}[|X_t^x|]$,

$$\begin{aligned}
\mathbb{E}[|X_t^x|] &= \mathbb{E}\left[\left|x + \int_0^t b(u, X_u^x, \mathbb{P}_{X_u^x}, \alpha_u)du + B_t\right|\right] \\
&\le |x| + \mathbb{E}\left[\int_0^t |b(u, X_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|du + |B_t|\right] \\
&= |x| + \int_0^t \mathbb{E}[|b(u, X_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|]du + \mathbb{E}\left[|B_t|\right] \\
&\le |x| + \int_0^t \big(\mathbb{E}[C_1(C(\omega) + |X_u^x| + \mathbb{E}[|X_u^x|])]\big)du + \mathbb{E}[|B_t|] \\
&\le |x| + \mathbb{E}[|B_t|] + C_1T\mathbb{E}[C(\omega)] + 2C_1\int_0^t \mathbb{E}[|X_u^x|]du \\
&\le |x| + \mathbb{E}[|B_t|] + C_1T\mathbb{E}[|C(\omega)|^2]^{\frac{1}{2}} + 2C_1\int_0^t \mathbb{E}[|X_u^x|]du \\
&\le |x| + \mathbb{E}[|B_t|] + C_1T\mathbb{E}[e^{|C(\omega)|^2}]^{\frac{1}{2}} + \int_0^t 2C_1\mathbb{E}[|X_u^x|]du \\
&\le |x| + C_{2,T} + 2C_1\int_0^t \mathbb{E}[|X_u^x|]du,
\end{aligned}$$

where $C_{2,T}$ is a constant depending on T. The inequality written above holds true since $\mathbb{E}[|B_t|]$ and $\mathbb{E}[e^{|C(\omega)|^2}]$ are finite quantities. Using the

Grönwall's inequality, we have a constant $C_{3,T}$ depending on T such that,

$$\mathbb{E}[|X_t^x|] \leq C_{3,T}\left(1+|x|\right),$$

consequently, we have,

$$\begin{aligned}|b(t,X_t^x,\mathbb{P}_{X_t^x},\alpha_t)| &\leq C_1(C(\omega)+|X_t^x|+\mathcal{K}(\mathbb{P}_{X_t^x},\delta_0)),\\ &\leq C_1(C(\omega)+|X_t^x|+\mathbb{E}[|X_t^x|]),\\ &\leq C_{4,T}(C(\omega)+|X_t^x|+1+|x|),\end{aligned}$$

next, we are going to use the estimate of $|b(t,X_t^x,\mathbb{P}_{X_t^x},\alpha_t)|$ to estimate X_t^x as follows:

$$\begin{aligned}|X_t^x| &= \left|x+\int_0^t b(u,X_u^x,\mathbb{P}_{X_u^x},\alpha_u)du+B_t\right|\\ &\leq |x|+\int_0^t |b(u,X_u^x,\mathbb{P}_{X_u^x},\alpha_u)|du+|B_t|\\ &\leq |x|+\int_0^t C_{4,T}(C(\omega)+|X_u^x|+1+|x|)du+|B_t|,\end{aligned}$$

we get after applying the Grönwall's ineqauality:

$$|X_t^x| \leq C_{5,T}(1+|x|+|B_t|+C(\omega)),$$

thus,

$$\begin{aligned}|b(t,X_t^x,\mathbb{P}_{X_t^x},\alpha_t)| &\leq C_{4,T}(C(\omega)+1+|x|+|B_t|),\\ &\leq C_{4,T}\left(C(\omega)+1+|x|+\sup_{u\in[0,T]}|B_u|\right),\end{aligned}$$

where $C_{4,T}$ has been updated by a new constant that we also call $C_{4,T}$. Therefore, with this estimate of $|b(t,X_t^x,\mathbb{P}_{X_t^x},\alpha_t)|$, we can clearly see that the estimates (3.34) and (3.35) hold true, therefore proving the lemma. □

Lemma 3.5.3. Let us consider $(X_t^{n,x})_{0\leq t\leq T, n\geq 1}$, the strong unique solutions of the following MFSDE,

$$dX_t^{n,x} = b_n(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}}, \alpha_t)dt + dB_t, \qquad X_0^{n,x} = x \in \mathbb{R}, \tag{3.36}$$

thus, for some constant C,

$$\sup_{n\geq 1} \sup_{0\leq t\leq T} \operatorname*{ess\,sup}_{x\in K} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}] \leq C, \tag{3.37}$$

for any compact set $K \in \mathbb{R}$ and $p \geq 2$.

Proof of Lemma 3.5.3. Let us first note that if we consider for a second that b is differentiable in the second argument, by taking the derivative w.r.t. the initial condition x in the MFSDE (3.32), $\partial_x X_t^x$ has the following expression,

$$\partial_x X_t^x = 1 + \int_0^t b'(s, X_s^x, \mathbb{P}_{X_s^x}, \alpha_s)\partial_x X_t^x + \partial_x b(s, z, \mathbb{P}_{X_s^x}, \alpha_s)|_{z=X_s^x} ds, \tag{3.38}$$

where b' is seen as the derivative of b w.r.t. the second variable. The solution to the above differential equation has the following representation,

$$\partial_x X_t^x = e^{\int_0^t b'(s,X_s^x,\mathbb{P}_{X_s^x},\alpha_s)ds} + \int_0^t e^{\int_u^t b'(r,X_r^x,\mathbb{P}_{X_r^x},\alpha_r)dr} \partial_x b(u, z, \mathbb{P}_{X_u^x}, \alpha_u)|_{z=X_u^x} du, \tag{3.39}$$

however, since b_1 is not differentiable in the second variable, we will have this representation instead,

$$\partial_x X_t^x = e^{-\int_0^t \int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x}) L^{X^x}(dr,dz) + \int_0^t b_2'(r,X_r^x,\alpha_r)dr}$$
$$+ \int_0^t e^{-\int_u^t \int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x}) L^{X^x}(dr,dz) + \int_u^t b_2'(r,X_r^x,\alpha_r)dr} \partial_x b(u, z, \mathbb{P}_{X_u^x}, \alpha_u)|_{z=X_u^x} du,$$

thus, we have,

$$
\begin{aligned}
&\mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \\
&= \mathbb{E}\Bigg[\Bigg|e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)dr} \\
&+ \int_0^t e^{-\int_u^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_u^t b_2'(r,X_r^{n,x},\alpha_r)dr}\partial_x b_n(u,z,\mathbb{P}_{X_u^{n,x}},\alpha_u)|_{z=X_u^{n,x}}du\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} \\
&\leq C_p\mathbb{E}\Bigg[\Bigg|e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} \\
&+ C_p\mathbb{E}\Bigg[\Bigg|\int_0^t e^{-\int_u^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_u^t b_2'(r,X_r^{n,x},\alpha_r)dr}\partial_x b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})|_{z=X_u^{n,x}}du\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} \\
&\leq C_p\mathbb{E}\Bigg[\Bigg|e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} \\
&+ C_p\int_0^t \mathbb{E}\Bigg[\Bigg|e^{-\int_u^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+\int_u^t b_2'(r,X_r^{n,x},\alpha_r)dr}\partial_x b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})|_{z=X_u^{n,x}}\Bigg|^{2p}\Bigg]^{\frac{1}{2p}}du \\
&\leq C_p\mathbb{E}\Bigg[e^{-4p\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+4p\int_0^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{4p}} \\
&+ C_p\int_0^t \mathbb{E}\Bigg[e^{-4p\int_u^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+4p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{4p}} \\
&\times \mathbb{E}[|\partial_x b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})|_{z=X_u^{n,x}}|^{4p}]^{\frac{1}{4p}}du \\
&\lesssim \sup_{0\leq s\leq T}\mathbb{E}\Bigg[e^{-4p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+4p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{4p}}\Bigg\{1 \\
&+ \int_0^t \mathbb{E}[|\partial_x b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})|_{z=X_u^{n,x}}|^{4p}]^{\frac{1}{4p}}du\Bigg\},
\end{aligned}
$$

next, we first separate the term $\mathbb{E}\Bigg[e^{-4p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+4p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{4p}}$ using the Minkowski inequality and we apply the Girsanov transform with a change of trajectory,

$$
\begin{aligned}
&\mathbb{E}\Bigg[e^{-4p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)+4p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{4p}} \\
&\leq \mathbb{E}\Bigg[e^{-8p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(dr,dz)}\Bigg]^{\frac{1}{8p}}\mathbb{E}\Bigg[e^{8p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\Bigg]^{\frac{1}{8p}}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left[e^{-8p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_r)dB_u\right)\right]^{\frac{1}{8p}} \\
&\quad \times \mathbb{E}\left[e^{8p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\right]^{\frac{1}{8p}}, \\
&\leq \mathbb{E}\left[e^{-16p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]^{\frac{1}{16p}}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_r)dB_u\right)^2\right]^{\frac{1}{16p}} \\
&\quad \times \mathbb{E}\left[e^{8p\int_s^t b_2'(r,X_r^{n,x},\alpha_r)dr}\right]^{\frac{1}{8p}} \\
&= T_1^n \times T_2^n \times T_3^n \qquad (3.40) \\
&< \infty
\end{aligned}
$$

the last line holds true because T_1^n is finite uniformly in n and x due to Lemma 3.5.1 , T_2^n is finite due to Lemma 3.3.2, and lastly T_3^n is also finite because the space derivative of b_2 is bounded by a random variable with finite exponential moments.

$\mathbb{E}[|\partial_x X_t^n|^{2p}]^{\frac{1}{2p}}$ now gives,

$$
\begin{aligned}
&\mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \\
&\leq C_p\left\{1+\int_0^t \mathbb{E}[|\partial_x b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})|_{z=X_u^{n,x}}|^{4p}]^{\frac{1}{4p}}du\right\} \\
&= C_p\left\{1+\int_0^t \mathbb{E}\left[\left|\lim_{x^\circ\to x}\frac{b_{1,n}(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x}})-b_{1,n}(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x^\circ}})}{x-x^\circ}\right|^{4p}\right]^{\frac{1}{4p}}du\right\} \\
&= C_p\left\{1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathbb{E}[|b_{1,n}(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x}})-b_{1,n}(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x^\circ}})|^{4p}]^{\frac{1}{4p}}du\right\},
\end{aligned}
$$

using the Lipshitz property of the map $\mu \mapsto b_1(t,z,\mu)$, and the Minkowski inequality, we get,

$$
\mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \leq C_p\left(1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathcal{K}(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x^\circ}})du\right),
$$

if we denote $\bar{H}(K)$, the closed convex hull of K, which is also compact since K is. Let us find an estimate for $\mathcal{K}(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x^\circ}})$ for any arbitrary x and x° in $\bar{H}(K)$,

$$
\begin{aligned}
&\mathcal{K}(\mathbb{P}_{X_u^{n,x}}, \mathbb{P}_{X_u^{n,x^\circ}}) \\
&= \sup_{h \in \mathrm{Lip1}(\mathbb{R})} \left| \int_{\mathbb{R}} h(y)(\mathbb{P}_{X_u^{n,x}} - \mathbb{P}_{X_u^{n,x^\circ}})(dy) \right|, \\
&\leq \mathbb{E}\Big[|X_u^{n,x} - X_u^{n,x^\circ}| \Big], \\
&\leq |x - x^\circ| + \mathbb{E}\Big[\Big| \int_0^u b_{1,n}(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, X_r^{n,x^\circ}, \mathbb{P}_{X_r^{n,x^\circ}}) dr \Big| \Big] \\
&\quad + \mathbb{E}\Big[\Big| \int_0^u b_2(r, X_r^{n,x}, \alpha_r) - b_2(r, X_r^{n,x^\circ}, \alpha_r) dr \Big| \Big],
\end{aligned}
$$

after applying the mean value theorem to the previous expression, we get,

$$
\begin{aligned}
&\mathbb{E}[|X_u^{n,x} - X_u^{n,x^\circ}|] \\
&\leq |x - x^\circ| + \mathbb{E}\Big[\Big| \int_0^u \Big(\int_0^1 (b'_{1,n}(r, X_r^{n,x+\lambda(x^\circ - x)}, \mathbb{P}_{X_u^{n,x+\lambda(x^\circ - x)}}) \partial_x X_r^{n,x+\lambda(x^\circ - x)} \\
&\quad + \partial_x b_{1,n}(r, z, \mathbb{P}_{X_r^{n,x+\lambda(x^\circ - x)}})|_{z = X_r^{n,x+\lambda(x_0 - x)}}) d\lambda \Big) (x - x^\circ) dr \Big| \Big] \\
&\quad + C \int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr \\
&\leq |x - x^\circ| + |x - x^\circ| \mathbb{E}\Big[\int_0^1 \Big| \int_0^u (b'_{1,n}(r, X_r^{n,x+\lambda(x^\circ - x)}, \mathbb{P}_{X_r^{n,x+\lambda(x^\circ - x)}}) \partial_x X_r^{n,x+\lambda(x^\circ - x)} \\
&\quad + \partial_x b_{1,n}(r, z, \mathbb{P}_{X_r^{n,x+\lambda(x^\circ - x)}})|_{z = X_r^{n,x+\lambda(x^\circ - x)}}) dr \Big| d\lambda \Big] + C \int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr \\
&\leq |x - x^\circ| + |x - x^\circ| \int_0^1 \mathbb{E}\Big[\Big| \int_0^u (b'_{1,n}(r, X_r^{n,x+\lambda(x^\circ - x)}, \mathbb{P}_{X_r^{n,x+\lambda(x_0 - x)}}) \partial_x X_r^{n,x+\lambda(x^\circ - x)} \\
&\quad + \partial_x b_{1,n}(r, z, \mathbb{P}_{X_r^{n,x+\lambda(x^\circ - x)}})|_{z = X_r^{n,x+\lambda(x^\circ - x)}}) dr \Big| \Big] d\lambda + C \int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr \\
&= |x - x^\circ| + |x - x^\circ| \int_0^1 \mathbb{E}\Big[\Big| \partial_x X_u^{n,x+\lambda(x^\circ - x)} - (1 - \lambda) \\
&\quad - \int_0^u b'_2(r, X_r^{n,x+\lambda(x^\circ - x)}, \alpha_r) \partial_x X_r^{n,x+\lambda(x^\circ - x)} dr \Big| \Big] d\lambda + C \int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr \\
&\leq |x - x^\circ| + |x - x^\circ| \int_0^1 \Big((1 - \lambda) + \mathbb{E}[|\partial_x X_u^{n,x+\lambda(x^\circ - x)}|] \\
&\quad + \mathbb{E}\Big[\Big| \int_0^u b'_2(r, X_r^{n,x+\lambda(x^\circ - x)}, \alpha_r) \partial_x X_r^{n,x+\lambda(x^\circ - x)} dr \Big| \Big] \Big) d\lambda + C \int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr
\end{aligned}
$$

$$\leq |x - x^\circ| + |x - x^\circ| \int_0^1 \bigg((1-\lambda) + \mathbb{E}\big[\big|\partial_x X_u^{n,x+\lambda(x^\circ - x)}\big|\big]$$
$$+ \int_0^u \mathbb{E}\big[\big|b_2'(r, X_r^{n,x+\lambda(x^\circ-x)}, \alpha_r)\partial_x X_r^{n,x+\lambda(x^\circ-x)}\big|\big] dr\bigg) d\lambda + C\int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr$$

here we now apply the Cauchy-Schwarz inequality to get:

$$\begin{aligned}
&\mathbb{E}[|X_u^{n,x} - X_u^{n,x^\circ}|] \\
&\leq |x - x^\circ| + |x - x^\circ| \int_0^1 \bigg((1-\lambda) + \mathbb{E}\big[\big|\partial_x X_u^{n,x+\lambda(x^\circ - x)}\big|\big] \\
&\quad + \int_0^u \mathbb{E}\big[\big|b_2'(r, X_r^{n,x+\lambda(x^\circ-x)}, \alpha_r)|^2\big]^{\frac{1}{2}} \mathbb{E}\big[\big|\partial_x X_r^{n,x+\lambda(x^\circ-x)}\big|^2\big]^{\frac{1}{2}} dr\bigg) d\lambda \\
&\quad + C\int_0^u \mathbb{E}[|X_r^{n,x} - X_r^{n,x^\circ}|] dr
\end{aligned}$$

applying the Grönwall's inequality yields:

$$\begin{aligned}
&\mathbb{E}[|X_u^{n,x} - X_u^{n,x^\circ}|] \\
&\leq C\bigg\{ |x - x^\circ| + |x - x^\circ| \int_0^1 \bigg((1-\lambda) + \mathbb{E}\big[\big|\partial_x X_u^{n,x+\lambda(x^\circ - x)}\big|^2\big]^{\frac{1}{2}} \\
&\quad + \int_0^u \mathbb{E}\big[\big|b_2'(r, X_r^{n,x+\lambda(x^\circ-x)}, \alpha_r)|^2\big]^{\frac{1}{2}} \mathbb{E}\big[\big|\partial_x X_r^{n,x+\lambda(x^\circ-x)}\big|^2\big]^{\frac{1}{2}} dr\bigg) d\lambda \bigg\}
\end{aligned}$$

since the space derivative of b_2 is bounded, we can write:

$$\begin{aligned}
&\mathbb{E}[|X_u^{n,x} - X_u^{n,x^\circ}|] \\
&\leq C|x - x^\circ| \bigg\{ 1 + \int_0^1 \bigg(\mathbb{E}\big[\big|\partial_x X_u^{n,x+\lambda(x^\circ - x)}\big|^2\big]^{\frac{1}{2}} + \int_0^u \mathbb{E}\big[\big|\partial_x X_r^{n,x+\lambda(x^\circ-x)}\big|^2\big]^{\frac{1}{2}} dr\bigg) d\lambda \bigg\} \\
&\leq C|x - x^\circ| \bigg\{ 1 + \operatorname*{ess\,sup}_{x \in \bar{H}(K)} \mathbb{E}\big[\big|\partial_x X_u^{n,x}\big|^2\big]^{\frac{1}{2}} \bigg\} \\
&= C\bigg\{ |x - x^\circ| + |x - x^\circ| \operatorname*{ess\,sup}_{x \in \bar{H}(K)} \mathbb{E}\big[\big|\partial_x X_u^{n,x}\big|^2\big]^{\frac{1}{2}} \bigg\}.
\end{aligned}$$

Therefore, we have:

$$\mathcal{K}(\mathbb{P}_{X_u^{n,x}}, \mathbb{P}_{X_u^{n,x^\circ}}) \leq C\bigg(|x - x^\circ| + |x - x^\circ| \operatorname*{ess\,sup}_{x \in \bar{H}(K)} \mathbb{E}[|\partial_x X_u^{n,x}|^2]^{\frac{1}{2}} \bigg), \quad (3.41)$$

combining all results gotten so far, everything shows that we can find a constant C independent on $n \geq 1$, $t \in [0,T]$ and $x \in \bar{H}(K)$ such that,

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\bar{H}(K)} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} &\leq C\left(1+\int_0^t \operatorname*{ess\,sup}_{x\in\bar{H}(K)} \mathbb{E}[|\partial_x X_u^{n,x}|^{2}]^{\frac{1}{2}} du\right) \\ &\leq C\left(1+\int_0^t \operatorname*{ess\,sup}_{x\in\bar{H}(K)} \mathbb{E}[|\partial_x X_u^{n,x}|^{2p}]^{\frac{1}{2p}} du\right), \end{aligned} \tag{3.42}$$

remember from the computations that we can always find a constant depending on n such that,

$$\mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \leq C_n\left(1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathcal{K}(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x^\circ}})du\right),$$

and from (3.43), we can say that,

$$\begin{aligned} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} &\leq C_n\left(1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathcal{K}(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x^\circ}})du\right) \\ &\leq C_n\left(1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathbb{E}[|X_u^{n,x}-X_u^{n,x^\circ}|]du\right) \\ &\leq C_n\left(1+\liminf_{x^\circ\to x}\frac{1}{|x-x^\circ|}\int_0^t \mathbb{E}[|X_u^{n,x}-X_u^{n,x^\circ}|^2]^{\frac{1}{2}}du\right) \\ &< C_{2,n} \text{ (coming from (3.43))}. \end{aligned}$$

Consequently, the map $t \mapsto \operatorname{ess\,sup}_{x\in\bar{H}(K)} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}}$ is integrable over $[0,T]$. We can then apply the lemma 5 in Jones (1964) to obtain,

$$\operatorname*{ess\,sup}_{x\in K} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \leq \operatorname*{ess\,sup}_{x\in\bar{H}(K)} \mathbb{E}[|\partial_x X_t^{n,x}|^{2p}]^{\frac{1}{2p}} \leq C + C^2\int_0^t e^{C(t-s)}ds < \infty,$$

which proves the lemma. □

Lemma 3.5.4. Assume that the drift b can be decomposed as in (3.3) and b_1, uniformly Lipschitz continuous in the measure variable (3.7). Consider $(X_t^x)_{0\leq t\leq T}$, the strong unique solution of the MFSDE (3.32). Hence, we can find a constant C such that,

$$\mathbb{E}[|X_t^x - X_t^y|^2]^{\frac{1}{2}} \leq C|x-y|, \tag{3.43}$$

for all $t \in [0, T]$ and $x, y \in K$.

Proof of Lemma 3.5.4.

$$\begin{aligned}
&\mathbb{E}[|X_t^{n,x} - X_t^{n,y}|^2]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\left[\left|x - y + \int_0^t \big(b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b_n(u, X_u^{n,y}, \mathbb{P}_{X_u^{n,y}}, \alpha_u)\big)du\right|^2\right]^{\frac{1}{2}}, \\
&\lesssim |x - y| + \mathbb{E}\left[\left|\int_0^t \big(b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b_n(u, X_u^{n,y}, \mathbb{P}_{X_u^{n,y}}, \alpha_u)\big)du\right|^2\right]^{\frac{1}{2}},
\end{aligned}$$

now, we use the mean value theorem for the map $x \mapsto b_1(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}})$ and we get,

$$\begin{aligned}
&\mathbb{E}[|X_t^{n,x} - X_t^{n,y}|^2]^{\frac{1}{2}} \\
&\lesssim |x - y| + \mathbb{E}\left[\left|\int_0^t \big(b_{1,n}(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}}) - b_{1,n}(u, X_u^{n,y}, \mathbb{P}_{X_u^{n,y}})\big)du\right|^2\right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}\left[\left|\int_0^t \big(b_2(u, X_u^{n,x}, \alpha_u) - b_2(u, X_u^{n,y}, \alpha_u)\big)du\right|^2\right]^{\frac{1}{2}} \\
&= |x - y| + \mathbb{E}\left[\left|\int_0^t \left(\int_0^1 \frac{\partial}{\partial y}\big[b_{1,n}(u, X_u^{n,y+\theta(x-y)}, \mathbb{P}_{X_u^{n,y+\theta(x-y)}})\big]d\theta\right)(x-y)du\right|^2\right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}\left[\left|\int_0^t \big(b_2(u, X_u^{n,x}, \alpha_u) - b_2(u, X_u^{n,y}, \alpha_u)\big)du\right|^2\right]^{\frac{1}{2}}
\end{aligned}$$

we apply the Minkowski inequality to have:

$$\begin{aligned}
&\mathbb{E}[|X_t^{n,x} - X_t^{n,y}|^2]^{\frac{1}{2}} \\
&\lesssim |x - y| + |x - y|\mathbb{E}\left[\left|\int_0^t \left(\int_0^1 \big(b'_{1,n}(u, X_u^{n,y+\theta(x-y)}, \mathbb{P}_{X_u^{n,y+\theta(x-y)}})\partial_y X_u^{n,y+\theta(x-y)}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \partial_y b_{1,n}(u, X_u^{n,y+\theta(x-y)}, \mathbb{P}_{X_u^{n,y+\theta(x-y)}})\big)d\theta\right)du\right|^2\right]^{\frac{1}{2}} \\
&\quad + \int_0^t \mathbb{E}\left[\big|b_2(u, X_u^{n,x}, \alpha_u) - b_2(u, X_u^{n,y}, \alpha_u)\big|^2\right]^{\frac{1}{2}}du,
\end{aligned}$$

we use the Fubini's theorem to switch the integrals,

$$
\begin{aligned}
&\mathbb{E}[|X_t^{n,x}-X_t^{n,y}|^2]^{\frac{1}{2}}\\
&\lesssim |x-y|+|x-y|\mathbb{E}\Big[\Big|\int_0^1\int_0^t\big(b_{1,n}'(u,X_u^{n,y+\theta(x-y)},\mathbb{P}_{X_u^{n,y+\theta(x-y)}})\partial_y X_u^{n,y+\theta(x-y)}\\
&\quad+\partial_y b_{1,n}(u,X_u^{n,y+\theta(x-y)},\mathbb{P}_{X_u^{n,y+\theta(x-y)}})\big)dud\theta\Big|^2\Big]^{\frac{1}{2}}+\int_0^t\mathbb{E}\big[\big|X_u^{n,x}-X_u^{n,y}\big|^2\big]^{\frac{1}{2}}du\\
&=|x-y|+|x-y|\mathbb{E}\Big[\Big|\int_0^1\Big(\partial_y X_t^{n,y+\theta(x-y)}-(1-\theta)\\
&\quad-\int_0^t b_2'(r,X_r^{n,y+\theta(x-y)},\alpha_r)\partial_y X_r^{n,y+\theta(x-y)}dr\Big)d\theta\Big|^2\Big]^{\frac{1}{2}}+\int_0^t\mathbb{E}\big[\big|X_u^{n,x}-X_u^{n,y}\big|^2\big]^{\frac{1}{2}}du\\
&\lesssim |x-y|+|x-y|\Big\{\mathbb{E}\Big[\Big|\int_0^1\Big(\partial_y X_t^{n,y+\theta(x-y)}-(1-\theta)\Big)d\theta\Big|^2\Big]^{\frac{1}{2}}\\
&\quad+\mathbb{E}\Big[\Big|\int_0^1\Big(\int_0^t b_2'(r,X_r^{n,y+\theta(x-y)},\alpha_r)\partial_y X_r^{n,y+\theta(x-y)}dr\Big)d\theta\Big|^2\Big]^{\frac{1}{2}}\Big\}\\
&\quad+\int_0^t\mathbb{E}\big[\big|X_u^{n,x}-X_u^{n,y}\big|^2\big]^{\frac{1}{2}}du\\
&\lesssim |x-y|+|x-y|\Big(\int_0^1\mathbb{E}[|\partial_y X_t^{n,y+\theta(x-y)}|^2]^{\frac{1}{2}}d\theta\\
&\quad+\int_0^1\int_0^t\mathbb{E}[|b_2'(r,X_r^{n,y+\theta(x-y)},\alpha_r)\partial_y X_r^{n,y+\theta(x-y)}|^2]^{\frac{1}{2}}drd\theta\Big)+\int_0^t\mathbb{E}\big[\big|X_u^{n,x}-X_u^{n,y}\big|^2\big]^{\frac{1}{2}}du,
\end{aligned}
$$

we next separate the expectation inside the double integral:

$$
\begin{aligned}
&\mathbb{E}[|X_t^{n,x}-X_t^{n,y}|^2]^{\frac{1}{2}}\\
&\lesssim |x-y|+|x-y|\Big(\int_0^1\mathbb{E}[|\partial_y X_t^{n,y+\theta(x-y)}|^2]^{\frac{1}{2}}d\theta\\
&\quad+\int_0^1\int_0^t\mathbb{E}[|b_2'(r,X_r^{n,y+\theta(x-y)},\alpha_r)|^4]^{\frac{1}{4}}\mathbb{E}[|\partial_y X_r^{n,y+\theta(x-y)}|^4]^{\frac{1}{4}}drd\theta\Big)\\
&\quad+\int_0^t\mathbb{E}\big[\big|X_u^{n,x}-X_u^{n,y}\big|^2\big]^{\frac{1}{2}}du
\end{aligned}
$$

we continue with the Grönwall's inequality:

$$
\begin{aligned}
&\mathbb{E}[|X_t^{n,x}-X_t^{n,y}|^2]^{\frac{1}{2}}\\
&\lesssim |x-y|+|x-y|\Big(\int_0^1\mathbb{E}[|\partial_y X_t^{n,y+\theta(x-y)}|^2]^{\frac{1}{2}}d\theta\\
&\quad+\int_0^1\int_0^t\mathbb{E}[|b_2'(r,X_r^{n,y+\theta(x-y)},\alpha_r)|^4]^{\frac{1}{4}}\mathbb{E}[|\partial_y X_r^{n,y+\theta(x-y)}|^4]^{\frac{1}{4}}drd\theta\Big),
\end{aligned}
$$

since the stochastic differential flow is integrable with finite $2p-th$ order moment, we can write

$$\mathbb{E}[|X_t^{n,x} - X_t^{n,y}|^2]^{\frac{1}{2}} \lesssim |x-y|,$$

which proves the lemma. □

Proposition 3.5.0.1. Assume that the drift b can be decomposed as in (3.3) such that its b_1 component admits a first differentiable with bounded derivative component and a second bounded component as written in (3.5). Consider $(X_t^x)_{t\in[0,T]}$, the strong solution of the MFSDE (3.32) with $V \subset \mathbb{R}$ be a bounded and open subset. Hence,

$$x \mapsto b(t, z, \mathbb{P}_{X_t^x}, \alpha_t) \in W^{1,p}(V). \tag{3.44}$$

Proof of Proposition 3.5.0.1. Consider $(b_n)_{n\geq 1}$, the sequence approximating the drift b as defined in the following:

$$b_n(t, z, \mu, \alpha) = \hat{b}_{1,n}(t, z, \mu) + \tilde{b}_1(t, z, \mu) + b_2(t, z, \alpha), \text{with } n \geq 1, \tag{3.45}$$

such that $\hat{b}_{1,n} \in L^\infty([0,T], C_b^{1,L}(\mathbb{R}, \mathcal{P}_1(\mathbb{R})))$, with

$\sup_{n\geq 1} ||\hat{b}_{1,n}||_\infty \leq C < \infty$, $||\cdot||_\infty$ denoting the supremum norm on all arguments, such that we have $\hat{b}_{1,n}$ that converges to $\hat{b}_1$ in every μ pointwise and almost everywhere in (t,z) w.r.t. the Lebesgue measure. $\mathcal{C}_b^{1,L}(\mathbb{R}, \mathcal{P}_1(\mathbb{R})$ denotes the space containing elements that are functions $g : \mathbb{R} \times \mathcal{P}_1(\mathbb{R})$ verifying the existence of a constant C such that the following properties are satisfied:

(i) $(z \mapsto g(z,\nu)) \in \mathcal{C}_{b,C}^{1,1}(\mathbb{R})$for all $\nu \in \mathcal{P}_1(\mathbb{R})$ and,

(ii) $(\nu \mapsto g(z,\nu)) \in \text{Lip}_C(\mathcal{P}_1(\mathbb{R}), \mathbb{R})$ for all $z \in \mathbb{R}$.

$(X_t^{n,x})_{t\in[0,T],n\geq 1}$ the corresponding sequence strong solutions of the following,

$$dX_t^{n,x} = b_n(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}}, \alpha_t)dt + dB_t, \qquad X_0^{n,x} = x \in \mathbb{R}, \tag{3.46}$$

let us first prove that the sequence $\{b_n\}_{n\geq 1}$ is weakly relatively compact in $W^{1,p}(V)$. Based on the proof of Lemma 3.5.2 and Lemma 3.5.3, we can say that:

$$\sup_{n\geq 1} \left\| b_n(t, z, \mathbb{P}_{X_t^{n,x}}, \alpha_t) \right\|_{W^{1,p}(V)} < \infty,$$

therefore, we have boundedness of the sequence $\{b_n\}_{n\geq 1}$ in $W^{1,p}(V)$ and consequently the sequence is weakly relatively compact based on Theorem 10.44 in Leoni (2009). Thus, there exists at least a sub-sequence $\{n_k\}_{k\geq 1}$ and $v \in W^{1,p}(V)$ such that we have the sub-sequence b_{n_k} converging weakly to v as k goes to infinity. Consider an arbitrary test-function $\varphi \in \mathcal{C}_0^\infty(V)$ and φ' its first derivative assuming φ is well-defined. Let,

$$\langle b_n, \varphi \rangle := \int_V b_n(t, z, \mathbb{P}_{X_t^{n,x}}, \alpha_t)\varphi(x)dx, \tag{3.47}$$

we have,

$$\begin{aligned}
\langle b_n - b, \varphi' \rangle &= \int_V (b_n(t, z, \mathbb{P}_{X_t^{n,x}}, \alpha_t) - b(t, z, \mathbb{P}_{X_t^x}, \alpha_t))\varphi'(x)dx, \\
&\leq \sup_{x\in\bar{V}} |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_1(t, z, \mathbb{P}_{X_t^x})| \int_V \varphi'(x)dx, \\
&\leq \sup_{x\in\bar{V}} |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_1(t, z, \mathbb{P}_{X_t^x})| \left(\int_V \varphi'(x)^p dx \right)^{\frac{1}{p}} \left(\int_V dx \right)^{\frac{1}{q}}, \\
&= \sup_{x\in\bar{V}} |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_1(t, z, \mathbb{P}_{X_t^x})| \, \|\varphi'\|_{L^p(V)} \, |V|^{\frac{1}{q}}, \\
&< \infty, \text{ based on Lemma 3.5.2},
\end{aligned}$$

where $\bar{V}$ represents the closure of V. We have shown in chapter 3 that there exists a sub-sequence $\{X^{n_k,x}\}_{k\geq 1}$ that strongly converges to X_t^x in

L^2. Consequently, we have,

$$
\begin{aligned}
&|b_n(t, z, \mathbb{P}_{X_t^{n,x}}, \alpha_t) - b(t, z, \mathbb{P}_{X_t^x}, \alpha_t)| \\
&\qquad = |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) + b_2(t, z, \alpha_t) - b_1(t, z, \mathbb{P}_{X_t^x}) - b_2(t, z, \alpha_t)| \\
&\qquad \leq |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_1(t, z, \mathbb{P}_{X_t^x})|, \\
&\qquad \leq |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_{1,n}(t, z, \mathbb{P}_{X_t^x})| + |b_{1,n}(t, z, \mathbb{P}_{X_t^x}) - b_1(t, z, \mathbb{P}_{X_t^x})|, \\
&\qquad = |b_{1,n}(t, z, \mathbb{P}_{X_t^{n,x}}) - b_{1,n}(t, z, \mathbb{P}_{X_t^x})| + |b_{1,n}(t, z, \mathbb{P}_{X_t^x}, \alpha_t) - b_1(t, z, \mathbb{P}_{X_t^x}, \alpha_t)|, \\
&\qquad \leq C\mathcal{K}(\mathbb{P}_{X_t^{n,x}}, \mathbb{P}_{X_t^x}) + |b_{1,n}(t, z, \mathbb{P}_{X_t^x}) - b_1(t, z, \mathbb{P}_{X_t^x})|,
\end{aligned}
$$

the convergence of $\mathcal{K}(\mathbb{P}_{X_t^{n,x}}, \mathbb{P}_{X_t^x})$ has been shown in (3.29) and $b_{1,n}$ converges to b_1 pointwise in every μ, therefore we can see that

$|b_n(t, z, \mathbb{P}_{X_t^{n,x}}, \alpha_t) - b(t, z, \mathbb{P}_{X_t^x}, \alpha_t)| \longrightarrow 0$ as $n \to \infty$. Hence,

$$
\begin{aligned}
&\lim_{k\to\infty} \langle b_{n_k} - b, \varphi' \rangle = 0, \\
&\Rightarrow \lim_{k\to\infty} \langle b_{n_k}, \varphi' \rangle - \langle b, \varphi' \rangle = 0, \\
&\Rightarrow \langle b, \varphi' \rangle = \lim_{k\to\infty} \langle b_{n_k}, \varphi' \rangle,
\end{aligned}
$$

since the function φ has compact support and vanishes at infinity, we have,

$$
\langle b, \varphi' \rangle = \lim_{k\to\infty} \langle b_{n_k}, \varphi' \rangle = - \lim_{k\to\infty} \langle b'_{n_k}, \varphi \rangle = -\langle v', \varphi \rangle,
$$

where b'_{n_k} is the first variation process of b_{n_k} and v', the first variation process of v. □

Lemma 3.5.5. Assume that the drift b can be decomposed as in (3.3) . Consider the strong solution of the MFSDE (3.32) denoted by $(X_t^x)_{t\in[0,T]}$. Furthermore, we take $\{b_n\}_{n\geq 1}$ as the sequence approximating the drift b as expressed in (3.45). Also, $(X_t^{n,x})_{n\geq 1}$ is the corresponding strong solution of the MFSDE (3.46). Consequently,

$$
\sup_{x\in K} \mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) dB_u\right)\right|^2\right]^{\frac{1}{2}}, \tag{3.48}
$$

tends to 0 as n tends to ∞, for any compact subset K of $\mathbb{R}$.

Lemma 3.5.6. Assume that the drift b can be decomposed as in (3.3). Consider the strong unique solution of the MFSDE (3.32) denoted by $(X_t^x)_{t\in[0,T]}$. In addition, we take $\{b_n\}_{n\geq 1}$ as the approximating sequence of b as expressed in (3.45). Also, $(X_t^{n,x})_{n\geq 1}$ is the corresponding strong solution of the MFSDE (3.46). Consequently,

$$\sup_{x\in K}\mathbb{E}\left[\left|e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(u,z,\mathbb{P}_{X_u^{n,x}})L^{B^x}(du,dz)} - e^{-\int_s^t\int_{\mathbb{R}} b_1(u,z,\mathbb{P}_{X_u^x})L^{B^x}(du,dz)}\right|^p\right]^{\frac{1}{p}}, \quad (3.49)$$

tends to 0 as n tends to ∞, for any compact subset K of $\mathbb{R}$.

Proof of Lemma 3.5.5. using the inequality $|e^v - e^w| \leq |e^v + e^w||v-w|$, we get,

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\right|^2\right]^{\frac{1}{2}}\\
&= \mathbb{E}\Bigg[\Bigg|e^{\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u - \frac{1}{2}\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2du}\\
&\quad - e^{\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u - \frac{1}{2}\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2du}\Bigg|^2\Bigg]^{\frac{1}{2}}\\
&\leq \mathbb{E}\Bigg[\Bigg|\int_0^T \left(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u) - b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)\right)dB_u\\
&\quad - \frac{1}{2}\int_0^T \left(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2 - b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2\right)du\Bigg|^2\\
&\quad \times\left(\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right) + \mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\right)^2\Bigg]^{\frac{1}{2}}\\
&\lesssim \mathbb{E}\Bigg[\left(\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right) + \mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\right)^2\\
&\quad \times\left|\int_0^T \left(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u) - b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)\right)dB_u\right|^2\Bigg]^{\frac{1}{2}}\\
&+ \mathbb{E}\Bigg[\left(\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right) + \mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\right)^2\\
&\quad \times\left|\int_0^T \left(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2 - b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2\right)du\right|^2\Bigg]^{\frac{1}{2}},
\end{aligned}$$

we apply the Cauchy-Schwarz inequality to get,

$$\mathbb{E}\bigg[\bigg|\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)-\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg|^2\bigg]^{\frac{1}{2}}$$
$$\leq C\mathbb{E}\bigg[\bigg(\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)+\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg)^4\bigg]^{\frac{1}{4}}$$
$$\times\mathbb{E}\bigg[\bigg|\int_0^T\big(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)\big)dB_u\bigg|^4\bigg]^{\frac{1}{4}}$$
$$+C\mathbb{E}\bigg[\bigg(\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)+\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg)^4\bigg]^{\frac{1}{4}}$$
$$\times\mathbb{E}\bigg[\bigg|\int_0^T\big(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2\big)du\bigg|^4\bigg]^{\frac{1}{4}},$$

we get after applying the estimate (3.15) of Lemma 3.3.2:

$$\mathbb{E}\bigg[\bigg|\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)-\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg|^2\bigg]^{\frac{1}{2}}$$
$$\leq C\mathbb{E}\bigg[\bigg|\int_0^T\big(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)\big)dB_u\bigg|^4\bigg]^{\frac{1}{4}}$$
$$+C\mathbb{E}\bigg[\bigg|\int_0^T\big(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2\big)du\bigg|^4\bigg]^{\frac{1}{4}},$$

next, we apply Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\bigg[\bigg|\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)-\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg|^2\bigg]^{\frac{1}{2}}$$
$$\leq C\mathbb{E}\bigg[\bigg|\int_0^T|b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)|^2du\bigg|^2\bigg]^{\frac{1}{4}}$$
$$+C\mathbb{E}\bigg[\bigg|\int_0^T\big(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)^2-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)^2\big)du\bigg|^4\bigg]^{\frac{1}{4}},$$

we now apply the Minkowski inequality,

$$\mathbb{E}\bigg[\bigg|\mathcal{E}\left(\int_0^T b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)dB_u\right)-\mathcal{E}\left(\int_0^T b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)dB_u\right)\bigg|^2\bigg]^{\frac{1}{2}}$$
$$\leq C\bigg(\int_0^T\mathbb{E}[|b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u)|^4]^{\frac{1}{2}}du\bigg)^{\frac{1}{2}}$$
$$+C\mathbb{E}\bigg[\bigg|\int_0^T(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)-b(u,B_u^x,\mathbb{P}_{X_u^x},\alpha_u))(b_n(u,B_u^x,\mathbb{P}_{X_u^{n,x}},\alpha_u)$$

$$
\begin{aligned}
&+ b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u))du\Big|^4\Big]^{\frac{1}{4}} \\
&\leq C\Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + C\mathbb{E}\Big[\Big|\int_0^T |b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|(|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)| \\
&\quad + |b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|)du\Big|^4\Big]^{\frac{1}{4}}, \\
&\leq C\Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + C\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4(|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)|^4 \\
&\quad + |b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4)]^{\frac{1}{4}} du, \\
&\leq C\Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + C\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{8}} \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)|^8 \\
&\quad + |b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{8}} du,
\end{aligned}
$$

using the estimate for the drift b in Lemma 3.5.2, we obtain,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{E}\Big(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\Big) - \mathcal{E}\Big(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\Big)\Big|^2\Big]^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^4]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + \int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{8}} du, \\
&\lesssim \Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}} du\Big)^{\frac{1}{2}} \\
&\quad + \int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^n}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{8}} du, \\
&\lesssim \Big(\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}} du\Big)^{\frac{1}{2}},
\end{aligned}
$$

next, using the triangle inequality and the assumption according to which b_1 has the Lipschitz continuity property in the measure variable (3.7), we have,

$$\begin{aligned}
&\mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{8}} \\
&\quad = \mathbb{E}[|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^{n,x}}) + b_2(u, B_u^x, \alpha_u) - b_1(u, B_u^x, \mathbb{P}_{X_u^x}) - b_2(u, B_u^x, \alpha_u)|^8]^{\frac{1}{8}} \\
&\quad \leq C\mathbb{E}[|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^{n,x}}) - b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^x})|^8]^{\frac{1}{8}} + \mathbb{E}[|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^x}) \\
&\qquad - b_1(u, B_u^x, \mathbb{P}_{X_u^x})|^8]^{\frac{1}{8}}, \\
&\quad \leq C\mathcal{K}(\mathbb{P}_{X_u^{n,x}}, \mathbb{P}_{X_u^x}) + \mathbb{E}[|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^x}) - b_1(u, B_u^x, \mathbb{P}_{X_u^x})|^8]^{\frac{1}{8}}, \\
&\quad \leq C\mathbb{E}[|X_u^{n,x} - X_u|] + \mathbb{E}[|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^x}) - b_1(u, B_u^x, \mathbb{P}_{X_u^x})|^8]^{\frac{1}{8}},
\end{aligned}$$

we now apply the Girsanov theorem to the first term,

$$\begin{aligned}
&\mathbb{E}[|X_u^{n,x} - X_u^x|] \\
&= \mathbb{E}\left[|B_t^x|\left(\mathcal{E}\left(\int_0^t b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^t b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right)\right] \\
&\leq \mathbb{E}[|B_t^x|^2]^{\frac{1}{2}}\mathbb{E}\left[\left(\mathcal{E}\left(\int_0^t b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^t b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right)^2\right]^{\frac{1}{2}} \\
&\leq C\mathbb{E}\left[\left(\mathcal{E}\left(\int_0^t b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^t b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

we obtain next,

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right|^2\right]^{\frac{1}{2}} \\
&\leq C\left(\int_0^T \mathbb{E}\left[\left(\mathcal{E}\left(\int_0^t b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^t b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right)^2\right]\right. \\
&\qquad \left. + \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}}du\right)^{\frac{1}{2}},
\end{aligned}$$

so, after squaring both size,

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right|^2\right] \\
&\leq C\int_0^T \mathbb{E}\left[\left(\mathcal{E}\left(\int_0^t b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^t b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right)^2\right] \\
&\qquad + \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}}du,
\end{aligned}$$

after applying the Grönwall's inequality, we get,

$$\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right|^2\right]$$
$$\leq C\int_0^T \mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}}du,$$

let us now evaluate $\mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}}$,

$$\mathbb{E}[|b_n(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)|^8]^{\frac{1}{4}}$$
$$= \left(\int_{-\infty}^{\infty} |b_n(u, z, \mathbb{P}_{X_u^x}, \alpha_u) - b(u, z, \mathbb{P}_{X_u^x}, \alpha_u)|^8 \frac{1}{\sqrt{2\pi u}} e^{-\frac{(z-x)^2}{2u}} dz\right)^{\frac{1}{4}}$$
$$= \left(\int_{-\infty}^{\infty} |b_{1,n}(u, z, \mathbb{P}_{X_u^x}) + b_2(u, z, \alpha_u) - b_1(u, z, \mathbb{P}_{X_u^x}) - b_2(u, z, \alpha_u)|^8 \frac{1}{\sqrt{2\pi u}} e^{-\frac{(z-x)^2}{2u}} dz\right)^{\frac{1}{4}}$$
$$= \left(\int_{-\infty}^{\infty} |b_{1,n}(u, z, \mathbb{P}_{X_u^x}) - b_1(u, z, \mathbb{P}_{X_u^x})|^8 \frac{1}{\sqrt{2\pi u}} e^{-\frac{z^2}{4u}} e^{-\frac{(z-2x)^2}{4u}} e^{\frac{x^2}{2u}} dz\right)^{\frac{1}{4}}$$
$$\leq \left(\int_{-\infty}^{\infty} |b_{1,n}(u, z, \mathbb{P}_{X_u^x}) - b_1(u, z, \mathbb{P}_{X_u^x})|^8 \frac{1}{\sqrt{2\pi u}} e^{-\frac{z^2}{4u}} e^{\frac{x^2}{2u}} dz\right)^{\frac{1}{4}}$$
$$= e^{\frac{x^2}{8u}} \left(\int_{-\infty}^{\infty} |b_{1,n}(u, z, \mathbb{P}_{X_u^x}) - b_1(u, z, \mathbb{P}_{X_u^x})|^8 \frac{1}{\sqrt{2\pi u}} e^{-\frac{z^2}{4u}} dz\right)^{\frac{1}{4}},$$

from Lemma 3.5.4, we see that the map $x \mapsto \mathbb{P}_{X_t^x}$ is a continuous map. Consequently, the image of the compact set K under that continuous map is also a compact set. Let us denote that image set K'. Therefore,

$$\sup_{x\in K} |b_{1,n}(u, z, \mathbb{P}_{X_u^x}) - b_1(u, z, \mathbb{P}_{X_u^x})| = \sup_{\mu\in K'} |b_{1,n}(u, z, \mu) - b_1(u, z, \mu)| \to 0 \text{ as } n \to \infty,$$

so, $\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(u, B_u^x, \mathbb{P}_{X_u^{n,x}}, \alpha_u)dB_u\right) - \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}, \alpha_u)dB_u\right)\right|^2\right]^{\frac{1}{2}}$ tends to 0 as n tends to ∞. □

Proof of Lemma 3.5.6. using the inequality $|e^v - e^w| \leq |e^v + e^w||v - w|$, and similar arguments as in previous proofs, we get:

$$
\begin{aligned}
&\mathbb{E}\left[\left|e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)} - e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right|^p\right]^{\frac{1}{p}}\\
&\le \mathbb{E}\Bigg[\left|\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) - \int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)\right|^p\\
&\quad \times \left(e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_u^{n,x}})L^{B^x}(dr,dz)} + e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right)^p\Bigg]^{\frac{1}{p}}\\
&\le \mathbb{E}\left[\left|\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) - \int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)\right|^{2p}\right]^{\frac{1}{2p}}\\
&\quad \times \mathbb{E}\left[\left(e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)} + e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right)^{2p}\right]^{\frac{1}{2p}}\\
&\le C\mathbb{E}\left[\left|\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) - \int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)\right|^{2p}\right]^{\frac{1}{2p}}\\
&\quad \times \left(\mathbb{E}\left[e^{-2p\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)}\right]^{\frac{1}{2p}} + \mathbb{E}\left[e^{-2p\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right]^{\frac{1}{2p}}\right),
\end{aligned}
$$

we apply Lemma 3.3.3 to get next,

$$
\begin{aligned}
&\mathbb{E}\left[\left|e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)} - e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right|^p\right]^{\frac{1}{p}}\\
&\le C\mathbb{E}\left[\left|\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz) - \int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)\right|^{2p}\right]^{\frac{1}{2p}},\\
&= C\mathbb{E}\left[\left|\int_s^t\int_{\mathbb{R}} \big(b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}}) - b_1(r,z,\mathbb{P}_{X_r^x})\big)L^{B^x}(dr,dz)\right|^{2p}\right]^{\frac{1}{2p}},
\end{aligned}
$$

Now, let $\hat{B}_t = B_{T-t}$ and the Brownian motion W_t, being the corresponding Brownian motion adapted to the natural filtration of $\hat{B}_t$. Using the identity in (3.22), we have,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t\int_{\mathbb{R}} \big(b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}}) - b_1(r,z,\mathbb{P}_{X_r^x})\big)L^{B^x}(dr,dz)\right|^{2p}\right]^{\frac{1}{2p}}\\
&= \mathbb{E}\Bigg[\Bigg|\int_s^t \big(b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})\big)dB_r\\
&\quad + \int_{T-t}^{T-s} \big(b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x})\big)dW_r\\
&\quad - \int_{T-t}^{T-s} \big(b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})\big)\frac{\hat{B}_r}{T-r}dr\Bigg|^{2p}\Bigg]^{\frac{1}{2p}}
\end{aligned}
$$

$$\lesssim \mathbb{E}\Big[\Big|\int_s^t \big(b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})\big)dB_r\Big|^{2p}\Big]^{\frac{1}{2p}}$$
$$+\,\mathbb{E}\Big[\int_{T-t}^{T-s} \big(b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x})\big)dW_r\Big|^{2p}\Big]^{\frac{1}{2p}}$$
$$+\,\mathbb{E}\Big[\Big|\int_{T-t}^{T-s} \big(b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})\big)\frac{\hat{B}_r}{T-r}dr\Big|^{2p}\Big]^{\frac{1}{2p}},$$

after applying the Burholder-Davis-Gundy inequality, the Minkowski inequality and the Cauchy-Schwarz inequality, we get,

$$\mathbb{E}\Big[\Big|\int_s^t\int_{\mathbb{R}} \big(b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}}) - b_1(r,z,\mathbb{P}_{X_r^x})\big)L^{B^x}(dr,dz)\Big|^{2p}\Big]^{\frac{1}{2p}}$$
$$\lesssim \mathbb{E}\Big[\Big|\int_s^t |b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^2 dr\Big|^{p}\Big]^{\frac{1}{2p}}$$
$$+\,\mathbb{E}\Big[\Big|\int_{T-t}^{T-s} |b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x}|^2 dr\Big|^{p}\Big]^{\frac{1}{2p}}$$
$$+\,\mathbb{E}\Big[\Big|\int_{T-t}^{T-s} (b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x}))\frac{\hat{B}_r}{\sqrt{T-r}}\frac{dr}{\sqrt{T-r}}\Big|^{2p}\Big]^{\frac{1}{2p}}$$
$$\lesssim \Big(\int_s^t \mathbb{E}[|b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^{2p}]^{\frac{1}{p}}dr\Big)^{\frac{1}{2}}$$
$$+\Big(\int_{T-t}^{T-s} \mathbb{E}[|b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x}|^{2p}]^{\frac{1}{p}}dr\Big)^{\frac{1}{2}}$$
$$+\Big(\int_{T-t}^{T-s} \mathbb{E}\Big[|b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^{2p}\Big|\frac{B_{T-r}}{\sqrt{T-r}}\Big|^{2p}\Big]^{\frac{1}{2p}}\frac{dr}{\sqrt{T-r}}\Big)$$
$$\lesssim \Big(\int_s^t \mathbb{E}[|b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^{2p}]^{\frac{1}{p}}dr\Big)^{\frac{1}{2}}$$
$$+\Big(\int_{T-t}^{T-s} \mathbb{E}[|b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x}|^{2p}]^{\frac{1}{p}}dr\Big)^{\frac{1}{2}}$$
$$+\Big(\int_{T-t}^{T-s} \mathbb{E}\big[|b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^{4p}\big]^{\frac{1}{4p}}\frac{dr}{\sqrt{T-r}}\Big)$$
$$\lesssim \mathbb{E}\Big[\Big|\int_s^t |b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^2 dr\Big|^{p}\Big]^{\frac{1}{2p}}$$
$$+\,\mathbb{E}\Big[\Big|\int_{T-t}^{T-s} |b_{1,n}(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(T-r,\hat{B}_r^x,\mathbb{P}_{X_r^x}|^2 dr\Big|^{p}\Big]^{\frac{1}{2p}}$$
$$+\int_{T-t}^{T-s} \mathbb{E}\big[|b_{1,n}(r,B_r^x,\mathbb{P}_{X_r^{n,x}}) - b_1(r,B_r^x,\mathbb{P}_{X_r^x})|^{4p}\big]^{\frac{1}{4p}}\frac{dr}{\sqrt{T-r}}$$

$$\lesssim \left| \int_s^t \mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{2p}]^{\frac{1}{p}} dr \right|^{\frac{1}{2}}$$
$$+ \left| \int_{T-t}^{T-s} \mathbb{E}[|b_{1,n}(T-r, \hat{B}_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(T-r, \hat{B}_r^x, \mathbb{P}_{X_r^x})|^{2p}]^{\frac{1}{p}} dr \right|^{\frac{1}{2}}$$
$$+ \int_{T-t}^{T-s} \mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}} \frac{dr}{\sqrt{T-r}},$$
$$= A_n^1 + A_n^2 + A_n^3,$$

the last challenge is now to show that $\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$ converges to 0 as n grows large.

$$\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$$
$$= \mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) + b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$$
$$\leq C\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}} + C\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x})$$
$$- b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}},$$

we now evaluate each term, the last term becomes,

$$\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$$
$$\leq e^{\frac{x^2}{8rp}} \left(\int_{-\infty}^{\infty} |b_{1,n}(r, z, \mathbb{P}_{X_r^x}) - b_1(r, z, \mathbb{P}_{X_r^x})|^{4p} \frac{1}{\sqrt{2\pi r}} e^{-\frac{z^2}{4r}} dz \right)^{\frac{1}{4p}},$$

which tends to 0 as n becomes large. The first term becomes,

$$\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$$
$$\leq C\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x})$$
$$\leq C\mathbb{E}[|X_r^{n,x} - X_r^x|]$$
$$= C\mathbb{E}\left[|B_r^x| \left| \mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) dB_r \right) - \mathcal{E}\left(\int_0^T b(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) dB_r \right) \right| \right]$$
$$\leq C\mathbb{E}[|B_r^x|^2]^{\frac{1}{2}} \mathbb{E}\left[\left| \mathcal{E}\left(\int_0^T b_n(r, B_r^x, \mathbb{P}_{X_r^{n,x}}, \alpha_r) dB_r \right) - \mathcal{E}\left(\int_0^T b(r, B_r^x, \mathbb{P}_{X_r^x}, \alpha_r) dB_r \right) \right|^2 \right]^{\frac{1}{2}},$$

from Lemma 3.5.5 and the fact that $\mathbb{E}[|B_r^x|^2]^{\frac{1}{2}}$ is finite, we see that,

$\mathbb{E}[|b_{1,n}(r, B_r^x, \mathbb{P}_{X_r^{n,x}}) - b_1(r, B_r^x, \mathbb{P}_{X_r^x})|^{4p}]^{\frac{1}{4p}}$ converges to 0 as n tends to ∞.

Also, it means that $\mathcal{K}(\mathbb{P}_{X_r^{n,x}}, \mathbb{P}_{X_r^x})$ tends to 0 as n grows large, therefore, $\mathbb{P}_{X_r^{n,x}}$ converges $\mathbb{P}_{X_r^x}$ as n tends to ∞. Consequently, the second term will be 0 as n becomes ∞. Combining everything together, we just showed that A_n^3 tends to 0 as n grows large. We can use the same exact approach to show that both A_n^1 and A_n^2 converges to 0 as n tends to ∞. Hence,

$$\sup_{x\in K}\mathbb{E}\left[\left|e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(dr,dz)} - e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(dr,dz)}\right|^p\right]^{\frac{1}{p}}$$

tends to 0 as n tends to ∞, which proves the lemma. □

Proof of Theorem 3.5.1. Let us consider the sequence $(b_n)_{n\geq 1}$ approximating the drift b as shown in (3.45), and $(X_t^{n,x})_{n\geq 0}$, the corresponding strong unique solutions of the MFSDE (4.1). We know from Bauer et al. (2018) that this solution is Sobolev differentiable and can be written explicitly as:

$$\mathcal{G}_{0,t}^{n,\alpha} := e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r} \tag{3.50}$$
$$+\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;X_s^{n,x})\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\mathrm{d}s.$$

We show that the representation in the theorem hold by showing that $(\mathcal{G}^n)_{n\geq 0}$ converges weakly in $L^2(V\times\Omega)$ to $\mathcal{G}$. Since $L^2(V\times\Omega)$ is spanned by the space

$$\left\{v\otimes\mathcal{E}\left(\int_0^T\dot{\varphi}_r\mathrm{d}B_r\right), \varphi\in C_b^1(\mathbb{R}), v\in C_0^\infty(V)\right\},$$

We need to show that

$$\int_V v(x)\mathbb{E}\Big[(\mathcal{G}_n-\mathcal{G})\mathcal{E}\Big(\int_0^T\dot{\varphi}_r\mathrm{d}B_r\Big)\Big]\mathrm{d}x \longrightarrow 0 \text{ as } n\to\infty, \tag{3.51}$$

We have

$$
\begin{aligned}
&\int_V v(x)\mathbb{E}\Big[(\mathcal{G}_n-\mathcal{G})\mathcal{E}\Big(\int_0^T \dot{\varphi}_r \mathrm{d}B_r\Big)\Big]\mathrm{d}x \\
&=\int_V v(x)\mathbb{E}\Big[\Big(e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r} \\
&\quad+\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;X_s^{n,x})\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\mathrm{d}s \\
&\quad-e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r} \\
&\quad-\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(dr,dz)+\int_s^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;X_s^x)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]\mathrm{d}s\Big)\mathcal{E}\Big(\int_0^T \dot{\varphi}_r \\
&=\int_V v(x)\mathbb{E}\Big[\Big(e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r} \\
&\quad+\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{X^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,X_r^{n,x},\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]|_{y=X_s^{n,x}} \\
&\quad-e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r} \\
&\quad-\int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(dr,dz)+\int_s^t b_2'(r,X_r^x,\alpha_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]|_{y=X_s^x}\mathrm{d}s\Big) \\
&\quad\times\mathcal{E}\Big(\int_0^T \dot{\varphi}_r \mathrm{d}B_r\Big)\Big]\mathrm{d}x,
\end{aligned}
$$

Let $\tilde{X}^{n,x}(t,\omega) = X^{n,x}(t,\omega+\varphi)$, $\tilde{\alpha}(\omega) = \alpha(\omega+\varphi)$ and applying the Cameron-Martin-Girsanov transform to obtain,

$$
\begin{aligned}
&\int_V v(x)\mathbb{E}\Big[(\mathcal{G}_n - \mathcal{G})\mathcal{E}\Big(\int_0^T \dot{\varphi}_r \mathrm{d}B_r\Big)\Big]\mathrm{d}x \\
&= \int_V v(x)\mathbb{E}\Big[\Big(e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r} \\
&+ \int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r} \\
&\times \bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]|_{y=\tilde{X}_s^{n,x}}\mathrm{d}s - e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r} \\
&- \int_0^t e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]|_{y=\tilde{X}_s^x}\mathrm{d}s\Big)\Big]\mathrm{d}x, \\
&= \int_V v(x)\mathbb{E}\Big[\Big(e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r} \\
&- e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t \tilde{b}_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\Big)\Big]\mathrm{d}x \\
&+ \int_V v(x)\int_0^t \mathbb{E}\Big[e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r} \\
&\times \bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]|_{y=\tilde{X}_s^{n,x}} \\
&- e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}_r^x}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]|_{y=\tilde{X}_s^x}\Big]\mathrm{d}s\mathrm{d}x \\
&= \int_V v(x)R_1^n\mathrm{d}x + \int_V v(x)R_2^n\mathrm{d}x.
\end{aligned}
$$

Let us check the convergence of R_1^n, we have:

$$
\begin{aligned}
R_1^n &= \mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r} \\
&- e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)+\int_0^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\Big] \\
&\leq \mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big|e^{\int_0^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r} - e^{\int_0^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\Big|^2\Big]^{\frac{1}{2}} \\
&+ \mathbb{E}\Big[e^{\int_0^t 2b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}r}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big|e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)} - e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{2}} \\
&= R_{11}^n \times R_{12}^n + R_{13}^n \times R_{14}^n. \qquad (3.52)
\end{aligned}
$$

Using the Girsanov transform and the Cauch-Schwarz inequality, Lemmas 3.3.2 and 3.5.1, we get

$$\begin{aligned}R_{11}^n =&\mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)}\Big]^{\frac{1}{2}}\\ \leq&\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T\big(b_n(r,B_r^x,\mathbb{P}_{X_r^{n,x}},\tilde{\alpha}_r)+\dot{\varphi}_r\big)\mathrm{d}B_r\Big)^2\Big]^{\frac{1}{4}}\mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} 4b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\Big]^{\frac{1}{4}}<\infty\end{aligned}$$

R_{13}^n is finite due to the boundedness of the space derivative of b_2. Next, we prove the convergence of R_{14}^n. We do so by first defining the following terms:

$$Z_1^n := e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)},\quad Z_1 := e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^{x}})L^{\tilde{X}^{x}}(\mathrm{d}r,\mathrm{d}z)}.$$

Again, we show convergence of Z_1^n to Z_1 by first showing weak convergence in L^2 and then showing convergence of the second moment n the Euclidiant norm.

As before, we show weak the convergence by showing the following result:

$$\Big|\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\dot{\varphi}_r dB_r\Big)(Z_1^n-Z_1)\Big]\Big|\to 0 \text{ as } n\to\infty.$$

So we have after applying the Cameron-Martin Girsanov theorem,

$$\begin{aligned}&\Big|\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\dot{\varphi}_{1,r}\mathrm{d}B_r\Big)(Z_1^n-Z_1)\Big]\Big|\\ =&\Big|\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\dot{\varphi}_{1,r}\mathrm{d}B_r\Big)\big(e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)}-e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^{x}})L^{\tilde{X}^{x}}(\mathrm{d}r,\mathrm{d}z)}\big)\Big]\Big|\\ =&\Big|\mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\hat{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)}-e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^{x}})L^{\hat{X}^{x}}(\mathrm{d}r,\mathrm{d}z)}\Big]\Big|\\ =&\Big|\mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\mathcal{E}\Big(\int_0^T\big(b_n(s,B_s^x,\mathbb{P}_{X_s^{n,x}},\hat{\alpha}_s)+\dot{\varphi}_s+\dot{\varphi}_{1,s}\big)\mathrm{d}B_s\Big)\\ &-e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^{x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\mathcal{E}\Big(\int_0^T\big(b(s,B_s^x,\mathbb{P}_{X_s^x},\hat{\alpha}_s)+\dot{\varphi}_s+\dot{\varphi}_{1,s}\big)\mathrm{d}B_s\Big)\Big]\Big|\end{aligned}$$

where $\hat{X}^x(\omega) = X^x(\omega + \varphi + \varphi_1)$, $\hat{\alpha}(\omega) = \alpha(\omega + \varphi + \varphi_1)$. Let us define:

$$\begin{aligned}
\hat{\mathcal{E}}_n(x) :=& \mathcal{E}\Big(\int_0^T \big(b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}, \hat{\alpha}_s) + \dot{\varphi}_s + \dot{\varphi}_{1,s}\big)\mathrm{d}B_s\Big),\\
\tilde{\mathcal{E}}_n(x) :=& \mathcal{E}\Big(\int_0^T \big(b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}, \tilde{\alpha}_s) + \dot{\varphi}_s\big)\mathrm{d}B_s\Big)\\
\hat{\mathcal{E}}(x) :=& \mathcal{E}\Big(\int_0^T \big(b(s, B_s^x, \mathbb{P}_{X_s^x}, \hat{\alpha}_s) + \dot{\varphi}_s + \dot{\varphi}_{1,s}\big)\mathrm{d}B_s\Big),\\
\tilde{\mathcal{E}}(x) :=& \mathcal{E}\Big(\int_0^T \big(b(s, B_s^x, \mathbb{P}_{X_s^x}, \tilde{\alpha}_s)\big)\mathrm{d}B_s\Big).
\end{aligned}$$

Hence:

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1 \dot{\varphi}_{1,r}\mathrm{d}B_r\Big)(Z_1^n - Z_1)\Big]\Big|\\
\leq& \mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\Big]^{\frac{1}{2}}\mathbb{E}\big[\big|\hat{\mathcal{E}}_n(x) - \hat{\mathcal{E}}(x)\big|^2\big]^{\frac{1}{2}}\\
&+\mathbb{E}\big[\hat{\mathcal{E}}(x)^2\big]^{\frac{1}{2}}\mathbb{E}\Big[\Big|e^{-\int_0^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)} - e^{-\int_0^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{2}},
\end{aligned}$$

The convergence of $\mathbb{E}\Big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\Big]^{\frac{1}{2}}\mathbb{E}\big[\big|\hat{\mathcal{E}}_n(x) - \hat{\mathcal{E}}(x)\big|^2\big]^{\frac{1}{2}}$ holds by Lemma 3.5.1 and the first claim of Lemma 3.5.5. The second term converges by Lemma 3.3.2 and the second claim of Lemma 3.5.5. Next, we show the convergence in the Euclidiant norm. More precisely, we prove that

$$\big|\mathbb{E}\big[|Z_1^n|^2\big] - \mathbb{E}\big[|Z_1|^2\big]\big| \to 0 \text{ as } n \to \infty.$$

Adding and substracting $e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\tilde{\mathcal{E}}(x)$ and using the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\big|\mathbb{E}\big[|Z_1^n|^2\big] - \mathbb{E}\big[|Z_1|^2\big]\big|\\
=&\big|\mathbb{E}\big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)}\big] - \mathbb{E}\big[e^{-\int_0^t\int_{\mathbb{R}} 2b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}^x}(\mathrm{d}r,\mathrm{d}z)}\big]\big|\\
=&\big|\mathbb{E}\big[e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\tilde{\mathcal{E}}_n(x) - e^{-\int_0^t\int_{\mathbb{R}} 2b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\tilde{\mathcal{E}}(x)\big]\big|\\
\leq&\big|\mathbb{E}\big[e^{-\int_0^t\int_{\mathbb{R}} 4b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\big]^{\frac{1}{2}}\mathbb{E}\big[\big|\tilde{\mathcal{E}}_n(x) - \tilde{\mathcal{E}}(x)\big|^2\big]^{\frac{1}{2}}\\
&+\mathbb{E}\big[\tilde{\mathcal{E}}(x)^2\big]^{\frac{1}{2}}\mathbb{E}\big[\big|e^{-\int_0^t\int_{\mathbb{R}} 2b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{B^x}(\mathrm{d}r,\mathrm{d}z)} - e^{-\int_0^t\int_{\mathbb{R}} 2b_1(r,z,\mathbb{P}_{X_r^x})L^{B^x}(\mathrm{d}r,\mathrm{d}z)}\big|^2\big]^{\frac{1}{2}}\big|.
\end{aligned}$$

Convergence follows using the similar arguments as before. Therefore, R_{14}^n

converges to 0 as n goes to ∞. The convergence of R_{12}^n follows from dominated convergence. Consequently, R_1^n converges as n grows large. Let us now study the convergence of R_2^n. Recall that

$$R_2^n$$
$$= \int_0^t \mathbb{E}\Big[\Big(e^{-\int_s^t\int_{\mathbb{R}} b_{1,n}(r,z,\mathbb{P}_{X_r^{n,x}})L^{\tilde{X}^{n,x}}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^{n,x},\tilde{\alpha}_r)\mathrm{d}r}\bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]|_{y=\tilde{X}_s^{n,x}}$$
$$- e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{\tilde{X}_r^x}(\mathrm{d}r,\mathrm{d}z)+\int_s^t b_2'(r,\tilde{X}_r^x,\tilde{\alpha}_r)\mathrm{d}u}\bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]|_{y=\tilde{X}_s^x}\Big)\Big]\mathrm{d}s.$$

Let us again take note of the following notation:

$$\begin{aligned}
\bar{e}_s(b_1,X^x) &:= e^{-\int_s^t\int_{\mathbb{R}} b_1(r,z,\mathbb{P}_{X_r^x})L^{X^x}(\mathrm{d}r,\mathrm{d}z)}, \quad \bar{e}_s(b_2,X^x) := e^{\int_s^t b_2'(r,X_r^x,\tilde{\alpha}_r)\mathrm{d}r}\\
\bar{\mathbb{E}}[\partial_\mu\tilde{b}\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}] &:= \bar{\mathbb{E}}[\partial_\mu b(s,\bar{X}_s^x,\mathbb{P}_{X_s^x},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]|_{y=\tilde{X}_s^x},\\
\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}] &:= \bar{\mathbb{E}}[\partial_\mu b_n(s,\bar{X}_s^{n,x},\mathbb{P}_{X_s^{n,x}},\bar{\alpha}_s;y)\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]|_{y=\tilde{X}_s^{n,x}}
\end{aligned}$$

Adding and substracting both $\bar{e}_s(b_1,\tilde{X}^x)\bar{e}_s(b_2,\tilde{X}^{n,x})\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]$ and $\bar{e}_s(\tilde{b}_2,\tilde{X}^x)\bar{e}_s(b_1,\tilde{X}^x)\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]$ gives

$$\begin{aligned}
R_2^n &= \int_0^t \mathbb{E}[\bar{e}_s(b_{1,n},\tilde{X}^{n,x})\bar{e}_s(\tilde{b}_2,\tilde{X}^{n,x})\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}] - \bar{e}_s(b_1,\tilde{X}^x)\bar{e}_s(b_2,X^x)\bar{\mathbb{E}}[\partial_\mu\tilde{b}\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]]\mathrm{d}s\\
&= \int_0^t \Big(\mathbb{E}\big[\big(\bar{e}_s(b_{1,n},\tilde{X}^{n,x}) - \bar{e}_s(b_1,\tilde{X}^x)\big)\bar{e}_s(\tilde{b}_2,\tilde{X}^{n,x})\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\big]\\
&\quad + \mathbb{E}\big[\big(\bar{e}_s(b_2,\tilde{X}^{n,x}) - \bar{e}_s(b_2,\tilde{X}^x)\big)\bar{e}_s(b_1,\tilde{X}^x)\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\big]\\
&\quad + \mathbb{E}\big[\big(\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}] - \bar{\mathbb{E}}[\partial_\mu\tilde{b}\bar{\mathcal{G}}_{r,s}^{\bar{\alpha}}]\big)\bar{e}_s(\tilde{b}_2,\tilde{X}^x)\bar{e}_s(b_1,\tilde{X}^x)\big]\Big)\mathrm{d}s = \int_0^t \big(R_{21}^n + R_{22}^n + R_{23}^n\big)\mathrm{d}s
\end{aligned}$$

let us evaluate each term:

$$\begin{aligned}
R_{21}^n &= \mathbb{E}\big[\big(\bar{e}_s(b_{1,n},\tilde{X}^{n,x}) - \bar{e}_s(b_1,\tilde{X}^x)\big)\bar{e}_s(b_2,\tilde{X}^{n,x})\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\big]\\
&\le \mathbb{E}\big[\big|\bar{e}_s(b_{1,n},\tilde{X}^{n,x}) - \bar{e}_s(b_1,\tilde{X}^x)\big|^2\big]^{\frac{1}{2}}\mathbb{E}\big[\big|\bar{e}_s(b_2,\tilde{X}^{n,x})\big|^4\big]^{\frac{1}{4}}\mathbb{E}\big[\big|\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\big|^4\big]^{\frac{1}{4}}
\end{aligned}$$

$\mathbb{E}\big[\big|\bar{e}_s(b_2,\tilde{X}^{n,x})\big|^4\big]^{\frac{1}{4}}$ and $\mathbb{E}\big[\big|\bar{\mathbb{E}}[\partial_\mu\tilde{b}_s^n\bar{\mathcal{G}}_{r,s}^{n,\bar{\alpha}}]\big|^4\big]^{\frac{1}{4}}$ are finite thanks to the boundedness of b_2' and $\partial_\mu\tilde{b}^n$ and Lemma 3.37. Convergence of $\mathbb{E}\big[\big|\bar{e}_s(b_{1,n},\tilde{X}^{n,x}) -$

$\bar{e}_s(b_1, \tilde{X}^x)\big|^2]^{\frac{1}{2}}$ follows using the same argument as in R^n_{14} whereas onvergence of R^n_{22} follows from dominated convergence and Lemma 4.3.1. We continue with R^n_{23}. Using the previous argiuments, we have

$$\begin{aligned}
R^n_{23} =&\mathbb{E}\big[\big(\bar{\mathbb{E}}[\partial_\mu \tilde{b}^n_s \bar{\mathcal{G}}^{n,\bar{\alpha}}_{r,s} - \partial_\mu \tilde{b}\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}]\big)\bar{e}_s(b_2, \tilde{X}^x)\bar{e}_s(b_1, \tilde{X}^x)\big] \\
\leq&\mathbb{E}\big[\bar{\mathbb{E}}[|\partial_\mu \tilde{b}^n_s \bar{\mathcal{G}}^{n,\bar{\alpha}}_{r,s} - \partial_\mu \tilde{b}\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}|^2]\big]^{\frac{1}{2}}\mathbb{E}\big[\bar{e}_s(b_2, \tilde{X}^x)^4\big]^{\frac{1}{4}}\mathbb{E}\big[\bar{e}_s(b_1, \tilde{X}^x)^4\big]^{\frac{1}{4}} \\
\leq&\mathbb{E}\big[\bar{\mathbb{E}}[|\partial_\mu \tilde{b}^n_s \bar{\mathcal{G}}^{n,\bar{\alpha}}_{r,s} - \partial_\mu \tilde{b}\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}|^2]\big]^{\frac{1}{2}}\mathbb{E}\big[\bar{e}_s(\tilde{b}_2, \tilde{X}^x)^4\big]^{\frac{1}{4}}\mathbb{E}\big[\bar{e}_s(b_1, B^x)^8\big]^{\frac{1}{8}} \\
&\times \mathbb{E}\Big[\mathcal{E}\Big(\int_0^T \big(b(s, B^x_u, \mathbb{P}_{X^x_u}, \tilde{\alpha}_u) + \dot{\varphi}_u\big)\mathrm{d}B_u\Big)^2\Big]^{\frac{1}{8}} \\
\leq&C\mathbb{E}\big[\bar{\mathbb{E}}[|\partial_\mu \tilde{b}^n_s \bar{\mathcal{G}}^{n,\bar{\alpha}}_{r,s} - \partial_\mu \tilde{b}\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}|^2]\big]^{\frac{1}{2}}.
\end{aligned}$$

Convergence of $\mathbb{E}\big[\bar{\mathbb{E}}[|\partial_\mu \tilde{b}^n_s \bar{\mathcal{G}}^{n,\bar{\alpha}}_{r,s} - \partial_\mu \tilde{b}\bar{\mathcal{G}}^{\bar{\alpha}}_{r,s}|^2]\big]^{\frac{1}{2}}$ can be proved using the same approach shown in (4.14). Therefore, R^n_2 converges to 0 as n tends to ∞. The proof is completed. □

Proof of Theorem 3.5.2. Let us consider $s, t \in [0, T]$, and two arbitrary element of K, x and z. $\{X^{n,x}\}_{n\geq 0}$ denotes the approximating sequence of solutions as presented in (3.46). Using the same approach as in the proof of Lemma 3.5.4, we can find that:

$$\mathbb{E}[|X^{n,x}_t - X^{n,z}_t|^2]^{\frac{1}{2}} \lesssim |x - z|,$$

as well, based on the proof of the estimate (3.37) of Lemma 3.5.3. Also, we have:

$$\begin{aligned}
\mathbb{E}[|X^{n,x}_t - X^{n,x}_{t'}|^2]^{\frac{1}{2}} &= \mathbb{E}\Bigg[\Bigg|\int_{t'}^t b_n(r, X^{n,x}_r, \mathbb{P}_{X^{n,x}_r}, \alpha_r)dr + B_t - B_{t'}\Bigg|^2\Bigg]^{\frac{1}{2}}, \\
&\lesssim \mathbb{E}\Bigg[\Bigg|\int_{t'}^t b_n(r, X^{n,x}_r, \mathbb{P}_{X^{n,x}_r}, \alpha_r)dr\Bigg|^2 + |B_t - B_{t'}|^2\Bigg]^{\frac{1}{2}}, \\
&\lesssim \mathbb{E}\Bigg[\Bigg|\int_{t'}^t b_n(r, X^{n,x}_r, \mathbb{P}_{X^{n,x}_r}, \alpha_r)dr\Bigg|^2\Bigg]^{\frac{1}{2}} + \mathbb{E}[|B_t - B_{t'}|^2]^{\frac{1}{2}},
\end{aligned}$$

we apply the Minkowski inequality to get:

$$\begin{aligned}\mathbb{E}[|X_t^{n,x} - X_{t'}^{n,x}|^2]^{\frac{1}{2}} &\lesssim \int_{t'}^{t} \mathbb{E}[|b_n(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)|^2]^{\frac{1}{2}} dr + \mathbb{E}[(B_t - B_{t'})^2]^{\frac{1}{2}}, \\ &\lesssim \int_{t'}^{t} \mathbb{E}[|b_n(r, X_r^{n,x}, \mathbb{P}_{X_r^{n,x}}, \alpha_r)|^2]^{\frac{1}{2}} dr + \mathbb{E}[(B_t - B_{t'})^2]^{\frac{1}{2}}, \\ &\leq C|t - t'|^{\frac{1}{2}},\end{aligned}$$

where C is greater than 0 and is a constant depending on T. The previous inequality holds true due to the estimate (3.34) of Lemma 3.5.2. We obtain afterwards:

$$\begin{aligned}\mathbb{E}[|X_t^{n,x} - X_{t'}^{n,z}|^2] &= \mathbb{E}[|X_t^{n,x} - X_t^{n,z} + X_t^{n,z} - X_{t'}^{n,z}|^2], \\ &\leq \mathbb{E}[|X_t^{n,x} - X_t^{n,z}|^2] + \mathbb{E}[|X_t^{n,z} - X_{t'}^{n,z}|^2], \\ &\leq C\big(|x - z|^2 + |t - t'|\big),\end{aligned}$$

knowing that $X^{n,x}$ converges to X^x in L^2 for at least a subsequence, we can use the Fatou's Lemma in Lemma 1.6.1 to obtain the desired result. □

3.6 Chapter Summary

In this chapter, we demonstrated that the strong solution of the MFSDE under study holds some properties among which the reresentation of a stochastic differential flow. Since in our case the controlled MFSDE has a non-smooth drift and is driven by a one-dimensional Brownian motion, we studied the representation of the stochastic (Sobolev) differential flow, via a time-space local time integration argument and we will use that representation to solve an optimal control problem where the state constraint is a MFSDE.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we finally study and characterize an optimal control for a MFSDE in which the measure variable is the law of the state process denoted by $\mathbb{P}_{X_t^x}$. After stating the assumptions we will use for the control problem, we will describe how an optimal control could be characterized for the system by studying necessary conditions for the existence of an optimal control.

4.2 Research Framework

We aim at optimizing the following performance functional:

$$J(\alpha) := \mathbb{E}\left[\int_0^T f(s, X_s^x, \mathbb{P}_{X_s^x}, \alpha_s)ds + g(X_T^x, \mathbb{P}_{X_T^x})\right],$$

subjected to:

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x}, \alpha_t)dt + dB_t, \qquad X_0^x = x, \quad t \in [0, T], \tag{4.1}$$

where,

- B_t is a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$,
- $\{\alpha_t\}_{t\geq 0}$ is a suitable control process adapted to the filtration $\mathcal{F}_t$ and takes values in a closed convex control space $E \in \mathbb{R}$, and all controls taken such that "(4.1) is uniquely solvable" constitute the set of admissible controls $\mathcal{D}$,
- X_t^x denotes the state of the system controlled by α_t,
- For all $(t, z, \mu, \alpha) \in [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \times E$,
  - f and g are continuously differentiable with bounded first derivatives,

– there exists a constant C such that for all (t, z, μ, α),

$$|f(t, z, \mu, \alpha)| + |g(z, \mu)| \leq C(1 + |z| + \mathcal{K}(\mu, \delta_0)), \tag{4.2}$$

– we denote by $\partial_z f$ and $\partial_z g$, the first derivative of f and g respectively with respect to the space variable,

- $\mathbb{P}_{X_t^x} \in \mathcal{P}_1(\mathbb{R})$ where,
  $\mathcal{P}_1(\mathbb{R}) = \left\{ \nu \middle| \nu \text{ probability measure on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ with } \int_{\mathbb{R}} |z| d\nu(z) < \infty \right\}$,

- The drift b can be decomposed in this form:

$$b(t, z, \mu, \alpha) = b_1(t, z, \mu) + b_2(t, z, \alpha), \tag{4.3}$$

  where,

  – $b_1 : [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ with b_1 bounded and adapted,

  – $b_2 : [0, T] \times \mathbb{R} \times E \to \mathbb{R}$ is bounded and differentiable in its second and third variable with bounded derivatives,

- the map $\mu \mapsto b_1(t, z, \mu)$ is Lipschitz continuous in the measure variable uniformly in $t \in [0, T]$ and $z \in \mathbb{R}$, i.e. for all $t \in [0, T]$, $z \in \mathbb{R}$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, there exists a constant C such that:

$$|b_1(t, z, \mu) - b_1(t, z, \nu)| \leq C\mathcal{K}(\mu, \nu), \tag{4.4}$$

problem 4.2.1. Find $\alpha^* \in \mathcal{A}$ such that

$$V(x) = \sup_{\alpha \in \mathcal{A}} J(\alpha) = J(\alpha^*). \tag{4.5}$$

The Hamiltonian H associated to the above control problem is defined as follows:

$$H(t,z,p,\mu,\alpha) = f(t,z,\mu,\alpha) + b(t,z,\mu,\alpha)p. \tag{4.6}$$

4.3 Stochastic Maximum Principle

The stochastic maximum principle is stated in the following theorem:

Theorem 4.3.1 (A necessary maximum principle)**.** *Suppose that the drift b is given as in (4.3). Let $(\hat{\alpha}, X^{x,\hat{\alpha}})$ be an optimal pair of the system (4.1) and (4.5). Then there exists an adapted process (adjoint process) $\hat{P}$ such that:*

1. *The following maximum principle holds*

$$\partial_\alpha H(t, X_t^{x,\hat{\alpha}}, \hat{P}_t, \mathbb{P}_{X_t^{x,\hat{\alpha}}}, \hat{\alpha}_t) \cdot (\gamma - \hat{\alpha}_t) \geq 0,\ \mathbb{P}\otimes \mathrm{d}t - a.s.\ \text{for all}\ \gamma \in \mathcal{A}. \tag{4.7}$$

2. *Let $\mathcal{G}^{\hat{\alpha}}$ be the well defined first variation process (in the Sobolev sense) of $X^{x,\hat{\alpha}}$ given by*

$$\begin{aligned}\mathcal{G}^{\hat{\alpha}}_{s,t} =& e^{-\int_s^t \int_{\mathbb{R}} b_1(u,z,\mathbb{P}_{X_u^{x,\hat{\alpha}}}) L^{X^{x,\hat{\alpha}}}(\mathrm{d}u,\mathrm{d}z) + \int_s^t \partial_z b_2(u, X_u^{x,\hat{\alpha}}, \hat{\alpha}_u)\mathrm{d}u} \\ &+ \int_s^t e^{-\int_r^t \int_{\mathbb{R}} b_1(u,z,\mathbb{P}_{X_u^{x,\hat{\alpha}}}) L^{X^{x,\hat{\alpha}}}(\mathrm{d}u,\mathrm{d}z) + \int_r^t \partial_z b_2(u, X_u^{x,\hat{\alpha}}, \hat{\alpha}_u)\mathrm{d}u} &(4.8)\\ &\times \tilde{\mathbb{E}}[\partial_\mu b(r, \tilde{X}_r^{x,\tilde{\hat{\alpha}}}, \mathbb{P}_{X_r^{x,\hat{\alpha}}}, \tilde{\hat{\alpha}}_r; X_r^{x,\tilde{\hat{\alpha}}})\tilde{\mathcal{G}}^{\tilde{\hat{\alpha}}}_{s,r}]\mathrm{d}r. &(4.9)\end{aligned}$$

Then

$$\begin{aligned}\hat{P}_t =& \mathbb{E}\Big[\mathcal{G}^{\hat{\alpha}}_{t,T}\partial_z g(X_T^{x,\hat{\alpha}}, \mathbb{P}_{X_T^{x,\hat{\alpha}}}) + \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^{x,\tilde{\hat{\alpha}}}, \mathbb{P}_{X_T^{x,\hat{\alpha}}}; X_T^{x,\hat{\alpha}})\tilde{\mathcal{G}}^{\tilde{\hat{\alpha}}}_{t,T}] \\ &+ \int_t^T \Big\{\mathcal{G}^{\hat{\alpha}}_{t,s}\partial_z f(s, X_s^{x,\hat{\alpha}}, \mathbb{P}_{X_s^{x,\hat{\alpha}}}, \hat{\alpha}_s) + \tilde{\mathbb{E}}[\partial_\mu f(s, \tilde{X}_s^{x,\tilde{\hat{\alpha}}}, \mathbb{P}_{X_s^{x,\hat{\alpha}}}, \tilde{\hat{\alpha}}_s; X_s^{x,\hat{\alpha}})\tilde{\mathcal{G}}^{\tilde{\hat{\alpha}}}_{t,s}]\Big\}\mathrm{d}s \Big| \mathcal{F}_t\Big]\end{aligned} \tag{4.10}$$

This whole chapter is entirely setup for the proof of Theorem 4.3.1. We consider $E \subset \mathbb{R}$, a closed convex subset of $\mathbb{R}$. A control is said to be admissible if it satisfies the following condition:

$$\mathcal{D} := \left\{ \alpha : [0,T] \times \Omega \to E, \text{ progressive (4.1) is uniquely solvable and } \sup_{0\leq t\leq T} \mathbb{E}[|\alpha_t|^4] < A, \right\} \tag{4.11}$$

where A is a constant such that $A > 0$. In addition, we define the sequence of smooth functions b_n expressed as follows:

$$b_n(t, X_t^{n,x,\alpha}, \mathbb{P}_{X_t^{n,x,\alpha}}, \alpha_t) = b_{1,n}(t, X_t^{n,x,\alpha}, \mathbb{P}_{X_t^{n,x,\alpha}}) + b_2(t, X_t^{n,x,\alpha}, \alpha_t),$$

where $b_{1,n} : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ are infinitely differentiable functions having a support that is compact and converging almost everywhere to b_1. $b_{1,n}$ can be a bounded sequence since b_1 is assumed to be bounded. The assumptions on b_n results in it being Lipschitz continuous, therefore making the sequence well-posed. We define in the same way the performance and the value function denoted by J_n and Q_n respectively, as follows:

$$J_n(\alpha) := \mathbb{E}\left[\int_0^T f(u, X_u^{n,x,\alpha}, \mathbb{P}_{X_u^{n,x,\alpha}}, \alpha_u)du + g(X_T^{n,x,\alpha}, \mathbb{P}_{X_T^{n,x,\alpha}})\right], \; Q_n = \sup_{\alpha\in\mathcal{D}} J_n(\alpha),$$

and,

$$dX_t^{n,x,\alpha} = b_n(t, X_t^{n,x,\alpha}, \mathbb{P}_{X_t^{n,x,\alpha}}, \alpha_t)dt + dB_t, \; t \in [0,T], \; X_0^{n,x,\alpha} = x.$$

In addition, the distance between two controls α^1 and α^2 will be expressed as:

$$\tau(\alpha^1, \alpha^2) = \sup_{0\leq t\leq T} \mathbb{E}\left[|\alpha_t^1 - \alpha_t^2|^4\right]^{\frac{1}{4}}.$$

The proof of Theorem 4.3.1 will consist in deriving an approximate control problem using the Ekeland variational principle and proving that the optimal control for the initial problem with Q_n is also "$\epsilon-$optimal" for the approximate control problem. Next, we will pass to the limit to show some form of convergence for the maximum principle. The whole proof will be demonstrated through the use of some technical lemmas.

Lemma 4.3.1. The following bounds hold true:

for every $\alpha^1, \alpha^2 \in \mathcal{D}$, we have:

1. $$\mathbb{E}\big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\big]^{\frac{1}{2}} \\ \leq C\Big\{\tau(\alpha^1,\alpha^2) + \Big(\int_0^t \frac{e^{-\frac{x^2}{8u}}}{(2\pi u)^{\frac{1}{8}}}\Big(\int_{\mathbb{R}} \big|b_{1,n}(u,z,\mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u,z,\mathbb{P}_{X_u^{x,\alpha^2}})\big|^8 e^{-\frac{z^2}{4u}}dz\Big)^{\frac{1}{4}} du\Big)^{\frac{1}{2}}\Big\}.$$

2. for every sequence $\{\alpha_n\}_{n\geq 0}$ in $\mathcal{D}$ converging to $\alpha \in \mathcal{D}$ in the norm $\sup_{0\leq t\leq T} \mathbb{E}[|\cdot|^4]^{\frac{1}{4}}$, we have:

   $$\sup_{0\leq t\leq T} \mathbb{E}\big[\big|X_t^{k,x,\alpha_n} - X_t^{k,x,\alpha}\big|^2\big]^{\frac{1}{2}} \text{ converges to 0 as } n \text{ grows to } \infty, \text{ for a given } k.$$

Proof of Lemma 4.3.1. We start with the proof of 1. in the lemma. We have,

$$\begin{aligned} &X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2} \\ &= \int_0^t \big(b_n(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}, \alpha_u^1) - b(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}, \alpha_u^2)\big)du, \\ &= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) + b_2(u, X_u^{n,x,\alpha^1}, \alpha_u^1) \\ &- b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du, \end{aligned}$$

after adding and substracting the following three terms: $b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})$,

$b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})$, and $b_2(u, X_u^{x,\alpha^2}, \alpha_u^1)$ and then applying the mean value theorem, we obtain:

$$\begin{aligned}
&X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2} \\
&= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) + b_2(u, X_u^{n,x,\alpha^1}, \alpha_u^1) \\
&\quad - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du, \\
&= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \big(b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{n,x,\alpha^1}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \bigg(\int_0^1 b'_{1,n}(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \mathbb{P}_{X_u^{x,\alpha^2}})d\theta\bigg)(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2})du \\
&\quad + \int_0^t \big(b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{n,x,\alpha^1}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^1)\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{x,\alpha^2}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du \\
&= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \bigg(\int_0^1 b'_{1,n}(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \mathbb{P}_{X_u^{x,\alpha^2}})d\theta\bigg)(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2})du \\
&\quad + \int_0^t \big(b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \bigg(\int_0^1 b'_2(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \alpha_u^1)d\theta\bigg)(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2})du \\
&\quad + \int_0^t \big(b_2(u, X_u^{x,\alpha^2}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du, \\
&= \int_0^t \big(b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \bigg(\int_0^1 \big(b'_{1,n}(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \mathbb{P}_{X_u^{x,\alpha^2}}) \\
&\quad + b'_2(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \alpha_u^1)\big)d\theta\bigg) \times (X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2})du \\
&\quad + \int_0^t \big(b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{x,\alpha^2}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big)du,
\end{aligned}
$$

therefore $X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}$ admits the representation:

$$
\begin{aligned}
&X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2} \\
&= \int_0^t e^{\int_s^t \int_0^1 \left(b'_{1,n}(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \mathbb{P}_{X_u^{x,\alpha^2}}) + b'_2(u, X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}), \alpha_u^1)\right) d\theta du} \\
&\quad \times \bigg(b_{1,n}(s, X_s^{n,x,\alpha^1}, \mathbb{P}_{X_s^{n,x,\alpha^1}}) - b_{1,n}(s, X_s^{n,x,\alpha^1}, \mathbb{P}_{X_s^{x,\alpha^2}}) + b_{1,n}(s, X_s^{x,\alpha^2}, \mathbb{P}_{X_s^{x,\alpha^2}}) \\
&\quad - b_1(s, X_s^{x,\alpha^2}, \mathbb{P}_{X_s^{x,\alpha^2}}) + b_2(s, X_s^{x,\alpha^2}, \alpha_s^1) - b_2(s, X_s^{x,\alpha^2}, \alpha_s^2)\bigg)ds
\end{aligned}
$$

squaring and applying the expectation to the power $\frac{1}{2}$ on both sides and the Cauchy-Schwarz inequality yields:

$$\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big]^{\frac{1}{2}}$$

$$\lesssim \mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t\int_0^1 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

$$\times\Big\{\mathbb{E}\Big[\Big|\int_0^t \big|b_{1,n}(u,X_u^{n,x,\alpha^1},\mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u,X_u^{n,x,\alpha^1},\mathbb{P}_{X_u^{x,\alpha^2}})\big|^2 du\Big|^2\Big]^{\frac{1}{4}}$$

$$+\mathbb{E}\Big[\Big|\int_0^t \big|b_{1,n}(u,X_u^{x,\alpha^2},\mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u,X_u^{x,\alpha^2},\mathbb{P}_{X_u^{x,\alpha^2}})\big|^2 du\Big|^2\Big]^{\frac{1}{4}}$$

$$+\mathbb{E}\Big[\Big|\int_0^t \big|b_2(u,X_u^{x,\alpha^2},\alpha_u^1) - b_2(u,X_u^{x,\alpha^2},\alpha_u^2)\big|^2 du\Big|^2\Big]^{\frac{1}{4}}\Big\}.$$

The next step is to show that

$$\mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t\int_0^1 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

is finite. So we have after applying the Jensen inequality:

$$\mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t\int_0^1 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

$$\leq \mathbb{E}\Big[\Big|\int_0^t\int_0^1 e^{\int_s^t 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)du} d\theta ds\Big|^2\Big]^{\frac{1}{4}}$$

we next apply the Minkowski inequality to have:

$$\mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t\int_0^1 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

$$\leq \Big(\int_0^t\int_0^1 \mathbb{E}\Big[e^{\int_s^t 4\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)du}\Big]^{\frac{1}{2}} d\theta ds\Big)^{\frac{1}{2}}.$$

Next, we evaluate $\mathbb{E}\Big[e^{\int_s^t 4\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)du}\Big]$. We first apply the Girsanov transform but before, let us introduce the following process $\{b_{n,t}^{\theta}\}_{t\in[0,T]}$ where $b_{n,u}^{\theta}$ can be found as follows:

$$
\begin{aligned}
&X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}) \\
&= x + B_u + \int_0^u b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2)dr + \theta\Big(\int_0^u \big(b_n(r, X_r^{n,x,\alpha^1}, \mathbb{P}_{X^{n,x,\alpha^1}}, \alpha_r^1) \\
&\quad - b(r, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}, \alpha_r^2)\big)dr\Big) \\
&= x + B_u + \int_0^u \big(\theta b_n(r, X_r^{n,x,\alpha^1}, \mathbb{P}_{X^{n,x,\alpha^1}}, \alpha_r^1) + (1-\theta) b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2)\big)dr \\
&= x + B_u + \int_0^u b_{n,r}^\theta dr.
\end{aligned}
$$

So we have:

$$
\begin{aligned}
&X_u^{x,\alpha^2} + \theta(X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}) \\
&= x + B_u + \int_0^u b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2)dr + \theta\Big(\int_0^u \big(b_n(r, X_r^{n,x,\alpha^1}, \mathbb{P}_{X^{n,x,\alpha^1}}, \alpha_r^1) \\
&\quad - b(r, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}, \alpha_r^2)\big)dr\Big) \\
&= x + B_u + \int_0^u \big(\theta b_n(r, X_r^{n,x,\alpha^1}, \mathbb{P}_{X^{n,x,\alpha^1}}, \alpha_r^1) + (1-\theta) b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2)\big)dr \\
&= x + B_u + \int_0^u b_{n,r}^\theta dr.
\end{aligned}
$$

which is finite due to the assumption on b_2', Lemma 3.5.1 and the boundedness of $b_{n,u}^\theta$. We go back to estimating $\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big]^{\frac{1}{2}}$. Applying the Minkowski's inequality gives:

$$
\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big]^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{n,x,\alpha^1}}) - b_{1,n}(u, X_u^{n,x,\alpha^1}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&+ \Big(\int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&+ \Big(\int_0^t \mathbb{E}\big[\big|b_2(u, X_u^{x,\alpha^2}, \alpha_u^1) - b_2(u, X_u^{x,\alpha^2}, \alpha_u^2)\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}},
\end{aligned}
$$

using the Lipschitz continuity of b_2, and the Lipschitz continuity of $b_{1,n}$ in the measure variable, we obtain:

$$
\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big]^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^t \mathcal{K}(\mathbb{P}_{X_u^{n,x,\alpha^1}}, \mathbb{P}_{X_u^{x,\alpha^2}})^2 du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^1 - \alpha_u^2\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^t \mathbb{E}\big[|X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}|\big]^2 du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^1 - \alpha_u^2\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^t \mathbb{E}\big[|X_u^{n,x,\alpha^1} - X_u^{x,\alpha^2}|^2\big] du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^1 - \alpha_u^2\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}},
\end{aligned}
$$

now let us apply the Grönwall's inequality after squaring both sides of the inequality and we obtain:

$$
\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big] \\
&\lesssim \int_0^t \mathbb{E}\big[\big|\alpha_u^1 - \alpha_u^2\big|^4\big]^{\frac{1}{2}} du + \int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du, \\
&\lesssim \tau(\alpha^1, \alpha^2)^2 + \int_0^t \mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]^{\frac{1}{2}} du
\end{aligned}
$$

next, we apply the Girsanov transform on $\mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big]$, and we get,

$$
\begin{aligned}
&\mathbb{E}\big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\big] \\
&= \mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2) dB_r\Big)\big|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\Big] \\
&\leq \mathbb{E}\Big[\mathcal{E}\Big(\int_0^T b(r, X_r^{x,\alpha^2}, \mathbb{P}_{X_r^{x,\alpha^2}}, \alpha_r^2) dB_r\Big)^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\big|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8\Big]^{\frac{1}{2}},
\end{aligned}
$$

since b is bounded, we do:

$$\begin{aligned}
&\mathbb{E}\Big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8\Big] \\
&\leq C\mathbb{E}\Bigg[\big|b_{1,n}(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, B_u^x, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8\Bigg]^{\frac{1}{2}} \\
&= C\bigg(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi u}} \big|b_{1,n}(u, z, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, z, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8 e^{-\frac{(z-x)^2}{2u}} dz\bigg)^{\frac{1}{2}},
\end{aligned}$$

since we also have:

$$e^{-\frac{(z-x)^2}{2u}} = e^{-\frac{z^2}{4u}} e^{-\frac{(z-2x)^2}{4u}} e^{-\frac{x^2}{2u}} \leq e^{-\frac{z^2}{4u}} e^{-\frac{x^2}{2u}},$$

we get,

$$\begin{aligned}
&\mathbb{E}\Big[\big|b_{1,n}(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, X_u^{x,\alpha^2}, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^4\Big] \\
&\leq C\bigg(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi u}} \big|b_{1,n}(u, z, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, z, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8 e^{-\frac{z^2}{4u}} e^{-\frac{x^2}{2u}} dz\bigg)^{\frac{1}{2}} \\
&= C\frac{e^{-\frac{x^2}{4u}}}{(2\pi u)^{\frac{1}{4}}}\bigg(\int_{\mathbb{R}} \big|b_{1,n}(u, z, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, z, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8 e^{-\frac{z^2}{4u}} dz\bigg)^{\frac{1}{2}},
\end{aligned}$$

Combining all results, we obtain at the end:

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{n,x,\alpha^1} - X_t^{x,\alpha^2}\big|^2\Big] \\
&\leq C\bigg\{\tau(\alpha^1, \alpha^2)^2 + \int_0^t \frac{e^{-\frac{x^2}{8u}}}{(2\pi u)^{\frac{1}{8}}}\bigg(\int_{\mathbb{R}} \big|b_{1,n}(u, z, \mathbb{P}_{X_u^{x,\alpha^2}}) - b_1(u, z, \mathbb{P}_{X_u^{x,\alpha^2}})\big|^8 e^{-\frac{z^2}{4u}} dz\bigg)^{\frac{1}{4}} du\bigg\}.
\end{aligned}$$

which proves the inequality (1.) of Lemma 4.3.1. We continue with the inequality (2.) of the lemma. We have:

$$
\begin{aligned}
&X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha} \\
&= \int_0^t \big(b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha}, \mathbb{P}_{X_u^{k,x,\alpha}}) + b_2(u, X_u^{k,x,\alpha^n}, \alpha_u^n) \\
&\quad - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big)du \\
&= \int_0^t \big(b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big)du \\
&\quad + \int_0^t \big(b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}}) - b_{1,k}(u, X_u^{k,x,\alpha}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du \\
&\quad + \int_0^t \big(b_2(u, X_u^{k,x,\alpha^n}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u^n)\big)du \\
&= \int_0^t \big(b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du \\
&\quad + \int_0^t \left(\int_0^1 (b'_{1,k}(u, X_u^{k,x,\alpha^n} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \mathbb{P}_{X_u^{k,x,\alpha}})d\theta\right)(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha})du \\
&\quad + \int_0^t \left(\int_0^1 (b'_2(u, X_u^{k,x,\alpha^n} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \alpha_u^n)d\theta\right)(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha})du \\
&\quad + \int_0^t \big(b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big)du, \\
&= \int_0^t \big(b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du \\
&\quad + \int_0^t \left(\int_0^1 (b'_{1,k}(u, X_u^{k,x,\alpha^n} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \mathbb{P}_{X_u^{k,x,\alpha}})\right. \\
&\quad \left. + b'_2(u, X_u^{k,x,\alpha^n} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \alpha_u^n))d\theta\right) \times (X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha})du \\
&\quad + \int_0^t \big(b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big)du,
\end{aligned}
$$

therefore $X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha}$ admits the representation:

$$
\begin{aligned}
&X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha} \\
&= \int_0^t e^{\int_s^t \int_0^1 \left(b'_{1,k}(u, X_u^{k,x,\alpha} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \mathbb{P}_{X_u^{k,x,\alpha}}) + b'_2(u, X_u^{k,x,\alpha^n} + \theta(X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}), \alpha_u^n)\right)d\theta du} \\
&\times \left(b_{1,k}(s, X_s^{k,x,\alpha^n}, \mathbb{P}_{X_s^{k,x,\alpha^n}}) - b_{1,k}(s, X_s^{k,x,\alpha^n}, \mathbb{P}_{X_s^{k,x,\alpha}}) + b_2(s, X_s^{k,x,\alpha}, \alpha_s^n) - b_2(s, X_s^{k,x,\alpha}, \alpha_s)\right)ds
\end{aligned}
$$

squaring and applying the expectation to the power $\frac{1}{2}$ on both sides and the Cauchy-Schwarz inequality yields:

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha}\big|^2\Big]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t \int_0^1 2\left(b'_{1,k}(u,X_u^{k,x,\alpha}+\theta(X_u^{k,x,\alpha^n}-X_u^{k,x,\alpha}),\mathbb{P}_{X_u^{k,x,\alpha}})+b'_2(u,X_u^{k,x,\alpha^n}+\theta(X_u^{k,x,\alpha^n}-X_u^{k,x,\alpha}),\alpha_u^n)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}} \\
&\quad \times \Big\{\mathbb{E}\Big[\Big|\int_0^t \big|b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big|^2 du\Big|^2\Big]^{\frac{1}{4}} \\
&\quad + \mathbb{E}\Big[\Big|\int_0^t \big|b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big|^2 du\Big|^2\Big]^{\frac{1}{4}}\Big\}.
\end{aligned}$$

One can show that

$$\mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t \int_0^1 2\left(b'_{1,k}(u,X_u^{k,x,\alpha}+\theta(X_u^{k,x,\alpha^n}-X_u^{k,x,\alpha}),\mathbb{P}_{X_u^{k,x,\alpha}})+b'_2(u,X_u^{k,x,\alpha^n}+\theta(X_u^{k,x,\alpha^n}-X_u^{k,x,\alpha}),\alpha_u^n)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

is finite using the same approach as we did when showing that

$$\mathbb{E}\Big[\Big|\int_0^t e^{\int_s^t \int_0^1 2\left(b'_{1,n}(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\mathbb{P}_{X_u^{x,\alpha^2}})+b'_2(u,X_u^{x,\alpha^2}+\theta(X_u^{n,x,\alpha^1}-X_u^{x,\alpha^2}),\alpha_u^1)\right)d\theta du} ds\Big|^2\Big]^{\frac{1}{4}}$$

is finite. So, using the Minkowski inequality, the Lipschitz continuity of $b_{1,k}$ in the measure variable and the Lipschitz continuity of b_2, we get:

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{k,x,\alpha_n} - X_t^{k,x,\alpha}\big|^2\Big]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\Big[\Big|\int_0^t \big|b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big|^2 du\Big|^2\Big]^{\frac{1}{4}} \\
&\quad + \mathbb{E}\Big[\Big|\int_0^t \big|b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big|^2 du\Big|^2\Big]^{\frac{1}{4}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}) - b_{1,k}(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha}})\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^t \mathbb{E}\big[\big|b_2(u, X_u^{k,x,\alpha}, \alpha_u^n) - b_2(u, X_u^{k,x,\alpha}, \alpha_u)\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}}, \\
&\lesssim \Big(\int_0^t \mathcal{K}(\mathbb{P}_{X_u^{k,x,\alpha^n}}, \mathbb{P}_{X_u^{k,x,\alpha}})^2 du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^n - \alpha_u\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big|\big]^2 du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^n - \alpha_u\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \mathbb{E}\big[\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big|^2\big] du\Big)^{\frac{1}{2}} + \Big(\int_0^t \mathbb{E}\big[\big|\alpha_u^n - \alpha_u\big|^4\big]^{\frac{1}{2}} du\Big)^{\frac{1}{2}}
\end{aligned}$$

after squaring both size and applying the Grönwall's inequality, we get:

$$\begin{aligned}\mathbb{E}\Big[\big|X_t^{k,x,\alpha_n} - X_t^{k,x,\alpha}\big|^2\Big] &\lesssim \int_0^t \mathbb{E}\big[\big|\alpha_u^n - \alpha_u\big|^4\big]^{\frac{1}{2}} du,\\ &= \int_0^t \big(\mathbb{E}\big[\big|\alpha_u^n - \alpha_u\big|^4\big]^{\frac{1}{4}}\big)^2 du,\\ &\lesssim \tau(\alpha^n, \alpha)^2,\end{aligned}$$

since every sequence $\{\alpha_n\}_{n\geq 0}$ in $\mathcal{D}$ converges to $\alpha \in \mathcal{D}$ in the norm $\sup_{0\leq t\leq T} \mathbb{E}[|\cdot|^4]^{\frac{1}{4}}$, we have:

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \mathbb{E}\Big[\big|X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha}\big|^2\Big]^{\frac{1}{2}} = 0, \text{ for a given } k,$$

which proves the lemma. □

Lemma 4.3.2. We consider a sequence $\{\alpha_n\}_{n\geq 0}$ in $\mathcal{D}$ converging to $\alpha \in \mathcal{D}$ in the norm $\sup_{0\leq t\leq T} \mathbb{E}[|\cdot|^4]^{\frac{1}{4}}$ for every $p \geq 1$. Hence, we assert the following:

1. $\lim\limits_{n\to\infty} \big|J_k(\alpha^n) - J_k(\alpha)\big| = 0$, for any given $k \in \mathbb{N}$, and also, the function J_k is continuous.

2. $\big|J_n(\alpha) - J(\alpha)\big| \leq \epsilon_n$, such that $\lim\limits_{n\to\infty} \epsilon_n = 0$.

Proof of Lemma 4.3.2. We start with expression (1.) of the lemma. We have:

$$\begin{aligned}&\big|J_k(\alpha^n) - J_k(\alpha)\big|\\ &= \Bigg|\mathbb{E}\Bigg[\int_0^T f(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}, \alpha_u^n)du + g(X_T^{k,x,\alpha^n}, \mathbb{P}_{X_T^{n,x,\alpha^n}})\\ &\quad - \int_0^T f(u, X_u^{k,x,\alpha}, \mathbb{P}_{X_u^{k,x,\alpha}}, \alpha_u)du - g(X_T^{k,x,\alpha}, \mathbb{P}_{X_T^{k,x,\alpha}})\Bigg]\Bigg|,\\ &\leq \mathbb{E}\Bigg[\int_0^T \big|f(u, X_u^{k,x,\alpha^n}, \mathbb{P}_{X_u^{k,x,\alpha^n}}, \alpha_u^n) - f(u, X_u^{k,x,\alpha}, \mathbb{P}_{X_u^{k,x,\alpha}}, \alpha_u)\big|du + \big|g(X_T^{k,x,\alpha^n}, \mathbb{P}_{X_T^{k,x,\alpha^n}})\\ &\quad - g(X_T^{k,x,\alpha}, \mathbb{P}_{X_T^{k,x,\alpha}})\big|\Bigg],\end{aligned}$$

using the Lipschitz continuity of f and g, we obtain,

$$
\begin{aligned}
&\big|J_k(\alpha^n) - J_k(\alpha)\big| \\
&\leq \mathbb{E}\Big[\int_0^T \big(\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big| + |\alpha_u^n - \alpha_u| + \mathcal{K}(\mathbb{P}_{X_u^{k,x,\alpha^n}}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du + \big|X_T^{k,x,\alpha^n} - X_T^{k,x,\alpha}\big| \\
&\quad + \mathcal{K}(\mathbb{P}_{X_T^{n,x,\alpha^n}}, \mathbb{P}_{X_T^{k,x,\alpha}})\Big] \\
&\leq \int_0^T \big(\mathbb{E}\big[\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big|\big] + \mathbb{E}[|\alpha_u^n - \alpha_u|] + \mathcal{K}(\mathbb{P}_{X_u^{k,x,\alpha^n}}, \mathbb{P}_{X_u^{k,x,\alpha}})\big)du \\
&\quad + \mathbb{E}\big[\big|X_T^{k,x,\alpha^n} - X_T^{k,x,\alpha}\big|\big] + \mathcal{K}\big(\mathbb{P}_{X_T^{k,x,\alpha^n}}, \mathbb{P}_{X_T^{k,x,\alpha}}\big), \\
&\leq \int_0^T \big(2\mathbb{E}\big[\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big|\big] + \mathbb{E}[|\alpha_u^n - \alpha_u|]\big)du + 2\mathbb{E}\big[\big|X_T^{k,x,\alpha^n} - X_T^{k,x,\alpha}\big|\big], \\
&\leq \int_0^T \big(2\mathbb{E}\big[\big|X_u^{k,x,\alpha^n} - X_u^{k,x,\alpha}\big|^2\big]^{\frac{1}{2}} + \mathbb{E}[|\alpha_u^n - \alpha_u|^4]^{\frac{1}{4}}\big)du + 2\mathbb{E}\big[\big|X_T^{k,x,\alpha^n} - X_T^{k,x,\alpha}\big|^2\big]^{\frac{1}{2}}, \\
&\leq C\bigg(\tau(\alpha^n, \alpha) + \sup_{0\leq t\leq T} \mathbb{E}\big[\big|X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha}\big|^2\big]^{\frac{1}{2}}\bigg)
\end{aligned}
$$

the results follows by expression (2.) of Lemma 4.3.1 and the convergence of $\{\alpha_n\}_{n\geq 0}$ to α. We now prove expression (2.) of Lemma 4.3.2. Using the same approach as in the previous proof, we have:

$$
\begin{aligned}
&\big|J_n(\alpha) - J(\alpha)\big| \\
&\leq \bigg|\mathbb{E}\Big[\int_0^T f(u, X_u^{n,x,\alpha}, \mathbb{P}_{X_u^{n,x,\alpha}}, \alpha_u)du + g(X_T^{n,x,\alpha}, \mathbb{P}_{X_T^{n,x,\alpha}}) \\
&\qquad + \int_0^T f(u, X_u^{x,\alpha}, \mathbb{P}_{X_u^{x,\alpha}}, \alpha_u)du + g(X_T^{x,\alpha}, \mathbb{P}_{X_T^{x,\alpha}})\Big]\bigg| \\
&\leq \int_0^T 2\mathbb{E}\big[\big|X_u^{n,x,\alpha} - X_u^{x,\alpha}\big|^2\big]^{\frac{1}{2}} du + 2\mathbb{E}\big[\big|X_T^{n,x,\alpha} - X_T^{x,\alpha}\big|^2\big]^{\frac{1}{2}} \\
&\leq C \sup_{0\leq t\leq T} \mathbb{E}\big[\big|X_t^{n,x,\alpha} - X_t^{x,\alpha}\big|^2\big]^{\frac{1}{2}} \\
&\leq C\bigg\{\sup_{0\leq t\leq T} \frac{e^{-\frac{x^2}{8t}}}{(2\pi t)^{\frac{1}{8}}}\bigg(\int_{\mathbb{R}} \big|b_{1,n}(t, z, \mathbb{P}_{X_t^{x,\alpha}}) - b_1(t, z, \mathbb{P}_{X_t^{x,\alpha}})\big|^4 e^{-\frac{z^2}{4t}} dz\bigg)^{\frac{1}{4}}\bigg\},
\end{aligned}
$$

using expression (1.) of Lemma 4.3.1, we have:

$$
\big|J_n(\alpha) - J(\alpha)\big| \leq \epsilon_n \text{ such that we have } \lim_{n\to\infty} \epsilon_n = 0.
$$

which proves the lemma. □

Lemma 4.3.3. Let $\{\alpha_n\}_{n\geq 0}$ be a sequence in $\mathcal{A}$ converging to $\alpha \in \mathcal{A}$. Denote by $\mathcal{G}^{\alpha}$ and $\mathcal{G}^{n,\alpha^n}$ the first variation process associated to $X^{x,\alpha}$ and X^{n,x,α^n}, respectively. Then for every $0 \leq s \leq t \leq T$

1. $\mathbb{E}\big[\big|\mathcal{G}^{\alpha}_{t,s} - \mathcal{G}^{n,\alpha^n}_{t,s}\big|^2\big]^{\frac{1}{2}}$ converges to 0 as n tends to ∞,
2. $\mathbb{E}\big[\big|P_t - P^n_t\big|\big]$ converges to 0 as n tends to ∞, where

$$\begin{aligned} P_t =&\mathbb{E}\Big[\mathcal{G}^{\alpha}_{t,T}\partial_z g(X^{x,\alpha}_T, \mathbb{P}_{X^{x,\alpha}_T}) + \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}^{x,\tilde{\alpha}}_T, \mathbb{P}_{X^{x,\alpha}_T}; X^{x,\alpha}_T)\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,T}] \\ &+ \int_t^T \Big\{\mathcal{G}^{\alpha}_{t,s}\partial_z f(s, X^{x,\alpha}_s, \mathbb{P}_{X^{x,\alpha}_s}, \alpha_s) \\ &+ \tilde{\mathbb{E}}[\partial_\mu f(s, \tilde{X}^{x,\tilde{\alpha}}_s, \mathbb{P}_{X^{x,\alpha}_s}, \tilde{\alpha}_s; X^{x,\alpha}_s)\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,s}]\Big\}\mathrm{d}s\Big|\mathcal{F}_t\Big], \end{aligned}$$

and P^n is defined similarly with $(X^{n,\alpha^n}, \mathcal{G}^{n,\alpha^n}, \alpha^n)$ instead of $(X^{\alpha}, \mathcal{G}^{\alpha}, \alpha)$.

Proof of Lemma 4.3.3. We start with the proof of expression (1) of the lemma. Using the notation $b_1(u, z, \mathbb{P}_{X^{x,\alpha}_u}) = b_{1,u}$, $b_2(u, X^{x,\alpha}_u, \alpha_u) = b_{2,u}$,

$b_{1,n}(u, z, \mathbb{P}_{X^{n,x,\alpha^n}_u}) = b^n_{1,u}$, $b_2(u, X^{n,x,\alpha^n}_u, \alpha^n_u) = b^n_{2,u}$, and $\partial_\mu b_u(r, \tilde{X}^{x,\tilde{\alpha}}_u, \mathbb{P}_{X^{x,\alpha}_u}, \tilde{\alpha}_u; X^{x,\alpha}_u) = \partial_\mu \tilde{b}_u$. Using triangular inequality, we have:

$$\begin{aligned} &\mathbb{E}\big[\big|\mathcal{G}^{\alpha}_{s,t} - \mathcal{G}^{n,\alpha^n}_{s,t}\big|^2\big]^{\frac{1}{2}} \\ &= \mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z) + \int_s^t \partial_z b_{2,u}\mathrm{d}u} + \int_s^t e^{-\int_r^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z) + \int_r^t \partial_z b_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}]\mathrm{d}r \\ &- e^{-\int_s^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z) + \int_s^t \partial_z b^n_{2,u}\mathrm{d}u} - \int_s^t e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z) + \int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r\tilde{\mathcal{G}}^{\tilde{\alpha}^n}_{s,r}]\mathrm{d}r\Big|^2\Big]^{\frac{1}{2}} \\ &\leq \mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z) + \int_s^t \partial_z b_{2,u}\mathrm{d}u} - e^{-\int_s^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z) + \int_s^t \partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{2}} \\ &+ \mathbb{E}\Big[\Big|\int_s^t \Big\{e^{-\int_r^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z) + \int_r^t \partial_z b_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}] \\ &- e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z) + \int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r\tilde{\mathcal{G}}^{\tilde{\alpha}^n}_{s,r}]\Big\}\mathrm{d}r\Big|^2\Big]^{\frac{1}{2}} = I_{1,n} + I_{2,n} \end{aligned}$$

Repeated use of Cauchy-Schwartz inequality give

$$
\begin{aligned}
I_{1,n} \leq & 4\mathbb{E}\Big[e^{8\int_s^t \partial_z b_{2,u} \mathrm{du}}\Big]^{\frac{1}{8}} \mathbb{E}\Big[\Big|e^{-\int_s^t \int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})} - e^{-\int_s^t \int_{\mathbb{R}} b_1^n L^{X^{n,x,\alpha^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{4}} \\
& \times \Big(\mathbb{E}\Big[e^{-4\int_s^t \int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})}\Big]^{\frac{1}{8}} + \mathbb{E}\Big[e^{-4\int_s^t \int_{\mathbb{R}} b_{1,u}^n L^{X^{n,x,\alpha^n}}(\mathrm{du},\mathrm{dz})}\Big]^{\frac{1}{8}}\Big) \\
& + 4\mathbb{E}\Big[e^{-8\int_s^t \int_{\mathbb{R}} b_{1,u}^n L^{X^{n,x,\alpha^n}}(\mathrm{du},\mathrm{dz})}\Big]^{\frac{1}{8}} \mathbb{E}\Big[\Big|e^{\int_s^t \partial_z b_{2,u} \mathrm{du}} - e^{\int_s^t \partial_z b_{2,u}^n \mathrm{du}}\Big|^2\Big]^{\frac{1}{4}} \\
& \times \Big(\mathbb{E}\Big[e^{4\int_s^t \partial_z b_{2,u} \mathrm{du}}\Big]^{\frac{1}{8}} + \mathbb{E}\Big[e^{4\int_s^t \partial_z b_{2,u}^n \mathrm{du}}\Big]^{\frac{1}{8}}\Big) \\
= & I_{11,n} \times I_{12,n}(I_{13,n} + I_{14,n}) + I_{15,n} \times I_{16,n}(I_{17,n} + I_{18,n}) \qquad (4.12)
\end{aligned}
$$

We can show as in Lemma 3.5.1 that $I_{11,n}, I_{13,n} \times I_{14,n}, I_{15,n}, I_{17,n} \times I_{18,n}$ are uniformly bounded. $I_{16,n}$ converges for at least a subsequence since $\partial_z b_2$ is continuous and bounded. The convergence of $I_{12,n}$ can be shown similarly to the proof of convergence of R^n_{14} expressed in (3.52).

$$
\begin{aligned}
& I_{2,n} \\
& \lesssim \int_s^t \mathbb{E}\Big[\Big|e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z b_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}] - e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z b^n_{2,u}\mathrm{d}u} \\
& \times \tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n \tilde{\mathcal{G}}^{\tilde{\alpha}^n}_{s,r}]\Big|^2\Big]^{\frac{1}{2}}\mathrm{d}r \\
& \lesssim \int_s^t \mathbb{E}\Big[\Big|e^{\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]\big(e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} - e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\big) \\
& + e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]\big(e^{\int_r^t \partial_z b_{2,u}\mathrm{d}u} - e^{\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\big) \\
& - e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z b_{2,u}\mathrm{d}u}\big(\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}] - \tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}]\big)\Big|^2\Big]^{\frac{1}{2}}\mathrm{d}r \\
& \lesssim \int_s^t \Big\{\mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^4] e^{4\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\big(e^{-2\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} + e^{-2\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\big)\Big]^{\frac{1}{4}} \\
& \times \mathbb{E}\Big[\Big|e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} - e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}} \\
& + \mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^4] e^{-4\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}\big(e^{2\int_r^t \partial_z b_{2,u}\mathrm{d}u} + e^{2\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\big)\Big]^{\frac{1}{4}} \\
& \times \mathbb{E}\Big[\Big|e^{\int_r^t \partial_z b_{2,u}\mathrm{d}u} - e^{\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}} \\
& + \mathbb{E}\Big[e^{-4\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)+4\int_r^t \partial_z b_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^2 + |\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}|^2]\Big]^{\frac{1}{4}} \\
& \times \mathbb{E}\big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r} - \partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}|^2]\big]^{\frac{1}{4}}\Big\}\mathrm{d}r \\
& \lesssim \int_s^t \Big\{\mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^{16}]\Big]^{\frac{1}{16}}\mathbb{E}\Big[\big(e^{-8\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} + e^{-8\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\big)\Big]^{\frac{1}{8}} \\
& \times \mathbb{E}\Big[e^{16\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\Big]^{\frac{1}{16}} \times \mathbb{E}\Big[\Big|e^{-\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} - e^{-\int_r^t\int_{\mathbb{R}} b^n_{1,u} L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}} \\
& + \mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^{16}]\Big]^{\frac{1}{16}}\mathbb{E}\Big[e^{-16\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}\Big]^{\frac{1}{16}}\mathbb{E}\Big[\big(e^{8\int_r^t \partial_z b_{2,u}\mathrm{d}u} + e^{8\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\big)\Big]^{\frac{1}{8}} \\
& \times \mathbb{E}\Big[\Big|e^{\int_r^t \partial_z b_{2,u}\mathrm{d}u} - e^{\int_r^t \partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}} + \mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r} - \partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}|^2]\Big]^{\frac{1}{4}} \\
& \times \mathbb{E}\Big[e^{-8\int_r^t\int_{\mathbb{R}} b_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}\Big]^{\frac{1}{8}}\mathbb{E}\Big[e^{8\int_r^t \partial_z b_{2,u}\mathrm{d}u}\big(\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}|^4 + |\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}|^4]\big)\Big]^{\frac{1}{8}}\Big\}\mathrm{d}r \\
& \lesssim \int_s^t \{I_{21,n} \times I_{22,n} \times I_{23,n} \times I_{24,n} + I_{21,n} \times I_{25,n} \times I_{26,n} \times I_{27,n} + I_{28,n} \times I_{210,n} \times I_{211,n}\}\mathrm{d}r.
\end{aligned}
$$

Boundedness of $I_{22,n}, I_{25,n}$, and $I_{210,n}$ follows by Lemma 3.5.1 after applying the Girsanov's transform. Boundedness of $I_{23,n}, I_{26,n}$, and $I_{211,n}$ hold true by assumption on the boundedness of $\partial_z b_2, \partial_\mu b$ and the integrability of the first variation process. We can see that $I_{21,n}$ is finite using the integrability of the first variation process shown in Lemma 3.5.3. Hence, we obtain:

$$
\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{G}^{\alpha}_{s,t}-\mathcal{G}^{n,\alpha^n}_{s,t}\big|^2\Big]^{\frac{1}{2}}\\
\lesssim&\mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_s^t\int_{\mathbb{R}} b_1^n L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}+\mathbb{E}\Big[\Big|e^{\int_s^t\partial_z b_{2,u}\mathrm{d}u}-e^{\int_s^t\partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}\\
&+\int_s^t\Big\{\mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_s^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}\\
&+\mathbb{E}\Big[\Big|e^{\int_r^t\partial_z b_{2,u}\mathrm{d}u}-e^{\int_r^t\partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}+\mathbb{E}\Big[\tilde{\mathbb{E}}[|\partial_\mu\tilde{b}^n_r\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}-\partial_\mu\tilde{b}_r\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}|^2]\Big]^{\frac{1}{4}}\Big\}\mathrm{d}r.
\end{aligned}
\tag{4.13}
$$

Let us consider the expression $\tilde{\mathbb{E}}[|\partial_\mu\tilde{b}^n_t\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,t}-\partial_\mu\tilde{b}_t\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,t}|^2]$

$$
\begin{aligned}
&\tilde{\mathbb{E}}[|\partial_\mu\tilde{b}^n_t\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,t}-\partial_\mu\tilde{b}_t\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,t}|^2]\\
\lesssim&\tilde{\mathbb{E}}\Big[\Big|\partial_\mu\tilde{b}_t e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+\int_s^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}-\partial_\mu\tilde{b}^n_t e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)+\int_s^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\Big|^2\Big]\\
&+\tilde{\mathbb{E}}\Big[\Big|\int_s^t\Big\{\partial_\mu\tilde{b}_t e^{-\int_r^t\int_{\mathbb{R}}\tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu\tilde{b}_r\tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}]\\
&-\partial_\mu\tilde{b}^n_t e^{-\int_r^t\int_{\mathbb{R}}\tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu\tilde{b}^n_r\tilde{\mathcal{G}}^{\tilde{\alpha}^n}_{s,r}]\Big\}\mathrm{d}r\Big|^2\Big]=J_{1,n}+J_{2,n}.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_{1,n}=&\tilde{\mathbb{E}}\Big[\Big|\partial_\mu\tilde{b}_t e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+\int_s^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}-\partial_\mu\tilde{b}^n_t e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)+\int_s^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\Big|^2\Big]\\
\leq&4\tilde{\mathbb{E}}\Big[(\partial_\mu\tilde{b}_t)^8e^{8\int_s^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}\Big]^{\frac{1}{8}}\tilde{\mathbb{E}}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_s^t\int_{\mathbb{R}}\tilde{b}^n_1L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}\\
&\times\Big(\tilde{\mathbb{E}}\Big[e^{-4\int_s^t\int_{\mathbb{R}}\tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)}\Big]^{\frac{1}{8}}+\tilde{\mathbb{E}}\Big[e^{-4\int_s^t\int_{\mathbb{R}}\tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\Big]^{\frac{1}{8}}\Big)\\
&+4\tilde{\mathbb{E}}\Big[e^{-8\int_s^t\int_{\mathbb{R}}\tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\Big]^{\frac{1}{8}}\tilde{\mathbb{E}}\Big[\Big|\partial_\mu\tilde{b}_t e^{\int_s^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}-\partial_\mu\tilde{b}^n_t e^{\int_s^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}\\
&\times\Big(\tilde{\mathbb{E}}\Big[(\partial_\mu\tilde{b}_t)^4e^{4\int_s^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}\Big]^{\frac{1}{8}}+\tilde{\mathbb{E}}\Big[(\partial_\mu\tilde{b}^n_t)^4e^{4\int_s^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\Big]^{\frac{1}{8}}\Big)\\
=&J_{11,n}\times J_{12,n}(J_{13,n}+J_{14,n})+J_{15,n}\times J_{16,n}(J_{17,n}+J_{18,n}).
\end{aligned}
$$

Convergence of $J_{1,n}$ holds similarly by applying the same arguments as with the convergence of $I_{1,n}$.

$$
\begin{aligned}
&J_{2,n} \\
&= \tilde{\mathbb{E}}\Big[\Big|\int_s^t \Big\{\partial_\mu \tilde{b}_t e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z \tilde{b}_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}] \\
&- \partial_\mu \tilde{b}^n_t e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}^n_{1,u} L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z \tilde{b}^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{\tilde{\alpha}^n}_{s,r}]\Big\}\mathrm{d}r\Big|^2\Big] \\
&\lesssim \int_s^t \tilde{\mathbb{E}}\Big[\Big|\partial_\mu \tilde{b}^n_t e^{\int_r^t \partial_z \tilde{b}^n_{2,u}\mathrm{d}u}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]\big(e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} - e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}^n_{1,u} L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\big) \\
&+ e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)}\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]\big(\partial_\mu \tilde{b}_t e^{\int_r^t \partial_z \tilde{b}_{2,u}\mathrm{d}u} - \partial_\mu \tilde{b}^n_t e^{\int_r^t \partial_z \tilde{b}^n_{2,u}\mathrm{d}u}\big) \\
&- \partial_\mu \tilde{b}_t e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+\int_r^t \partial_z \tilde{b}_{2,u}\mathrm{d}u}\big(\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}] - \tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}]\big)\Big|^2\Big]\mathrm{d}r \\
&\lesssim \int_s^t \Big\{|\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]|^2 \tilde{\mathbb{E}}\Big[\Big|\partial_\mu \tilde{b}^n_t e^{\int_r^t \partial_z \tilde{b}^n_{2,u}\mathrm{d}u}\big(e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)} - e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}^n_{1,u} L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\big)\Big|^2\Big] \\
&+ |\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r}]|^2 \tilde{\mathbb{E}}\Big[\Big|e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)}\big(\partial_\mu \tilde{b}_t e^{\int_r^t \partial_z \tilde{b}_{2,u}\mathrm{d}u} - \partial_\mu \tilde{b}^n_t e^{\int_r^t \partial_z \tilde{b}^n_{2,u}\mathrm{d}u}\big)\Big|^2\Big] \\
&+ |\tilde{\mathbb{E}}[\partial_\mu \tilde{b}^n_r \tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{s,r} - \partial_\mu \tilde{b}_r \tilde{\mathcal{G}}^{\tilde{\alpha}}_{s,r}]|^2 \tilde{\mathbb{E}}\Big[(\partial_\mu \tilde{b}_t)^2 e^{-2\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)+2\int_r^t \partial_z \tilde{b}_{2,u}\mathrm{d}u}\Big]\Big\}\mathrm{d}r
\end{aligned}
$$

$$\lesssim \int_s^t \Big\{|\tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r^n \tilde{\mathcal{G}}_{s,r}^{n,\tilde{\alpha}^n}]|^2 \tilde{\mathbb{E}}\Big[(\partial_\mu \tilde{b}_t^n)^8 e^{8\int_r^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\Big]^{\frac{1}{4}} \tilde{\mathbb{E}}\Big[e^{-4\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})}$$
$$+ e^{-4\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u}^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big]^{\frac{1}{4}} \tilde{\mathbb{E}}\Big[\Big|e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})} - e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u}^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{2}}$$
$$+ |\tilde{\mathbb{E}}[\partial_\mu \tilde{b}_r^n \tilde{\mathcal{G}}_{s,r}^{n,\tilde{\alpha}^n}]|^2 \tilde{\mathbb{E}}\Big[e^{-8\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{du},\mathrm{dz})}\Big]^{\frac{1}{4}} \tilde{\mathbb{E}}\big[(\partial_\mu \tilde{b}_t)^4 e^{4\int_r^t \partial_z \tilde{b}_{2,u} \mathrm{du}}$$
$$+ (\partial_\mu \tilde{b}_t^n)^4 e^{4\int_r^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\big]^{\frac{1}{4}} \tilde{\mathbb{E}}\big[\big|\partial_\mu \tilde{b}_t e^{\int_r^t \partial_z \tilde{b}_{2,u} \mathrm{du}} - \partial_\mu \tilde{b}_t^n e^{\int_r^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\big|^2\big]^{\frac{1}{2}}$$
$$+ \tilde{\mathbb{E}}[|\partial_\mu \tilde{b}_r^n \tilde{\mathcal{G}}_{s,r}^{n,\tilde{\alpha}^n} - \partial_\mu \tilde{b}_r \tilde{\mathcal{G}}_{s,r}^{\tilde{\alpha}}|^2] \tilde{\mathbb{E}}\Big[(\partial_\mu \tilde{b}_t)^2 e^{-2\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{du},\mathrm{dz}) + 2\int_r^t \partial_z \tilde{b}_{2,u} \mathrm{du}}\Big]\Big\} \mathrm{d}r.$$

Using similarly arguments as in $I_{2,n}$ and applying Gronwall's lemma, we obtain

$$\tilde{\mathbb{E}}[|\partial_\mu \tilde{b}_t^n \tilde{\mathcal{G}}_{s,t}^{n,\tilde{\alpha}^n} - \partial_\mu \tilde{b}_t \tilde{\mathcal{G}}_{s,t}^{\tilde{\alpha}}|^2]$$
$$\lesssim \tilde{\mathbb{E}}\Big[\Big|e^{-\int_s^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{du},\mathrm{dz})} - e^{-\int_s^t \int_{\mathbb{R}} \tilde{b}_1^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{4}} + \tilde{\mathbb{E}}\Big[\Big|\partial_\mu \tilde{b}_t e^{\int_s^t \partial_z \tilde{b}_{2,u} \mathrm{du}}$$
$$- \partial_\mu \tilde{b}_t^n e^{\int_s^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\Big|^2\Big]^{\frac{1}{4}} + \int_s^t \Big\{\tilde{\mathbb{E}}\Big[\Big|e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})} - e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u}^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{2}}$$
$$+ \tilde{\mathbb{E}}\big[\big|\partial_\mu \tilde{b}_t e^{\int_r^t \partial_z \tilde{b}_{2,u} \mathrm{du}} - \partial_\mu \tilde{b}_t^n e^{\int_r^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\big|^2\big]^{\frac{1}{2}} + \tilde{\mathbb{E}}[|\partial_\mu \tilde{b}_r^n \tilde{\mathcal{G}}_{s,r}^{n,\tilde{\alpha}^n} - \partial_\mu \tilde{b}_r \tilde{\mathcal{G}}_{s,r}^{\tilde{\alpha}}|^2]\Big\} \mathrm{d}r$$
$$\lesssim \tilde{\mathbb{E}}\Big[\Big|e^{-\int_s^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{du},\mathrm{dz})} - e^{-\int_s^t \int_{\mathbb{R}} \tilde{b}_1^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{4}} + \tilde{\mathbb{E}}\Big[\Big|\partial_\mu \tilde{b}_t e^{\int_s^t \partial_z \tilde{b}_{2,u} \mathrm{du}}$$
$$- \partial_\mu \tilde{b}_t^n e^{\int_s^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\Big|^2\Big]^{\frac{1}{4}} + \int_s^t \Big\{\tilde{\mathbb{E}}\Big[\Big|e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u} L^{X^{x,\alpha}}(\mathrm{du},\mathrm{dz})} - e^{-\int_r^t \int_{\mathbb{R}} \tilde{b}_{1,u}^n L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{du},\mathrm{dz})}\Big|^2\Big]^{\frac{1}{2}}$$
$$+ \tilde{\mathbb{E}}\big[\big|\partial_\mu \tilde{b}_t e^{\int_r^t \partial_z \tilde{b}_{2,u} \mathrm{du}} - \partial_\mu \tilde{b}_t^n e^{\int_r^t \partial_z \tilde{b}_{2,u}^n \mathrm{du}}\big|^2\big]^{\frac{1}{2}}\Big\} \mathrm{d}r \tag{4.14}$$

Substituting (4.14) into (4.13) gives

$$
\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{G}^{\alpha}_{s,t}-\mathcal{G}^{n,\alpha^n}_{s,t}\big|^2\Big]^{\frac{1}{2}}\\
&\lesssim \mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_s^t\int_{\mathbb{R}} b_1^n L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}+\mathbb{E}\Big[\Big|e^{\int_s^t\partial_z b_{2,u}\mathrm{d}u}-e^{\int_s^t\partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}\\
&+\int_s^t\Big\{\mathbb{E}\Big[\Big|e^{-\int_s^t\int_{\mathbb{R}} b_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_s^t\int_{\mathbb{R}} b^n_{1,u}L^{X^{n,x,\alpha^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}\\
&+\mathbb{E}\Big[\Big|e^{\int_r^t\partial_z b_{2,u}\mathrm{d}u}-e^{\int_r^t\partial_z b^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}\Big\}\mathrm{d}r\\
&+\int_s^t\Big\{\mathbb{E}\Big[\tilde{\mathbb{E}}\Big[\Big|e^{-\int_r^t\int_{\mathbb{R}} \tilde{b}_{1,u}L^{\tilde{X}^{x,\tilde{\alpha}}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_r^t\int_{\mathbb{R}} \tilde{b}^n_1 L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{4}}\Big]^{\frac{1}{4}}\\
&+\mathbb{E}\Big[\tilde{\mathbb{E}}\Big[\Big|\partial_\mu\tilde{b}_r e^{\int_r^t\partial_z\tilde{b}_{2,u}\mathrm{d}u}-\partial_\mu\tilde{b}^n_r e^{\int_r^t\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\Big|^2\Big]^{\frac{1}{4}}\Big]^{\frac{1}{4}}\Big\}\mathrm{d}r\\
&+\int_s^t\int_s^r\Big\{\mathbb{E}\Big[\tilde{\mathbb{E}}\Big[\Big|e^{-\int_v^r\int_{\mathbb{R}} \tilde{b}_{1,u}L^{X^{x,\alpha}}(\mathrm{d}u,\mathrm{d}z)}-e^{-\int_v^r\int_{\mathbb{R}} \tilde{b}^n_{1,u}L^{\tilde{X}^{n,x,\tilde{\alpha}^n}}(\mathrm{d}u,\mathrm{d}z)}\Big|^2\Big]^{\frac{1}{2}}\Big]^{\frac{1}{4}}\\
&+\mathbb{E}\Big[\tilde{\mathbb{E}}\big[\big|\partial_\mu\tilde{b}_r e^{\int_v^r\partial_z\tilde{b}_{2,u}\mathrm{d}u}-\partial_\mu\tilde{b}^n_r e^{\int_v^r\partial_z\tilde{b}^n_{2,u}\mathrm{d}u}\big|^2\big]^{\frac{1}{2}}\Big]^{\frac{1}{4}}\Big\}\mathrm{d}v\mathrm{d}r\\
&=I_{12,n}+I_{16,n}+\int_s^t\{I_{24,n}+I_{27,n}+I_{281,n}+I_{282,n}\}\mathrm{d}r+\int_s^t\int_s^r\{I_{283,n}+I_{284,n}\}\mathrm{d}v\mathrm{d}r
\end{aligned}
$$

Convergence of $I_{24,n}, I_{12,n}$ and $I_{283,n}$ holds by using the same arguments used in showing the convergence of R^n_{14} . Convergence of $I_{16,n}, I_{27,n}, I_{282,n}$ and $I_{284,n}$ follows by dominated convergence since $\partial_z b^n_2$ and $\partial_\mu\tilde{b}_r$ are continuous and bounded by assumption which concludes the convergence of $I_{214,n}$, proving the convergence of $\mathbb{E}\big[\big|\mathcal{G}^{\alpha}_{s,t}-\mathcal{G}^{n,\alpha^n}_{s,t}\big|^2\big]^{\frac{1}{2}}$.

Let us now prove (2):

$$
\begin{aligned}
&\mathbb{E}\big[\big|P_t-P^n_t\big|\big]\\
=&\mathbb{E}\Big\{\Big|\mathbb{E}\Big[\mathcal{G}^{\alpha}_{t,T}\partial_z g_T+\tilde{\mathbb{E}}[\partial_\mu\tilde{g}_T\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,T}]+\int_t^T\big(\mathcal{G}^{\alpha}_{t,s}\partial_z f_s+\tilde{\mathbb{E}}[\partial_\mu\tilde{f}_s\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,s}]\big)\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&-\mathbb{E}\Big[\mathcal{G}^{n,\alpha^n}_{t,T}\partial_z g^n_T+\tilde{\mathbb{E}}[\partial_\mu\tilde{g}^n_T\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{t,T}]+\int_t^T\big(\mathcal{G}^{n,\alpha^n}_{t,s}\partial_z f^n_s+\tilde{\mathbb{E}}[\partial_\mu\tilde{f}^n_s\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{t,T}]\big)\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big|\Big\}\\
\leq&\mathbb{E}\big[\big|\mathcal{G}^{\alpha}_{t,T}\partial_z g_T-\mathcal{G}^{n,\alpha^n}_{t,T}\partial_z g^n_T\big|\big]+\mathbb{E}\big[\big|\tilde{\mathbb{E}}[\partial_\mu\tilde{g}_T\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,T}-\partial_\mu\tilde{g}^n_T\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{t,T}]\big|\big]\\
&+\int_t^T\mathbb{E}\big[\big|\mathcal{G}^{\alpha}_{t,s}\partial_z f_s-\mathcal{G}^{n,\alpha^n}_{t,s}\partial_z f^n_s\big|\big]\mathrm{d}s+\int_t^T\mathbb{E}\big[\big|\tilde{\mathbb{E}}[\partial_\mu\tilde{f}_s\tilde{\mathcal{G}}^{\tilde{\alpha}}_{t,T}-\partial_\mu\tilde{f}^n_s\tilde{\mathcal{G}}^{n,\tilde{\alpha}^n}_{t,T}]\big|\big]\mathrm{d}s.
\end{aligned}
$$

Adding and substracting $\partial_z g_T \mathcal{G}_{t,T}^{n,\alpha^n}$ and $\partial_z f_s \mathcal{G}_{t,s}^{n,\alpha^n}$ and using the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
&\mathbb{E}\big[|P_t - P_t^n|\big] \\
&\lesssim \mathbb{E}\Big[\big|\partial_z g_T\big|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\big|\mathcal{G}_{t,T}^{\alpha} - \mathcal{G}_{t,T}^{n,\alpha^n}\big|^2\Big]^{\frac{1}{2}} + \mathbb{E}\Big[\big|\mathcal{G}_{t,T}^{n,\alpha^n}\big|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\big|\partial_z g_T - \partial_z g_T^n\big|^2\Big]^{\frac{1}{2}} \\
&+ \mathbb{E}[\tilde{\mathbb{E}}[(\partial_\mu \tilde{g}_T^n)^2 |\tilde{\mathcal{G}}_{t,T}^{n,\tilde{\alpha}^n} - \tilde{\mathcal{G}}_{t,T}^{\tilde{\alpha}}|]^{\frac{1}{2}}] + \mathbb{E}[\tilde{\mathbb{E}}[|\tilde{\mathcal{G}}_{t,T}^{\tilde{\alpha}}|^2]^{\frac{1}{2}} \tilde{\mathbb{E}}[|\partial_\mu \tilde{g}_T^n - \partial_\mu \tilde{g}_T|^2]^{\frac{1}{2}}] \\
&+ \int_t^T \mathbb{E}\Big[\big|\partial_z f_s\big|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\big|\mathcal{G}_{t,s}^{\alpha} - \mathcal{G}_{t,s}^{n,\alpha^n}\big|^2\Big]^{\frac{1}{2}} \mathrm{d}s + \int_t^T \mathbb{E}\Big[\big|\mathcal{G}_{t,s}^{n,\alpha^n}\big|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\big|\partial_z f_s - \partial_z f_s^n\big|^2\Big]^{\frac{1}{2}} \mathrm{d}s \\
&+ \int_t^T \{\mathbb{E}[\tilde{\mathbb{E}}[|\partial_\mu \tilde{f}_s^n|^2]^{\frac{1}{2}} \tilde{\mathbb{E}}[|\tilde{\mathcal{G}}_{s,r}^{n,\tilde{\alpha}^n} - \tilde{\mathcal{G}}_{s,r}^{\tilde{\alpha}}|^2]^{\frac{1}{2}}] + \mathbb{E}[\tilde{\mathbb{E}}[|\tilde{\mathcal{G}}_{s,r}^{\tilde{\alpha}}|^2]^{\frac{1}{2}} \tilde{\mathbb{E}}[|\partial_\mu f_s^n - \partial_\mu f_s|^2]^{\frac{1}{2}}]\} \mathrm{d}s.
\end{aligned}
$$

Convergence follows from Lipschitz continuity and boundedness of $\partial_z g, \partial_\mu g, \partial_z f$ and $\partial_\mu f$, the integrability of the first variation process, and the convergence result presented in Lemma 4.3.1. Hence, we have proven the second claim of Lemma 4.3.3. □

The next part develops the proof of Theorem 4.3.1. But before, let us state the Ekeland's variational principle:

Theorem 4.3.2 (Ekeland's Variational Principle). *Consider a complete metric space (W, d) and $L : W \to \mathbb{R} \cup \{+\infty\}$, a function bounded from below which is lower semi-continuous and not equal to $+\infty$. Given $\varepsilon > 0$, and $a \in W$ such that:*

$$L(a) \leq \inf_W L + \varepsilon,$$

so, we can find some point $e \in W$ such that for every $\delta > 0$:

1. $L(e) \leq L(a)$,
2. $d(a, e) \leq \delta$,
3. $\forall m \neq e, L(m) > L(e) - \frac{\varepsilon}{\delta} d(e, m)$,

Ekeland (1979).

Let us now prove Theorem 4.3.1:

Proof of Theorem 4.3.1. Fix $n \geq 1$ and let $\hat{\alpha}$ be an optimal control. Using Lemma 4.3.2, we have:

$$J_k(\alpha^n) - J_k(\alpha) \leq C\Big(\tau(\alpha^n, \alpha) + \sup_{0\leq t\leq T} \mathbb{E}\big[\big|X_t^{k,x,\alpha^n} - X_t^{k,x,\alpha}\big|^2\big]^{\frac{1}{2}}\Big),$$

and thus the function J_k is continuous on the metric space $(\mathcal{A}, \tau)$ and there exists ϵ_n such that

$$J_n(\hat{\alpha}) - J(\hat{\alpha}) \geq -\epsilon_n, \text{ and } J(\alpha) - J_n(\alpha) \geq -\epsilon_n \text{ for all } \alpha \in \mathcal{A}. \tag{4.15}$$

Therefore, adding the 2 terms in (4.15) gives

$$J_n(\hat{\alpha}) - J(\hat{\alpha}) + J(\alpha) - J_n(\alpha) \geq -2\epsilon_n.$$

Then maximizing both sides of the inequality over admissible controls α gives

$$J_n(\hat{\alpha}) \geq \sup_{\alpha\in\mathcal{A}} J_n(\alpha) - 2\epsilon_n.$$

Using Theorem 4.3.2 with the following correspondence: $a = \hat{\alpha}, W = \mathcal{A}, L = J_n, e = \hat{\alpha}^n, m = u(\text{such that } u \neq \hat{\alpha}^n), \varepsilon = 2\epsilon_n$ and $\delta = (2\epsilon_n)^{\frac{1}{2}}$, there exists an admissible control $e = \hat{\alpha}^n$ such that $\tau(\hat{\alpha}^n, \hat{\alpha}) \leq (2\epsilon_n)^{\frac{1}{2}}$, and we have:

$$(1)\tau(\hat{\alpha}^n, \hat{\alpha}) \leq (2\epsilon_n)^{\frac{1}{2}}$$

$$(2)J_n(\hat{\alpha}^n) \geq J_n(\hat{\alpha})$$

$$(3)J_n(\alpha) \leq J_n(\hat{\alpha}^n) + \frac{2\epsilon_n}{(2\epsilon_n)^{\frac{1}{2}}}\tau(\hat{\alpha}^n, \alpha) \text{ for all } \alpha \in \mathcal{A}$$

Thus

$$J_n(\hat{\alpha}^n) \geq J_n(\alpha) - \frac{2\epsilon_n}{(2\epsilon_n)^{\frac{1}{2}}}\tau(\hat{\alpha}^n, \alpha) = J_n(\alpha) - (2\epsilon_n)^{\frac{1}{2}}\tau(\hat{\alpha}^n, \alpha).$$

so that if we define $J_n^{\epsilon}(\alpha) := J_n(\alpha) - (2\epsilon_n)^{\frac{1}{2}}\tau(\hat{\alpha}^n, \alpha)$, it follows that $\hat{\alpha}^n$ is optimal for the control problem with the performance functional given by $J_n^\epsilon(\alpha)$. Let us consider γ, an arbitrary admissible control and ϵ a fixed constant. Since $\mathcal{A}$ is a convex set, we have that for $\lambda := \gamma - \hat{\alpha}^n$, $\hat{\alpha}^n + \epsilon\lambda \in \mathcal{A}$. Therefore, the smoothness of b_n implies the Gâteau differentiability of J_n and its Gâteau derivative in the direction of η is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon} J_n(\hat{\alpha}^n + \epsilon\lambda)|_{\epsilon=0} =& \mathbb{E}\Big[\int_0^T \Big\{\partial_z f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n) S_u^n \\ &+ \tilde{\mathbb{E}}[\partial_\mu f(u, \tilde{X}_u^{n,x,\tilde{\hat{\alpha}}^n}, \mathbb{P}_{X_u^{n,x,\tilde{\hat{\alpha}}^n}}, \tilde{\hat{\alpha}}_u^n; X_u^{n,x,\hat{\alpha}^n}) \tilde{S}_u^n] \\ &+ \partial_\alpha f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n)\lambda_u\Big\} \mathrm{d}u + \partial_z g(X_T^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}}) S_T^n \\ &+ \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^{n,x,\tilde{\hat{\alpha}}^n}, \mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}}; X_T^{n,x,\hat{\alpha}^n}) \tilde{S}_T^n]\Big]\end{aligned}$$

where S^n is the solution to the following SDE:

$$\begin{aligned}\mathrm{d}S_t^n =& \Big(\partial_z b_n(t, X_t^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_t^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_t^n) S_t^n + \tilde{\mathbb{E}}[\partial_\mu b(t, \tilde{X}_t^{n,x,\tilde{\hat{\alpha}}^n}, \mathbb{P}_{X_t^{n,x,\hat{\alpha}^n}}, \tilde{\hat{\alpha}}_t^n; X_t^{n,x,\hat{\alpha}^n}) \tilde{S}_t^n] \\ &+ \partial_\alpha b_n(t, X_t^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_t^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_t^n)\lambda_t\Big) \mathrm{d}t, \qquad S_0^n = 0. \end{aligned} \tag{4.16}$$

Reverse triangular inequality yields:

$$\frac{d}{d\epsilon}\tau(\hat{\alpha}^n + \epsilon\lambda, \alpha)_{|_{\lambda=0}} = \lim_{\epsilon\to 0} \frac{\tau(\hat{\alpha}^n + \epsilon\lambda, \alpha) - \tau(\hat{\alpha}^n, \alpha)}{\epsilon} \geq - \sup_{0\leq t\leq T} \mathbb{E}[|\lambda_t|^4]^{\frac{1}{4}}.$$

It follows from its definition and the above that J_n^ϵ is Gâteau differentiable and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon}J_n^\epsilon(\hat{\alpha}^n+\epsilon\lambda)_{|\epsilon=0} =& \frac{\mathrm{d}}{\mathrm{d}\epsilon}J_n(\hat{\alpha}^n+\epsilon\lambda)_{|\epsilon=0}-(2\epsilon_n)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\tau(\hat{\alpha}^n,\hat{\alpha}^n+\epsilon\lambda)_{|\epsilon=0}\\
\leq& \mathbb{E}\Big[\int_0^T\Big\{\partial_z f(u,X_u^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\hat{\alpha}_u^n)S_u^n \qquad (4.17)\\
&+\tilde{\mathbb{E}}[\partial_\mu f(u,\tilde{X}_u^{n,x,\tilde{\hat{\alpha}}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\tilde{\hat{\alpha}}_u^n;X_u^{n,x,\hat{\alpha}^n})\tilde{S}_u^n]\\
&+\partial_\alpha f(u,X_u^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\hat{\alpha}_u^n)\lambda_u\Big\}\mathrm{d}u+\partial_z g(X_T^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}})S_T^n\\
&+\tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^{n,x,\tilde{\hat{\alpha}}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}};X_T^{n,x,\hat{\alpha}^n})\tilde{S}_T^n]\Big]+(2\epsilon_n)^{\frac{1}{2}}\sup_{0\leq t\leq T}\mathbb{E}[|\lambda_t|^4]^{\frac{1}{4}}\\
\leq& \mathbb{E}\Big[\int_0^T\Big\{\partial_z f(u,X_u^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\hat{\alpha}_u^n)S_u^n \qquad (4.18)\\
&+\tilde{\mathbb{E}}[\partial_\mu f(u,\tilde{X}_u^{n,x,\tilde{\hat{\alpha}}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\tilde{\alpha}_u^{n,x,\hat{\alpha}^n};X_u^{n,x,\hat{\alpha}^n})\tilde{S}_u^n]\\
&+\partial_\alpha f(u,X_u^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}},\hat{\alpha}_u^n)\lambda_u\Big\}\mathrm{d}u+\partial_z g(X_T^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}})S_T^n\\
&+\tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^{n,x,\tilde{\hat{\alpha}}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}};X_t^{n,x,\hat{\alpha}^n})\tilde{S}_T^n]\Big]+C_M(2\epsilon_n)^{\frac{1}{2}}, \qquad (4.19)
\end{aligned}$$

where C_M is a constant depending on the value of M the upper bound in (4.11). Since we are working with smooth functions, the couple $(\hat{P}_t^n,\hat{Q}_t^n)$ is solution of the following backward stochastic differential equation:

$$\begin{aligned}
\mathrm{d}P_t^n =& -\Big(\partial_z H_n(t,X_t^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_t^{n,x,\hat{\alpha}^n}},\hat{\alpha}_t^n,P_t^n,Q_t^n)\\
&+\tilde{\mathbb{E}}[\partial_\mu H_n(t,\tilde{X}_t^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_t^{n,x,\hat{\alpha}^n}},\tilde{\hat{\alpha}}_t^n,\tilde{P}_t^n,\tilde{Q}_t^n;X_t^{n,x,\hat{\alpha}^n})]\Big)\mathrm{d}t+Q_t^n\mathrm{d}B_t,\\
P_T^n =& \partial_z g(X_T^{n,x,\hat{\alpha}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}})+\tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^{n,x,\tilde{\hat{\alpha}}^n},\mathbb{P}_{X_T^{n,x,\hat{\alpha}^n}};X_T^{n,x,\hat{\alpha}^n})],
\end{aligned}$$

where

$$H_n(t,z,p,\mu,\alpha)=f(t,z,\mu,\alpha)+b_n(t,z,\mu,\alpha)p.$$

Using the Itô's product rule and the Fubini's theorem (see for example Buckdahn et al. (2017); Carmona & Delarue (2018)), we have

$$\mathbb{E}[P_T^n S_T^n] = \mathbb{E}\Big[\int_0^T S_u^n\Big(-\partial_z f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n) \tag{4.20}$$
$$- \tilde{\mathbb{E}}[\partial_\mu f(u, \tilde{X}_u^{n,x,\tilde{\hat{\alpha}}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \tilde{\hat{\alpha}}_u^n; X_u^{n,x,\hat{\alpha}^n})]\Big)\mathrm{d}u\Big]$$
$$+ \mathbb{E}\Big[\int_0^T P_u^n \partial_\alpha b_n(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n)\lambda_u \mathrm{d}u\Big]. \tag{4.21}$$

Rearranging, using Fubini's theorem and (4.20), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J_n^\epsilon(\hat{\alpha}^n + \epsilon\lambda)_{|\epsilon=0}$$
$$\leq \mathbb{E}\Big[\int_0^T \Big\{\partial_z f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n) S_u^n + \tilde{\mathbb{E}}[\partial_\mu f(u, \tilde{X}_u^{n,x,\tilde{\hat{\alpha}}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \tilde{\hat{\alpha}}_u^n; X_u^{n,x,\hat{\alpha}^n})\tilde{S}_u^n]$$
$$+ \partial_\alpha f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n)\lambda_u\Big\}\mathrm{d}u\Big] + \mathbb{E}\Big[P_T^n S_T^n\Big] + C_A(2\epsilon_n)^{\frac{1}{2}},$$
$$= \mathbb{E}\Big[\int_0^T \Big(\partial_\alpha f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n)\lambda_u + P_u^n \partial_\alpha b_n(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n)\lambda_u\Big)\mathrm{d}u\Big]$$
$$+ C_A(2\epsilon_n)^{\frac{1}{2}},$$
$$= \mathbb{E}\Big[\int_0^T \partial_\alpha H_n(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n, P_u^n)\lambda_u \mathrm{d}u\Big] + C_A(2\epsilon_n)^{\frac{1}{2}}.$$

Therefore

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}\epsilon} J_n^\epsilon(\hat{\alpha}^n + \epsilon\lambda)_{|\epsilon=0} \leq \mathbb{E}\Big[\int_0^T \partial_\alpha H_n(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n, P_u^n)\lambda_u \mathrm{d}u\Big] + C_A(2\epsilon_n)^{\frac{1}{2}},$$

The expression can be reduced to

$$\Big(\partial_\alpha f(u, X_u^{n,x,\hat{\alpha}^n}, \mathbb{P}_{X_u^{n,x,\hat{\alpha}^n}}, \hat{\alpha}_u^n) + \partial_\alpha b_2(u, X_u^{n,x,\hat{\alpha}^n}, \hat{\alpha}_u^n)\hat{P}_u^n\Big) \cdot (\gamma - \hat{\alpha}_u^n) \geq 0, \ \mathbb{P} \otimes dt - a.s.$$

We know from Lemma 4.3.1 and Lemma 4.3.3 that for every $0 \leq u \leq T$, $X_u^{n,x,\hat{\alpha}^n}$ (resp. P_u^n) converges to $X_u^{x,\hat{\alpha}}$ (resp. P_u) $\mathbb{P}-$a.s. as $n \to \infty$, with $\hat{\alpha}^n$ also converging to $\hat{\alpha}$. Passing to the limit yields:

$$\Big(\partial_\alpha f(u, X_u^{x,\hat{\alpha}}, \mathbb{P}_{X_u^{x,\hat{\alpha}}}, \hat{\alpha}_u) + \partial_\alpha b_2(u, X_u^{x,\hat{\alpha}}, \hat{\alpha}_u)\hat{P}_u\Big) \cdot (\gamma - \hat{\alpha}_u) \geq 0, \ \mathbb{P} \otimes \mathrm{d}t - a.s.$$

The result follows. □

4.4 Chapter Summary

The main aim of this chapter was to derive the stochastic maximum principle for an optimal control problem consisting in maximizing a cost functional subjected to a MFSDE driven by a one dimensional Brownian motion and having anon-smooth drift. The proof consists in approximating the original control problem into an approximate one using the Ekeland's variational principle and later on show convergence of the maximum principle.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

In this chapter, we summarize the work that has been developed in the thesis. We will provide an overview of what has been done associated with conclusions, and recommendations for future research work going in the same direction.

5.1 Overview

The main objective of this thesis was to solve an optimal control problem where the state process is a mean-field stochastic differential equation with an irregular drift coefficient. With a mean-field stochastic differential equation having a drift that is non smooth and depending on the measure variable, we show convergence of an approximate sequence of solutions to the solution of the original mean-field stochastic differential equation, which is known to exist and to be unique depending on how we choose the random argument α to be. The control problem is therefore defined with a performance functional and a state dynamics which is the mean-field stochastic differential equation that was studied earlier.

5.2 Summary

This thesis evolves mainly around providing necessary conditions for optimality of a system driven by a stochastic differential equation of mean-field type. This works provides a detailed analysis on the properties of the solutions of the mean-field stochastic differential equation. The first property was developed in the third chapter where we have shown compactness of the approximating sequence of solutions and convergence of the sequence to the solution.

The second property of the solution was detailed in the fourth chapter in which we prove that the solution of the stochastic differential equation of mean-field type admits a Sobolev differentiable flow and the first variation process can be expressed in terms of time-space local time. Proving the representation of the stochastic differential flow of the solution is done through the proof of some technical lemma involving convergence and estimates.

In the last chapter, we derive the stochastic maximum principle using a variational approach as it has been done in the literature with works going in the same direction. However, one difference is that the adjoint process is indeed of first order, but is expressed explicitly using the stochastic differential flow property of the solution that was explained in the fourth chapter.

5.3 Conclusions

The stochastic maximum principle appears to be an excellent method if the objective is to solve a control problem using a probabilistic approach. This approach is developed on an original framework with a contribution to the literature of stochastic control theory. The contribution being the development of an approach for the optimization of systems driven a stochastic differential equation of mean-field type, and in addition with a drift coefficient that is neither differentiable nor Lipschitz. The drift coefficient is assumed to be measurable and of at most linear growth. One key aspect of the work is the use of the idea of weak differentiability in the initial condition in order to bypass the difficulty created by the non differentiability of the drift coefficient.

5.4 Recommendations

One important recommendation is related to the solution of the mean-field stochastic differential equation under study. In this work, it is important to choose the control α_t for which the state process has a unique solution. As mentioned earlier, if one takes $\alpha_t = \alpha(t, X_t)$, where α is a bounded and measurable function, then the state has a unique solution. However if $\{\alpha_t\}_{t\geq 0}$ is simply an $\{\mathcal{F}_t\}_{t\geq 0}-$adapted process, then existence and uniqueness of solution of such SDE is still open. It will also be interesting to apply to apply our results to some concrete example and solve the problem by some numerical methods.

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