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Rational Approximants to Holomorphic Functions in n-Dimensions

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We present a general method of calculating rational approximants to holomorphic function in n-dimensions. We establish analogs of some well-known properties of Padé approximants.

1. INTRODUCTION

The problem of constructing convergent rational approximants in one variable to a Taylor series representation of a holomorphic function is well understood. However its generalization to several variables is still far from clear, although some attempts have been made by Lutterodt [1], Chisholm [2], and others. In this paper we present an extension of the method of constructing rational approximants to holomorphic functions in several variables. The approach here is based on an earlier work (Lutterodt [3]), and differs somewhat from that of John and Lutterodt [4], which is specially tailored for numerical work on a computer.

In Section 2 we discuss notation and some preliminary definitions. In Section 3 we introduce a definition of a rational approximant to a holomorphic function in several variables, and we discuss the setting up of equations. In Section 4 some analogs of properties known in the one dimensional Padé case are examined. Finally, in Section 5 we conclude with a brief discussion.

2. PRELIMINARIES

We shall assume the existence of a holomorphic function over some domain (by ***domain*** we mean an open connected domain) $\mathscr{D} \subset \mathbb{C}^n$ and in some neighborhood U_w of a point $w = (w_1, ..., w_n) \in U_w \subset \mathscr{D}$ we define a Taylor development of the holomorphic function f by

$$f(\zeta) = \sum_{\lambda=0}^{\infty} c_\lambda (\zeta - w)^\lambda, \tag{2.1}$$

where

$$c_\lambda = c_{\lambda_1 \cdots \lambda_n} = \frac{1}{\lambda!} \frac{\partial^{|\lambda|}}{\partial \zeta^\lambda} f(\zeta)_{\,|\zeta = w}, \qquad c_{0 \cdots 0} \neq 0,$$

and

$$\zeta = (z_1, \ldots, z_n) \in U_w, \qquad (\zeta - w)^\lambda = (z_1 - w_1)^{\lambda_1} \cdots (z_n - w_n)^{\lambda_n},$$

$$\sum_{\lambda=0}^{\infty} \equiv \sum_{\lambda_1=0}^{\infty} \cdots \sum_{\lambda_n=0}^{\infty}, \qquad \frac{1}{\lambda!} \frac{\partial^{|\lambda|}}{\partial \zeta^\lambda} = \frac{1}{\lambda_1! \cdots \lambda_n!} \left(\frac{\partial^{\lambda_1 + \cdots + \lambda_n}}{\partial z_1^{\lambda_1} \cdots \partial z_n^{\lambda_n}} \right).$$

We let

$$P_\mu(\zeta) = \sum_{\alpha=0}^{\mu} a_\alpha (\zeta - w)^\alpha \qquad \text{and} \qquad Q_\nu(\zeta) = \sum_{\beta=0}^{\nu} b_\beta (\zeta - w)^\beta$$

be relatively prime over the domain $\mathscr{D}$ except for the variety Δ on which both polynomials vanish. The notations $\sum_{\alpha=0}^{\mu}$, a_α, etc., are as follows:

$$\sum_{\alpha=0}^{\mu} \equiv \sum_{\alpha_1=0}^{\mu_1} \cdots \sum_{\alpha_n=0}^{\mu_n}, \qquad \sum_{\beta=0}^{\nu} \equiv \sum_{\beta_1=0}^{\nu_1} \cdots \sum_{\beta_n=0}^{\nu_n},$$

$$a_\alpha \equiv a_{\alpha_1 \cdots \alpha_n}, \qquad b_\beta \equiv b_{\beta_1 \cdots \beta_n}, \qquad \text{and} \qquad b_{0 \cdots 0} \neq 0.$$

μ, ν represent the maximum "degrees" of the polynomials $P_\mu(\zeta)$ and $Q_\nu(\zeta)$, respectively. We introduce a rational function

$$R_{\mu\nu}(\zeta) = \frac{P_\mu(\zeta)}{Q_\nu(\zeta)} \tag{2.2}$$

wherever it has a determined value, finite or infinite, and ignore the variety of its indeterminate points given by

$$\Delta = \{\zeta \in \mathbb{C}^n : P_\mu(\zeta) = 0 \text{ and } Q_\nu(\zeta) = 0\}.$$

$R_{\mu\nu}(\zeta)$ has at most

$$N_P = \prod_{j=1}^{n} (\mu_j + 1) + \prod_{j=1}^{n} (\nu_j + 1)$$

parameters in a's and b's to be determined by setting up a system of linear equations. The setting up of the linear system of equations to determine the parameters of $R_{\mu\nu}(\zeta)$ forms the main theme of this paper.

Defining rational approximants in several variables requires certain preliminary definitions, which are given below. We first define a difference function

$$F(\zeta) = f(\zeta) - R_{\mu\nu}(\zeta), \tag{2.3}$$

and we define the Taylor coefficients of this difference function by

$$d_\lambda = \frac{1}{\lambda!} \frac{\partial^{|\lambda|}}{\partial \zeta^\lambda} F(\zeta)_{|\zeta=w} . \tag{2.4}$$

The suffixes $\lambda = (\lambda_1 ,..., \lambda_n)$, where $\lambda_i = 0, 1,..., i = 1,..., n$, span an n-dimensional infinite lattice Λ. By using the bijective map $\lambda \to (1/\lambda!) (\partial^{|\lambda|}/\partial \zeta^\lambda)$ we induce a lattice structure for the coefficients d_λ , which we call S^Λ.

DEFINITION 2.1. A finite subset A of S^Λ that is such that $d_{0,\ldots,0} \in A$ is said to have the ***inclusion property*** if whenever $d_\lambda \in A$, then $d_\gamma \in A$ for $0 < \gamma < \lambda$, i.e., $0 \leqslant \gamma_i \leqslant \lambda_i$, $i = 1,..., n$.

In geometric terms, what the above definition asserts is that when a finite subset A of S^Λ has the inclusion property and a coefficient d_λ is in A, then a hyperrectangular subset of S^Λ emanating from the origin with d_λ as its furthermost "corner" also lies in A.

The inclusion property insures that there are no gaps in the coefficients that belong to A. A similar property is implied tacitly in Padé approximant theory.

DEFINITION 2.2. A finite subset $A \subset S^\Lambda$ is said to be ***null*** if every coefficient in A is zero.

As in the Padé approximant case, we choose the total number of equations to be $N_e = N_p - 1$ so that it is one less than the total number of parameters to be solved for. This choice, of course, enables us to fix one of the parameters of $R_{\mu\nu}$, which in turn enables us to convert a certain homogeneous system of equations into an inhomogeneous one and hence to find conditions for a unique solution.

It is convenient to characterize finite subsets of S^Λ containing N_e of the d-coefficients that have the inclusion property with the labels μ and ν. Such a subset will be written $A^{\mu\nu}$, where μ refers to the $\prod_{j=1}^n (\mu_j + 1)$ part of N_e and ν to the $\prod_{j=1}^n (\nu_j + 1)$ part. It should be noted that there is no such unique finite subset $A^{\mu\nu}$ of S^Λ, for there are many suitable boundaries in S^Λ that could correspond to a form of $A^{\mu\nu}$. We shall denote such boundaries by $L_{\mu\nu}$, and the isomorphism between S^Λ and Λ implies that the finite subsets $E^{\mu\nu} \subset \Lambda$, corresponding to $A^{\mu\nu}$ in S^Λ, have the same boundaries $L_{\mu\nu}$.

3. DEFINITION AND CONSTRUCTION

In this section we give a formal definition of a rational approximant in several variables to a Taylor development of a holomorphic function f in some neighborhood of a point w in $\mathbb{C}^n$.

DEFINITION 3.1. The rational function $R_{\mu\nu}(\zeta)$ is said to be a rational approximant to $f(\zeta)$ in some neighborhood U_w of a point $w \in \mathscr{D}$ if a finite subset $A^{\mu\nu}$ of S^{Λ}, containing N_e d-coefficients with respect to a chosen boundary $L_{\mu\nu}$, is null.

The above definition is equivalent to the form

$$\frac{\partial^{|\lambda|}}{\partial \zeta^{\lambda}} (f(\zeta) - R_{\mu\nu}(\zeta)) \Big|_{\zeta=w} = 0 \qquad \forall \lambda \in E^{\mu\nu}. \tag{3.1}$$

This equation provides us with a nonlinear system of equations to solve in order to determine the parameters of $R_{\mu\nu}$. We therefore need to linearize (3.1) into a form where we can then solve the linear equations involving some of the parameters of $R_{\mu\nu}$. To establish that such a linearization of (3.1) is possible we need a generalized Leibnitz theorem, which we state as follows:

LEMMA (Leibnitz). *Suppose f and $g \in C^{\infty}(\mathscr{D})$, $\mathscr{D}\,(\subset \mathbb{C}^n)$ is some finite domain; then at each point $w \in \mathscr{D}$ we have for $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \geqslant 0$, $i = 1, \ldots, n$, where σ_i's are integers,*

$$\frac{\partial^{|\sigma|}}{\partial \zeta^{\sigma}} (f(\zeta)\, g(\zeta)) \Big|_{\zeta=w} = \left\{ \sum_{r=0}^{\sigma} \prod_{j=1}^{n} \binom{\sigma_j}{r_j} \frac{\partial^{|r|}}{\partial \zeta^{r}} f(\zeta) \frac{\partial^{|\sigma-r|}}{\partial \zeta^{\sigma-r}} g(\zeta) \right\} \Big|_{\zeta=w}.$$

The proof is by induction. Although we state the above lemma for f and $g \in C^{\infty}(\mathscr{D})$, we shall in fact be applying it to cases where f and g are both holomorphic, and therefore the result is valid.

The following theorem establishes the linearization of (3.1).

THEOREM 1. *If $R_{\mu\nu}(\zeta)$ is a rational approximant to $f(\zeta)$ in the sense of Definition* 3.1, *then*

$$\frac{\partial^{|\lambda|}}{\partial \zeta^{\lambda}} (f(\zeta) - R_{\mu\nu}(\zeta)) \Big|_{\zeta=w} = 0 \qquad \forall \lambda \in E^{\mu\nu} \tag{3.1}$$

is equivalent to

$$\frac{\partial^{|\lambda|}}{\partial \zeta^{\lambda}} (Q_{\nu}(\zeta) f(\zeta) - P_{\mu}(\zeta)) \Big|_{\zeta=w} = 0 \qquad \forall \lambda \in E^{\mu\nu}. \tag{3.2}$$

Proof.

(i) (3.1) $\Rightarrow$ (3.2).

Write

$$Q_{\nu}(\zeta) f(\zeta) - P_{\mu}(\zeta) = Q_{\nu}(\zeta)\,[f(\zeta) - R_{\mu\nu}(\zeta)]; \tag{3.3}$$

then by the lemma we get

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda}(Q_\nu(\zeta)f(\zeta) - P_\mu(\zeta))\Big|_{\zeta=w}$$

$$= \left\{ \sum_{r=0}^{\min(\lambda,\nu)} \prod_{j=1}^{n} \binom{\nu_j}{r_j} \frac{\partial^{|r|}}{\partial \zeta^r} Q(\zeta) \frac{\partial^{|\lambda-r|}}{\partial \zeta^{\lambda-r}} (f(\zeta) - R_{\mu\nu}(\zeta)) \right\}_{|\zeta=w}$$

$$= 0 \qquad \forall \lambda \in E^{\mu\nu} \qquad \text{by (3.1) and Definition 2.1.}$$

This establishes (i).

(ii) (3.1) ⇐ (3.2).

$Q_\nu(w) = b_{0\cdots 0} \neq 0 \Rightarrow \exists$ some neighborhood U_1 ($\subset U_w$) of w, where $Q_\nu(\zeta) \neq 0$ for $\zeta \in U_1$. We can then write

$$\frac{1}{Q_\nu(\zeta)}\{Q_\nu(\zeta)f(\zeta) - P_\mu(\zeta)\} = f(\zeta) - R_{\mu\nu}(\zeta), \tag{3.4}$$

and by the lemma we get

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda}(f(\zeta) - R_{\mu\nu}(\zeta))\Big|_{\zeta=w}$$

$$= \left\{ \sum_{r=0}^{\min(\lambda,\nu)} \prod_{j=1}^{n} \binom{\nu_j}{r_j} \frac{\partial^{|r|}}{\partial \zeta^r} \left(\frac{1}{Q_\nu(\zeta)}\right) \frac{\partial^{|\lambda-r|}}{\partial \zeta^{\lambda-r}} (Q_\nu(\zeta)f(\zeta) - P_\mu(\zeta)) \right\}_{|\zeta=w}$$

$$= 0 \qquad \forall \lambda \in E^{\mu\nu}$$

by (3.2) and Definition 2.1. This establishes (ii) and hence the equivalence between (3.1) and (3.2).

Now, using the form (3.2), we set up the following system of linear equations with the a's and the b's as the unknowns. In order to make the systems of equations tally with those of Padé approximants, we assign subset $E' = \{\lambda \in E^{\mu\nu} \mid 0 \leqslant \lambda_i \leqslant \mu_i, i = 1,\ldots, n\}$ to the inhomogeneous part of the equations from (3.2) involving both the b's and a's. That is:

$$\sum_{r=0}^{\min(\lambda,\nu)} b_r c_{\lambda-r} = a_\lambda, \qquad \lambda \in E'. \tag{3.5}$$

This comprises a system of $\prod_{j=1}^{n} (\mu_j + 1)$ linear equations. The homogeneous system of $\prod_{j=1}^{n} (\nu_j + 1) - 1$ linear equations involving only the b's are the following:

$$\sum_{r=0}^{\min(\lambda,\nu)} b_r c_{\lambda-r} = 0, \qquad \lambda \in E^{\mu\nu}\backslash E'. \tag{3.6}$$

As already mentioned in Section 2, in order to solve for the a and b parameters we fix one of these, for convenience one of the b parameters, so as to transform the homogeneous system of equations into an inhomogeneous system. As in Padé approximant theory, we fix $b_{0\cdots 0}$, normalizing it to unity ($b_{0\cdots 0} = 1$). Equation (3.6) then becomes

$$\sum_{r\neq 0}^{\min(\lambda,\nu)} b_r c_{\lambda-r} = -c_\lambda , \qquad \lambda \in E^{\mu\nu}\backslash E'. \tag{3.7}$$

Here $r \neq 0$ means in $r = (r_1, ..., r_n)$ we exclude the case

$$r_1 = r_2 = \cdots = r_n = 0,$$

where

$$0 \leqslant r_i \leqslant \min(\lambda_i , \nu_i), \qquad i = 1,..., n.$$

Equation (3.7) can be solved uniquely for the remaining b parameters after fixing $b_{0\cdots 0} = 1$ provided the determinant of the matrix of the c-coefficients in (3.7) does not vanish. Since the c's in the determinant array depend on $\lambda \in E^{\mu\nu}\backslash E'$ and therefore on the boundary $L_{\mu\nu}$, the choice of boundary becomes important. (A full discussion on boundaries is to be presented in a separate paper shortly.) If a unique solution of the b's exists under the above stipulation, then by substitution into (3.5) we can evaluate the a's. That is, we get a representation of the rational approximants $R_{\mu\nu}$ in terms of c-coefficients labeled by $\lambda \in E^{\mu\nu}\backslash E'$ depending on the boundary $L_{\mu\nu}$. Thus the existence and the uniqueness of the representation of $R_{\mu\nu}$ with respect to $L_{\mu\nu}$ is determined by the nonvanishing condition of the determinant of the c-coefficients in Eq. (3.7).

4. Some Properties

In this section we attempt to establish "analogs" of certain well-known properties of Padé approximants. These properties are established by means of three theorems, which are stated and proved below.

Theorem 2. *Suppose a holomorphic function $f(\zeta)$ over $\mathscr{D}$ has a formal power series in some neighborhood U_w of $w \in \mathscr{D}$ ($\subset \mathbb{C}^n$). And suppose the inverse function exists in the neighborhood of w. If $R_{\mu\nu}$ is a rational approximant to f in the sense of Definition* 3.1 *with respect to some boundary $L_{\mu\nu}$, then the inverse of $R_{\mu\nu}$ is a rational approximant to the inverse of f with respect to the same boundary $L_{\mu\nu}$.*

Proof. From Eq. (3.1) we get

$$f(w) = R_{\mu\nu}(w) = c_{0\cdots 0} \neq 0,$$

and therefore $R_{\mu\nu}^{-1}(w)$ exists. Thus in some neighborhood of w (which can be chosen for simplicity to coincide with U_w) $R_{\mu\nu}^{-1}(\zeta)$ exists. We now write

$$\frac{1}{f(\zeta)} - \frac{1}{R_{\mu\nu}(\zeta)} = \frac{R_{\mu\nu}(\zeta) - f(\zeta)}{f(\zeta)\, R_{\mu\nu}(\zeta)} = T(\zeta)\,(f(\zeta) - R_{\mu\nu}(\zeta)), \tag{4.1}$$

where

$$T(\zeta) = \frac{-1}{f(\zeta)\, R_{\mu\nu}(\zeta)}. \tag{4.2}$$

Using the lemma on the R.H.S. of (4.1) and applying (3.1), we find

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda} \{T(\zeta)\,[f(\zeta) - R_{\mu\nu}(\zeta)]\}_{|\zeta=w} = 0, \qquad \lambda \in E^{\mu\nu};$$

this implies

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda} \left\{\frac{1}{f(\zeta)} - \frac{1}{R_{\mu\nu}(\zeta)}\right\}_{|\zeta=w} = 0, \qquad \lambda \in E^{\mu\nu}. \tag{4.3}$$

Thus from Definition 3.1, $R_{\mu\nu}^{-1}(\zeta)$ is a rational approximant to $f^{-1}(\zeta)$ in some neighborhood of w and with respect to the same boundary $L_{\mu\nu}$.

Remark. What the above theorem asserts seems analogous to the well-known statement that the inverse of a Padé approximant is the Padé approximant of the inverse, except, of course, that in several variables the boundary reference is important.

The next property holds for multidiagonal rational approximants only. By a multidiagonal rational approximant we mean the special case for which the rational function $R_{\mu\nu}$ is such that $\mu = \nu$, i.e., $\mu_i = \nu_i$, $i = 1,..., n$. An even more ad hoc case, which we refer to as equimultidiagonal rational approximant, is the one for which when $\mu = \nu$ in $R_{\mu\nu}$, then $\mu_i = \mu$, $i = 1,..., n$.

THEOREM 3. *Suppose a holomorphic function $f(\zeta)$ over a domain has a formal power series in some neighborhood U_w of $w \in \mathscr{D}$. If $R_{\mu\mu}(\zeta)$ is a multidiagonal rational approximant to $f(\zeta)$ with respect to the boundary $L_{\mu\mu}$, then under the bilinear transformation $f \to ((af + b)/(cf + d))$ $ad - bc \neq 0$ and $cf(w) + d \neq 0$, the rational function*

$$\frac{aP_\mu(\zeta) + bQ_\mu(\zeta)}{cP_\mu(\zeta) + dQ_\mu(\zeta)} = \frac{aR_{\mu\mu}(\zeta) + b}{cR_{\mu\mu}(\zeta) + d}$$

is a multidiagonal rational approximant to the meromorphic function $(af(\zeta) + b)/(cf(\zeta) + d)$ with respect to the same boundary $L_{\mu\mu}$ in the neighborhood U_w.

Proof. Since $R_{\mu\mu}(w) = f(w)$ from (3.1), therefore

$$cR_{\mu\mu}(w) + d = cf(w) + d \neq 0.$$

Hence $(cR_{\mu\mu}(w) + d)^{-1}$ exists and in some neighborhood U_w of w $(cR_{\mu\mu}(\zeta) + d)^{-1}$ exists, so that

$$\frac{aR_{\mu\mu}(\zeta) + b}{cR_{\mu\mu}(\zeta) + d} = \frac{aP_\mu(\zeta) + bQ_\mu(\zeta)}{cP_\mu(\zeta) + dQ_\mu(\zeta)}$$

is well-defined in U_w. We now write

$$\frac{af(\zeta) + b}{cf(\zeta) + d} - \frac{aP_\mu(\zeta) + bQ_\mu(\zeta)}{cP_\mu(\zeta) + dQ_\mu(\zeta)} = T(\zeta)\,(Q_\mu(\zeta)\,f(\zeta) - P_\mu(\zeta)), \tag{4.4}$$

where

$$T(\zeta) = \frac{ad - bc}{(cf(\zeta) + d)\,(cP_\mu(\zeta) + dQ_\mu(\zeta))}. \tag{4.5}$$

Applying the lemma to the R.H.S. of (4.4) and making use of (3.3), we get

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda}\{T(\zeta)\,(Q_\mu(\zeta)\,f(\zeta) - P_\mu(\zeta))\}\,|_{\zeta=w} = 0, \qquad \forall\lambda \in E^{\mu\mu},$$

which implies

$$\frac{\partial^{|\lambda|}}{\partial \zeta^\lambda}\left\{\frac{af(\zeta) + b}{cf(\zeta) + d} - \frac{aP_\mu(\zeta) + bQ_\mu(\zeta)}{cP_\mu(\zeta) + dQ_\mu(\zeta)}\right\}_{|\zeta=w} = 0, \qquad \forall\lambda \in E^{\mu\mu}.$$

Thus from Definition 3.1 the result follows.

The next theorem establishes a kind of "invariance" under a certain type of projective map of a polycylindrical neighborhood U_w into itself, i.e., $\phi\colon U_w \to U_w$, where ϕ is defined by

$$\eta = \phi(\zeta) = (u_1, \ldots, u_n)$$

and

$$u_i - w_i = \frac{\alpha_i(z_i - w_i)}{\gamma_i(z_i - w_i) + \delta_i}, \qquad i = 1, \ldots, n,$$

with $\alpha_i\delta_i \neq 0$ and $\gamma_i \neq 0$, $i = 1, \ldots, n$, also $\gamma_i(z_i - w_i) + \delta_i \neq 0$, $i = 1, \ldots, n$. Here $\alpha_i, \gamma_i, \delta_i \in \mathbb{C}$, $i = 1, \ldots, n$ (see [5]).

THEOREM 4. *Let $f(\zeta)$ be a holomorphic function over a domain $\mathscr{D} \subset \mathbb{C}^n$, with a power series representation in some neighborhood U_w of w. Let ϕ be a projective map as defined above. If $R_{\mu\mu}(\zeta)$ is a multidiagonal rational approxi-*

mant in the sense of Definition 3.1 *to* $f(\zeta)$ w.r.t. $L_{\mu\mu}$, *then so is* $R_{\mu\mu}(\eta)$ *to* $f(\eta)$ w.r.t. $L_{\mu\mu}$.

Proof. $R_{\mu\mu}(\zeta)$ is a multidiagonal rational approximant to $f(\zeta)$ in U_w w.r.t. $L_{\mu\mu}$ means that

$$\frac{\partial^{|\lambda|}}{\partial\zeta^{\lambda}}\left\{f(\zeta)-\frac{P_\mu(\zeta)}{Q_\mu(\zeta)}\right\}_{|\zeta=w}=0, \qquad \forall\lambda\in E^{\mu\mu}. \tag{4.6}$$

Under the projective mapping ϕ: $\zeta\to\eta$ we get

$$\frac{\partial^{|\lambda|}}{\partial\eta^{\lambda}}=\left(\frac{\gamma\zeta+\delta}{\alpha\delta}\right)^{\lambda}\sum_{\sigma=1}^{\lambda} m_\sigma(\gamma\zeta+\delta)^{\sigma}\frac{\partial^{|\sigma|}}{\partial\zeta^{\sigma}}, \tag{4.7}$$

where

$$\begin{aligned}\left(\frac{\gamma\zeta+\delta}{\alpha\delta}\right)^{\lambda}&=\prod_{i=1}^{n}\left(\frac{\gamma_i(z_i-w_i)+\delta_i}{\alpha_i\delta_i}\right)^{\lambda_i},\\ (\gamma\zeta+\delta)^{\sigma}&=\prod_{i=1}^{n}(\gamma_i(z_i-w_i)+\delta_i)^{\sigma_i},\end{aligned} \tag{4.8}$$

and $m_\sigma\equiv m_{\sigma_1\cdots\sigma_n}$, which is a multiple of positive integers; $\sum_{\sigma=1}^{\lambda}$ and $(\partial^{|\sigma|})/(\partial\zeta^{\sigma})$ are as introduced earlier in Section 1. Now

$$\begin{aligned}&\frac{\partial^{|\lambda|}}{\partial\eta^{\lambda}}\left(f(\eta)-\frac{P_\mu(\eta)}{Q_\mu(\eta)}\right)_{|\eta=w}\\ &=\left(\frac{\gamma\zeta+\delta}{\alpha\delta}\right)^{\lambda}\sum_{\sigma=0}^{\lambda} m_\sigma(\gamma\zeta+\delta)^{\sigma}\frac{\partial^{|\sigma|}}{\partial\zeta^{\sigma}}\left(g(\zeta)-\frac{\bar{P}_\mu(\zeta)}{\bar{Q}_\mu(\zeta)}\right)_{|\zeta=w},\end{aligned} \tag{4.9}$$

where $f(\phi(\zeta))=g(\zeta)$ is holomorphic in U_w and has a power series representation in ζ in U_w, and

$$\frac{P_\mu(\eta)}{Q_\mu(\eta)}=\frac{\prod_{i=1}^{n}(\gamma_i(z_i-w_i)+\delta_i)\,\bar{P}_\mu(\zeta)}{\prod_{i=1}^{n}(\gamma_i(z_i-w_i)+\delta_i)\,\bar{Q}_\mu(\zeta)}=\frac{\bar{P}\,(\zeta)}{\bar{Q}_\mu(\zeta)}$$

$$(\gamma_i(z_i-w_i)+\delta_i\neq 0,\ i=1,\dots,n).$$

By (4.6) and the inclusion property, the R.H.S. of (4.9) is made to vanish $\forall\lambda\in E^{\mu\mu}$. Hence the result follows.

5. Discussion

In our construction of rational approximants in n-dimensions presented in this paper it has been convenient to use rational functions constructed from polynomials with hyperrectangular boundaries in several variables. The use of these polynomials enabled us to determine the limits of the summations in Eqs. (3.5) and (3.6), which were generalizations of the Padé set of equations. Use of nonhyperrectangular boundaries for the polynomials would require further insistence on a form of inclusion property for the coefficients of these polynomials. A more general approach along these lines was briefly discussed by John and Lutterodt (4), although the emphasis in that paper was on the computational aspect of the problem. It was also shown in the same paper how this more general use of "nonrectangular" polynomials is easily adaptable to a computer language. The numerical importance of rational approximants in several variables has been discussed by several authors recently, and a number of algorithms have been worked out (e.g., see [6]).

An area with a lot of scope for further development is the use of rational approximants as a means of analytically continuing holomorphic functions outside neighborhoods where they may have Taylor expansions. Some numerical work on analytic continuation was considered by Lutterodt (3) in relation to a problem in mathematical physics.

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