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Positive periodic solutions for neutral functional differential systems

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Abstract

We study the existence of positive periodic solutions of a system of neutral differential equations. In the process we construct two mappings in which one is a contraction and the other compact. A Krasnoselskii's fixed point theorem is then used in the analysis.

Key words : *Krasnoselskii, Neutral Functional differential System, Positive periodic solutions*

AMS subject classifications: *34K20, 45J05, 45D05.*

1. Introduction

In this paper we use a fixed point theorem due to Krasnoselskii to study the existence of positive periodic solutions of the system of neutral differential equations

$$(1.1)\ \frac{d}{dt}x(t) = A(t)x(t-\tau(t)) - C(t)\frac{d}{dt}x(t-\tau(t)) - F(t, x(t-\tau(t))),$$

where

$C(t) = \text{diag}[c_1(t), c_2(t), ..., c_n(t)]$, $A(t) = \text{diag}[a_1(t), a_2(t), ..., a_n(t)]$, and $F(t, x(t-\tau(t))) = [f_1(t, x_1(t-\tau(t))), f_2(t, x_2(t-\tau(t))), ..., f_n(t, x_n(t-\tau(t)))]^T$.

The scalar version of (1.1) arises in food-limited population models ([3],[4]-[7],[8],[9]) and blood cell models [2]. Recently, Raffoul, [21] obtained sufficient conditions for the existence of positive periodic solutions for the scalar neutral nonlinear differential equation

$$(1.2)\qquad x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t-g(t))).$$

In the current paper we extend the results in [21] to systems of equations. It must be noted that if $\tau(t) = 0$ in the first term on the right hand side of (1.1) and $n = 1$, then (1.1) reduces to (1.2). Thus, even for $n = 1$ our results obtained in this paper are more general than that obtained in [21]. Let $\mathbf{R}_+ = [0, +\infty)$. For each $x = (x_1, x_2, x_3, ..., x_n)^T \in \mathbf{R}^n$, the norm of x is defined as $|x| = \sum_{j=1}^{n} |x_j|$. $\mathbf{R}^n_+ = \{(x_1, x_2, x_3, ..., x_n)^T \in \mathbf{R}^n : x_j \geq 0,\ j = 1, 2, 3, ..., n\}$. We say that x is "positive" whenever $x \in \mathbf{R}^n_+$.

In this paper we make the following assumptions.

(H1) There exist constants $\sigma_j > 0$ such that $\sigma_j < c_j(t),\ j = 1, ..., n$, for all $t \in [0, \omega]$.

(H2) There exist constants α_j, such that $||c_j|| \leq \alpha_j,\ j = 1, 2, ..., n$.

(H3) There exist continuous functions $h_j : \mathbf{R} \to \mathbf{R},\ j = 1, ..., n$ such that

$$(1.3)\qquad h_j(t+\omega) = h_j(t), \quad \int_0^\omega h_j(s)ds > 0.$$

(H4) $0 < h_j(t) < 1$ for all $t \in [0, \omega],\ j = 1, ..., n$.

(H5) $\tau'(t) > 1$ for all $t \in \mathbf{R}$.

2. Preliminaries

Let $\mathbf{S}_\omega = \{\phi \in C(\mathbf{R}, \mathbf{R}^n) \ : \ \phi(t+\omega) = \phi(t) \text{ for } t \in \mathbf{R} \ \}$, be endowed with the usual linear structure as well as the norm

$$||\phi|| = \sum_{j=1}^{n} |\phi_j|_0, \text{ for } \phi = (\phi_1, \phi_2, ..., \phi_n) \in \mathbf{S}_\omega,$$

where

$$|\phi_j|_0 = \sup_{t\in\mathbf{R}} |\phi_j(t)| = \sup_{t\in[0,\omega]} |\phi_j(t)|, \ j = 1, ..., n.$$

Then $\mathbf{S}_\omega$ is a Banach space.

We assume that all functions in (1.1) are continuous with respect to their arguments.
We also assume that for all $t \in \mathbf{R}$,

$$a_j(t+\omega) = a_j(t), \ j = 1, 2, ..., n \tag{2.1}$$

$$f_j(t+\omega, .) = f_j(t, .), \ j = 1, 2, ..., n \tag{2.2}$$

$$\tau(t+\omega) = \tau(t) \tag{2.3}$$

$$c_j(t+\omega) = c_j(t), \ j = 1, 2, ..., n \tag{2.4}$$

Let

$$G_j(t, u) = \frac{e^{\int_u^t h_j(s)ds}}{1 - e^{-\int_0^\omega h_j(s)ds}}, \ j = 1, 2, ...n. \tag{2.5}$$

Set

$$G(t, u) = \text{diag}[G_1(t, u), G_2(t, u), ..., G_n(t, u)]. \tag{2.6}$$

Also, let

$$M_j = \frac{e^{\int_0^{2\omega} |h_j(s)|ds}}{1 - e^{-\int_0^{\omega} h_j(s)ds}}, \; j = 1, 2, ...n$$

and

$$m_j = \frac{e^{-\int_0^{2\omega} |h_j(s)|ds}}{1 - e^{-\int_0^{\omega} h_j(s)ds}}, \; j = 1, 2, ...n.$$

It is easy to see that for all $(t, s) \in [0, 2\omega] \times [0, 2\omega]$,

$$m_j \leq G_j(t, s) \leq M_j.$$

It is clear that $G_j(t + \omega, s + \omega) = G_j(t, s)$ and so $G(t + \omega, s + \omega) = G(t, s)$ for all $(t, s) \in \mathbf{R}^2$.

Let $\gamma = \max_{t \in \mathbf{R}} \Big[\tau'(t) - 1\Big]^{-1}$ and $\gamma_* = \min_{t \in \mathbf{R}} \Big[\tau'(t) - 1\Big]^{-1}$.

For the next lemma we consider

$$x_j'(t) = a_j(t)x_j(t - \tau(t)) - c_j(t)x_j'(t - \tau(t)) - f_j(t, x_j(t - \tau(t))),$$
$$j = 1, 2, ...n. \tag{2.7}$$

Lemma 2.1. *Suppose (2.1)-(2.4) hold. Suppose also that $\tau'(t) \neq 1$ for all $t \in \mathbf{R}$. If $x(t) \in \mathbf{S}_\omega$, then $x_j(t)$ is a solution of (2.7) if and only if*

$$x_j(t) = \frac{c_j(t)}{\tau'(t) - 1} x_j(t - \tau(t)) + \int_t^{t+\omega} G_j(t, s)[f_j(s, x_j(s - \tau(s))$$
$$+ h_j(s)x_j(s) - r_j(s)x_j(s - \tau(s)) - a_j(s)x_j(s - \tau(s))]ds, \tag{2.8}$$

where $G_j(t, u)$ is defined by (2.5) and

$$r_j(s) = \frac{\Big(c_j'(s) - c_j(s)h_j(s)\Big)\Big(1 - \tau'(s)\Big) + \tau''(s)c_j(s)}{(1 - \tau'(s))^2}. \tag{2.9}$$

Proof.

Multiplying both sides of (2.7) by $e^{-\int_0^t h_j(s)ds}$ and then integrating from t to $t+\omega$ gives

$$\begin{aligned} x_j(t+\omega)e^{-\int_0^{t+\omega} h_j(s)ds} - x_j(t)e^{-\int_0^t h_j(s)ds} &= \int_t^{t+\omega} \Big[a_j(s)x_j(s-\tau(s)) \\ &\quad - h_j(s)x_j(s) - c_j(s)x_j'(s-\tau(s)) \\ &\quad - f_j(s, x_j(s-\tau(s)))\Big] e^{-\int_0^s h_j(u)du} ds. \end{aligned}$$

By dividing both sides of the above equation by $e^{-\int_0^t h_j(s)ds}$ and using the fact that $x_j(t+T) = x_j(t)$, in the above equation gives

$$\begin{aligned} x_j(t)\Big[e^{-\int_0^\omega h_j(u)du} - 1\Big] &= \int_t^{t+\omega} \Big[a_j(s)x_j(s-\tau(s)) \\ &\quad - h_j(s)x_j(s) - c_j(s)x_j'(s-\tau(s)) \\ &\quad - f_j(s, x_j(s-\tau(s)))\Big] e^{\int_s^t h_j(u)du} ds. \end{aligned} \tag{2.10}$$

Rewrite

$$\begin{aligned} &\int_t^{t+\omega} c_j(s)x_j'(s-\tau(s))e^{\int_s^t h_j(u)du} ds \\ &= \int_t^{t+\omega} \frac{c_j(s)x_j'(s-\tau(s))(1-\tau'(s))}{(1-\tau'(s))} e^{\int_s^t h_j(u)du} ds. \end{aligned}$$

Integration by parts on the above integral with

$$U = \frac{c_j(u)}{1-\tau'(u)} e^{\int_s^t h_j(u)du}, \quad \text{and} \quad dV = x_j'(s-\tau(s))(1-\tau'(s))ds$$

gives

$$\begin{aligned} &\int_t^{t+\omega} c_j(s)x'(s-\tau(s))e^{\int_s^t h_j(u)du} ds \\ &= \tfrac{c_j(t)}{1-\tau'(t)} x_j(t-\tau(t)) \Big[e^{-\int_0^\omega h_j(u)du} - 1\Big] - \int_t^{t+\omega} r_j(s)e^{\int_s^t h_j(u)du} x_j(s-\tau(s))ds. \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10) and dividing through by $e^{-\int_0^\omega h_j(u)du} - 1$ we obtain,

$$x_j(t) = \frac{c_j(t)}{\tau'(t)-1}x_j(t-\tau(t)) + \int_t^{t+\omega} G_j(t,s)[f_j(s,x_j(s-\tau(s))) + h_j(s)x_j(s) - r_j(s)x_j(s-\tau(s)) - a_j(s)x_j(s-\tau(s))]ds.$$

This completes the proof.

We next state Krasnoselskii's Theorem which is the main mathematical tool in this paper in the following lemma.

Lemma 2.3 (Krasnoselskii's) Let $\mathbf{M}$ be a closed convex nonempty subset of a Banach space $(\mathbf{S}_\omega, ||.||)$. Suppose that J and D map $\mathbf{M}$ into $\mathbf{S}_\omega$ such that
(i) $x, y \in \mathbf{M}$, implies $Jx + Dy \in \mathbf{M}$,
(ii) D is continuous and $D\mathbf{M}$ is contained in a compact set,

(iii) J is a contraction mapping.
Then there exists $z \in \mathbf{M}$ with $z = Jz + Dz$.

3. Main Results

For some non-negative constant L and a positive constant K define the set

$$\mathbf{M} = \{\phi \in \mathbf{S}_\omega \ : \ L \le ||\phi|| \le K \text{ with } \frac{L}{n} \le |\phi_j|_0 \le \frac{K}{n},\ j = 1, 2, ..., n.\},$$

which is a closed convex and bounded subset of the Banach space $\mathbf{S}_\omega$. We also assume that for all $s \in \mathbf{R}, \rho \in \mathbf{M}$

$$\frac{(1-\sigma_j\gamma_*)L}{m_j\omega n} \le f_j(s,\rho_j) + h_j(s)\rho_j - r_j(s)\rho_j - a_j(s)\rho_j \le \frac{(1-\alpha_j\gamma)K}{M_j\omega n}$$

(3.1)

where $j = 1, 2, ...n$.
Define the map $D : \mathbf{M} \to \mathbf{S}_\omega$ by

$$
\begin{aligned}
(D\varphi)(t) &= \int_t^{t+\omega} G(t,s)[F(s,\varphi(s-\tau(s))) + H(s)\varphi(s) \\
&\quad - R(s)\varphi(s-\tau(s)) - A(s)(s)\varphi(s-\tau(s))]ds,
\end{aligned} \tag{3.2}
$$

where $(D\varphi) = (D\varphi_1, D\varphi_2, ..., D\varphi_n)^T$, $H(s) = \text{diag}[h_1(s), ..., h_n(s)]$ and $R(s) = \text{diag}[r_1(s), ..., r_n(s)]$.

Also, define $J : \mathbf{M} \to \mathbf{S}_\omega$ by

$$
(J\varphi)(t) = \frac{1}{\tau'(t) - 1} C(t)\varphi(t - \tau(t)), \tag{3.3}
$$

where $(J\varphi) = (J\varphi_1, J\varphi_2, ..., J\varphi_n)^T$.

Lemma 3.1. *Suppose that (2.1)-(2.4), (3.1), (H1), (H2), (H3) and (H5) hold. Then the operator D is completely continuous on $\mathbf{M}$.*

Proof. For $t \in [0, T]$ and for $\varphi \in \mathbf{M}$, we have by (3.1) that

$$
\begin{aligned}
|(D\varphi_j)(t)| &\le |\int_t^{t+\omega} G_j(t,s)[f_j(s,\varphi_j(s-\tau(s))) + h_j(s)\varphi_j(s) \\
&\quad - r_j(s)\varphi_j(s-\tau(s)) - a_j(s)(s)\varphi_j(s-\tau(s))]ds| \\
&\le M_j\omega \frac{(1-\alpha_j\gamma)K}{M_j\omega n} = \frac{(1-\alpha_j\gamma)K}{n}.
\end{aligned}
$$

It follows that

$$
|(D\varphi_j)|_0 \le \frac{(1-\alpha_j\gamma)K}{n}.
$$

Thus,

$$
\begin{aligned}
||(D\varphi)|| &= \sum_{j=1}^{n} |(D\varphi_j)|_0 \\
&\le \sum_{j=1}^{n} \frac{(1-\alpha^*)K}{n},
\end{aligned}
$$

where $\alpha^* = \min_{1 \le j \le n}\Big(\alpha_j \gamma\Big)$. It therefore follows that

$$||(D\varphi)|| \leq K.$$

This shows that $D(\mathbf{M})$ is uniformly bounded.

We will next show that $D(\mathbf{M})$ is equi-continuous. Let $\varphi \in \mathbf{M}$. Then differentiating (3.2) with respect to t gives

$$\begin{aligned}(D\varphi_j)'(t) &= \Big[G_j(t, t+\omega) - G_j(t,t)\Big]\Big[f_j(t, \varphi_j(t-\tau(t))) + h_j(t)\varphi_j(t) \\ &\quad -r_j(t)\varphi_j(t-\tau(t)) - a_j(t)\varphi_j(t-\tau(t))\Big] + h_j(t)(D\varphi_j)(t). \end{aligned} \tag{3.4}$$

Thus

$$|(D\varphi_j)'(t)| \leq \frac{(1-\alpha_j\gamma)KM_j}{\omega n} + ||h_j||\frac{(1-\alpha_j\gamma)K}{n}.$$

It follows that

$$|(D\varphi_j)'|_0 \leq \frac{(1-\alpha_j\gamma)KM_j}{\omega n} + |h_j|_0\frac{(1-\alpha_j\gamma)K}{n}.$$

Hence

$$\begin{aligned} ||(D\varphi)'|| &= \sum_{j=1}^{n} |(D\varphi_j)|_0 \\ &\leq \sum_{j=1}^{n}\Big[\frac{(1-\alpha^*)KM}{\omega n} + ||h||\frac{(1-\alpha^*)K}{n}\Big] \\ &\leq \frac{(1-\alpha^*)KM}{\omega} + ||h||(1-\alpha^*)K, \end{aligned}$$

where $M = \max\{M_1, M_2, ..., M_n\}$. Thus showing that $D(\mathbf{M})$ is equicontinuous. Then using Ascoli-Arzela theorem we obtain that D is a compact map. Due to the continuity of all the terms in (3.2), we have that D is continuous.

Lemma 3.2 Suppose that (H2) and (H5) hold. Then the operator J is a contraction.

Proof. For $\varphi, \psi \in \mathbf{M}$

$$|(J\varphi_j) - (J\psi_j)|_0 \quad \leq \quad \alpha_j\gamma|\varphi_j - \psi_j|_0$$

Hence,

$$\begin{aligned} ||(J\varphi_j) - (J\psi_j)|| &\leq \sum_{j=1}^{n} |(J\varphi_j) - (J\psi_j)|_0 \\ &\leq \sum_{j=1}^{n} \alpha_j\gamma|\varphi_j - \psi_j|_0 \\ &\leq \alpha \sum_{j=1}^{n} |\varphi_j - \psi_j|_0 = \alpha||\varphi - \psi||, \end{aligned}$$

where $\alpha = \max\{\alpha_1\gamma, ..., \alpha_n\gamma\}$. This completes the proof of lemma 3.2.

Theorem 3.3 Suppose (H1), (H2), (H3), (H4), (H5), and (3.1) hold. Also suppose that the hypotheses of Lemma 3.2 and Lemma 3.3 hold. Then (1.1) has a positive periodic solution x satisfying $L \leq ||x|| \leq K$.

Proof. Let $\varphi, \psi \in \mathbf{M}$. Then

$$\begin{aligned} (J\psi_j)(t) + (D\varphi_j)(t) &= \frac{1}{\tau'(t) - 1} c_j(t)\psi_j(t - \tau(t)) \\ &\quad + \int_t^{t+\omega} G_j(t,s)[f_j(s, \varphi_j(s - \tau(s))) + h_j(s)\varphi_j(s) \\ &\quad - r_j(s)\varphi_j(s - \tau(s)) - a_j(s)\varphi_j(s - \tau(s))]ds \\ &\leq \alpha_j\gamma\frac{K}{n} + M_j \int_t^{t+\omega} [f_j(s, \varphi_j(s - \tau(s))) + h_j(s)\varphi_j(s) \\ &\quad - r_j(s)\varphi_j(s - \tau(s)) - a_j(s)\varphi_j(s - \tau(s))]ds \\ &\leq \alpha_j\gamma\frac{K}{n} + M_j\omega\frac{(1 - \alpha_j\gamma)K}{M_j n\omega} \\ &\leq \frac{K}{n}. \end{aligned}$$

Thus,

$$(J\varphi)(t) + (H\psi)(t) \quad \leq \quad \sum_{j=1}^{n} \frac{K}{n} = K.$$

On the other hand,

$$\begin{aligned}(J\psi_j)(t) + (D\varphi_j)(t) &= \frac{1}{\tau'(t) - 1} c_j(t)\psi_j(t - \tau(t)) \\ &\quad + \int_t^{t+\omega} G_j(t,s)[f_j(s, \varphi_j(s - \tau(s))) + h_j(s)\varphi_j(s) \\ &\quad - r_j(s)\varphi_j(s - \tau(s)) - a_j(s)\varphi_j(s - \tau(s))]ds \\ &\geq \sigma_j\gamma_* \frac{L}{n} + m_j \int_t^{t+\omega} [f_j(s, \varphi_j(s - \tau(s))) + h_j(s)\varphi_j(s) \\ &\quad - r_j(s)\varphi_j(s - \tau(s)) - a_j(s)\varphi_j(s - \tau(s))]ds \\ &\geq \sigma_j\gamma_* \frac{L}{n} + m_j\omega \frac{(1 - \sigma_j\gamma_*)L}{m_j n\omega} \\ &\geq \frac{L}{n}.\end{aligned}$$

Thus,

$$(J\varphi)(t) + (H\psi)(t) \geq \sum_{j=1}^{n} \frac{L}{n} = L.$$

This completes the proof of theorem 3.3.

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