UNIVERSITY OF CAPE COAST

EIGENVALUES OF SUM OF HERMITIAN MATRICES

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BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Charles Addai

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Professor (Mrs.) Natalia Mensah

ABSTRACT

The research deals with a systematic provision of mathematical algorithms to solve eigenvalue inequalities bounding the sum of eigenvalues of Hermitian matrices. We study the problem of defining the set of eigenvalues of sum of Hermitian matrices and then investigate the relationship existing between the eigenvalues of $n_i \times n_j$ Hermitian matrices A and B to the eigenvalues of their summand $H = A + B$. The recent solution appears to deal with the situation of eigenvalue inequalities where $r = n - 2$. We modified a number of numerical methods to formulate inequalities connecting sum of r eigenvalues for H in relation to sum of r eigenvalues for both A and B depending on the parameters for $r < n$ where $r = n - 1, n - 3$. Lastly, the research showed the solution of possible eigenvalues of sum of Hermitian matrices of size H_r^n for $n \leq 12$. Other gaping issues were also specified.

KEY WORDS

Dimension

Eigenvalues

Hermitian matrices

Inequalities

Numerical algorithm

Parameters

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DEDICATION

To my lovely wife, my children and my parents

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CHAPTER ONE

INTRODUCTION

Background to the Study

Over the years, there has been much substantial interest in eigenvalues of sum of Hermitian matrices (Chang, 1999). In matrix analysis and most specifically Hermitian matrices, it is seen in sciences and physics in particular that eigenvalue issues are among the most significant mathematical challenges. This is because of its application in quantum mechanics due to the requirement that the result of a physical measurement must be a real quantity, eigenvalues are therefore used to represent the needed physical quantities (Saleem, 2015). Thus given a system of Hermitian matrices A, B and H, we can find the possible set of eigenvalues $\gamma\mu$ of $H = A + B$ where $(\lambda\mu\alpha_1, \lambda\mu\alpha_2, ..., \lambda\mu\alpha_n)$ and $(\lambda\mu\beta_1, \lambda\mu\beta_2, ..., \lambda\mu\beta_n)$ are eigenvalues of A and B respectively (Fiedler, 1971). In terms of equality, these condition can be completely described by Knutson and Tao (2001) as

$$\sum_{i=1}^{n} \lambda\mu\alpha_i + \sum_{j=1}^{n} \lambda\mu\beta_j = \sum_{k=1}^{n} \gamma\mu_k \tag{1.1}$$

and in the form of linear inequality

$$\sum_{i=1}^{n} \lambda\mu\alpha_i + \sum_{j=1}^{n} \lambda\mu\beta_j \geq \sum_{k=1}^{n} \gamma\mu_k \tag{1.2}$$

where (i, j, k) are increasing sequence of integers. The following are some inequalities in the special instance of (i, j, k) for Herm (n) matrices A, B and $H = A + B$.

i. For all i and j such that $i + j - 1 \leq n$,

$$\lambda\alpha_i(A) + \lambda\beta_j(B) \geq \gamma_{i+j-1}(H)$$

(Weyl, 1912).

ii. For all indices

$$1 \leq l_1 < l_2 < ... < l_r \leq n,$$

and

$$1 \leq j_1 < j_2 < ... < j_r \leq n,$$

where

$$i_r + j_r - r \leq n$$

$$\sum_{s=1}^{r} \lambda\alpha_{i_s}(A) + \sum_{s=1}^{r} \lambda\beta_{j_s}(B) \geq \sum_{s=1}^{r} \gamma_{i_s+j_s-s}(H)$$

(Thompson & Freede, 1971)

Finally, Horn (1962) conjectured a complete set H_r^n of triples of indices (i, j, k) of cardinality r such that $i = i_1, i_2, ..., i_r$, $j = j_1, j_2, ..., j_r$ and $k = k_1, k_2, ..., k_r$ which are subsets of $1, 2, ..., n$ of $r < n$ by

$$\sum_{i=1}^{r} \lambda\alpha_i + \sum_{j=1}^{r} \lambda\beta_j \geq \sum_{k=1}^{r} \gamma_k$$

Then $(i, j, k) \in H_r^n$ such that

$$\sum_{x=1}^{s} i_x + \sum_{y=1}^{s} j_y \geq \sum_{z=1}^{s} k_z + \frac{s(s+1)}{2}$$

for any $s < r$ and for triples of indices $x_1, x_2, ..., x_s, y_1, y_2, ..., y_s$ and $z_1, z_2, ..., z_s$ which are subsets of $1, 2, ..., r \in H_s^r$.

Eigenvalue analysis have many application in quantum mechanics and also in oil companies where eigenvalue analysis are used to explore land for oil and as well give a good indication of the location of oil reserves (Saleem, 2015). Thus eigenvalues of Hermitian matrices have been of great interest not to the field of algebra alone but to the field of physics and the field of engineering as well (Weisstein, 2002).

Many effort have been employed at solving eigenvalues in relation to the sum of Hermitian matrices by several other researchers (Amir-Moez, 1956; Fulton, 1998; Horn, 1962; Thompson & Freede, 1971). In recent times, however, the case of eigenvalues in relation to the sum of Hermitian matrices has been virtually solved providing possible inequalities bounding sums of r eigenvalues for $C = A+B$ such that r is less than n, where $r = n-2$ for $(i, j, k) \in T_r^n$ (Taylor, 2015). This research will therefore focus on the eigenvalues of $H = A + B$ in relation to the eigenvalues of A and B for the set of integers $(i, j, k) \in H_r^n$ where $r = n - 1, n - 3$.

Eigenvalues are a particular collection of scalars connected to a sequence of linear equations $Ax = \lambda x$ that are sometimes also known as characteristics roots, characteristics values, (Ferrar, 1972), proper values, or latent roots (Marcus & Minc, 1988), where the number or scalar value λ is an eigenvalue of A. The process of identifying the eigenvalues and eigenvectors of a matrix system is highly useful in areas such as physics and engineering where matrix diagonalization is the equivalent and arises and appears in programs like stability analysis and the physics of rotating bodies (Weisstein, 2002). Each eigenvalue is paired with a corresponding eigenvector (Dobson & Cox, 1999). Given an $n \times n$ matrix M, a scalar λ is an eigenvalue of M if and only if there is a nonzero vector x, called an eigenvector such that $(\lambda I$ - $M)$x = 0, where I is an identity matrix. Since x is non-zero, the eigenvalues of matrix M are the root of det$(\lambda I$ - $M) = 0$, which is a polynomial in λ.

Matrix is a rectangular array of numbers, symbols and expressions arranged in rows and columns. $A := (a_{ij})_{m\times n}$ defines an $m \times n$ matrix A with each entry in the matrix $A[ij]$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Some programming languages also begin the convention at zero, such that we have $0 \leq i \leq m$ - 1 and $0 \leq j \leq n$ - 1.

Below are some types of matrices;

1. Square matrix; $A \in \mathbf{C}^{n\times n}$
2. Hermitian matrix; $A \in \mathbf{C}^{n\times n}, A = \bar{A}^T$
3. Anti Hermitian matrix; $A = -\bar{A}^T$ (Strang, 2006).

A symmetric matrix is a square matrix which is equal to its transpose, $A = A^T$ (Lipovetsky, 2013). Only square matrix is symmetric since equal matrices have equal dimensions. Let $A = (a_{ij})$ be an $n_i \times n_j$ square matrix, then A is symmetric if $A = A^T$ for all $(a_{ij}) \in \mathbf{R}$. A and A^T have the same eigenvalues and eigenvectors and eigenvectors corresponding to distinct eigenvalues are orthogonal. The theorems below show the relationship between eigenvalues, eigenvectors and symmetric matrices.

Theorem 1.1

All eigenvalues of a real symmetric matrix are real.

Theorem 1.2

Eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal (Weiss, 2019).

Suppose an eigenvalue $\lambda\alpha$ has a multiplicity x, then we can find a set of x orthonormal eigenvectors $(\mu_1, \mu_2, ..., \mu_n)$ for $\lambda\alpha$. If $A \in R^{n\times n}$ is a symmetric matrix then there is an orthonormal set of eigenvectors $(\mu_1, \mu_2, ..., \mu_n)$ corresponding to eigenvalues $(\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n)$. Thus we have a spectral decomposition $V^TAV = d$ where $V = \mu_1, \mu_2, ..., \mu_n$ is an orthogonal matrix such that $V^T = V^{-1}$ and $d = diag(\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n)$ is diagonal.

According to Weisstein (2001), Hermitian matrix is a complex square matrix that is equal to its own conjugate transpose. Hermitian matrices are the complex extension of real symmetric matrices. Let $A = (a_{ij}) \in Mn(\mathbf{C})$, then A is Hermitian if $A = (\bar{A})^T$ (i.e. $a_{ij} = \bar{a}_{ji}$ for all $(a_{ij}) \in \mathbf{C}$ except $(i \neq j) \in \mathbf{R}$. The conjugate transpose of A given by A^*, is obtained by taken the complex conjugate of its transpose (i.e $A^* = (\bar{A}^T = \bar{A^T})$.

A matrix $X = (x_{ij}) \in \mathbf{R}$ for all indexes (i, j) is Hermitian if and only if it is symmetric with respect to its entries at the main diagonal (Johnson, Kroschel, & Omladic, 2004). The sum of two Hermitian matrices X, Y is Hermitian and their product is Hermitian if the two matrices commute, thus if $XY = YX$. Hermitian matrices are normal matrices. A matrix X is said to be normal if $X\bar{X}^T = \bar{X}^T X$. By the spectral theorem, any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. Thus for every Hermitian matrix $A \in Mn(\mathbf{C})$, we have; $\langle Pq, r\rangle = \langle q, P^* r\rangle = \langle q, Pr\rangle$, for all $q.r \in \mathbf{C}^n$. This implies that $\langle q, Pq\rangle = \langle Pq, q\rangle = \langle q, \bar{P}q\rangle$, for all $q \in \mathbf{C}^n$ and hence $\langle q, Pq\rangle$ is necessarily a real number (Taylor, 2015).

Theorem 1.3

All eigenvalues of a Hermitian matrix are real.

Proof: Let $P = (p_{ij}) \in Mn(\mathbf{C})$ such that $P \in \text{Herm}(n)$ with $\lambda_\alpha r = Pr$ where λ_α is the eigenvalue of P and $(r \neq 0)$, then we have;

$\lambda_\alpha\langle r, r\rangle = \langle r, \lambda_\alpha r\rangle = \langle r, Pr\rangle = \langle Pr, r\rangle = \langle \lambda_\alpha r, r\rangle = \bar{\lambda}_\alpha\langle r, r\rangle$. Since the inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{C}^{\mathbf{n}\times\mathbf{n}}$ is positive definite, then $\lambda_\alpha = \bar{\lambda}_\alpha$ is a real number (Taylor, 2015).

Matrices X and Y show example of complex and real Hermitian matrices respectively;

$$X = \begin{bmatrix} 1 & 3+2i & 4 \\ 3-2i & 3 & -i \\ 4 & i & 5 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Proposition 1.4

The trace of a square matrix A (i.e. $A \in Mn(\mathbf{C})$ is the sum of entries in its main diagonal, that is

$$tr(A) = \sum_{i=1}^{n} \lambda\alpha_{ij}, i = j$$

Theorem 1.5

If $(\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n)$ are the eigenvalues of $A \in Mn(\mathbf{C})$, then

$$tr(A) = \sum_{i=1}^{n} \lambda\alpha_i$$

Theorem 1.6

Every Hermitian matrix A is similar to a diagonal matrix D (Singh, 2021).

Proof: Let $A = a_{ij} \in Mn(\mathbf{C})$ where $A \in$ Herm (n). Then there exist a diagonal matrix D and an invertible matrix P such that $D = PAP^{-1}$. Hence

$$\sum \lambda\alpha_i(A) = tr(A) = tr(AP^{-1}P) = tr(PAP^{-1}) = tr(D)$$

(Taylor, 2015). Thus, the trace of a diagonal matrix D is the sum of its eigenvalues.

Theorem 1.7

The sum of any two Hermitian matrices is Hermitian.

Proof: Let $A, B \in$ Herm (n) and $H = A + B$. From the trace inequality $tr(A) + tr(B) = tr(A + B)$ (Zhang, 2011). This follows from Proposition 1.4 that

$$\sum_{i=1}^{n} \lambda\mu\alpha_i(A) + \sum_{j=1}^{n} \lambda\mu\beta_j(B) = \sum_{k=1}^{n} \gamma\mu_k(H)$$

where $H = A + B$

Statement of the Problem

This research seeks a systematic way to solve the inequalities bounding the sum of eigenvalues of Hermitian matrices $H = A + B$ for some $r < n$. We define and determine the eigenvalues of set of all possible $\gamma\mu$ of $n \times n$ Hermitian matrices $H = A + B$ for $n \leq 12$. Thus, the eigenvalues of sum of Hermitian matrices H in relation to the eigenvalues of A and B. In specific terms, the research problem can be stated as follows;

1. Given $n \times n$ Hermitian matrices A and B with corresponding eigenvalues $(\lambda\mu\alpha_1, \lambda\mu\alpha_2, ..., \lambda\mu\alpha_n)$ and $(\lambda\mu\beta_1, \lambda\mu\beta_2, ..., \lambda\mu\beta_n)$ and $H \in$ Herm (n) with eigenvalues $(\gamma\mu_1, \gamma\mu_2, ..., \gamma\mu_n)$ where $H = A + B$, we generate possible inequalities of size H_r^n for $1 \leq r < n \leq 12$.

2. Given $n \times n$ Hermitian matrices A, B and $H \in$ Herm (n), where $H = A + B$, we find the set of all possible $\gamma\mu$ of H for $1 \leq r < n \leq 12$.

Purpose of the Study

The aim of this research is to establish the eigenvalues of sum of Hermitian matrices for set $\gamma\mu$ of $H = A + B$ for $1 \leq r < n \leq 12$. The research will formulate all possible inequalities

$$\sum_{s=1}^{r} \gamma\mu_{k_s} \leq \sum_{s=1}^{r} \lambda\mu\alpha_{i_s} + \sum_{s=1}^{r} \lambda\mu\beta_{j_s}$$

for H_r^n denoting the set of triples (i, j, k) and establish a large class of such inequalities for $1 \leq r < n \leq 12$. A necessary condition for $H = A + B$ with the above given eigenvalues is that

$$\sum_{k=1}^{n} \gamma\mu_k = \sum_{i=1}^{n} \lambda\mu\alpha_i + \sum_{j=1}^{n} \lambda\mu\beta_j$$

Naturally, the next level is to find the inequalities bounding for $H = A + B$, as was posed by (Horn, 1962). It turns out that the necessary condition for the presence of Hermitian matrices with prescribed eigenvalues also involve a set of inequalities.

Research Objectives

In this research, we will review previous results obtained by Taylor (2015) in respect to the eigenvalues of sum of Hermitian matrices showing the size of T_r^n for $1 \leq r < n \leq 11$. Thus our specific objectives include;

1. Formulate all possible inequalities;

$$\sum_{s=1}^{r} \gamma\mu_{k_s} \leq \sum_{s=1}^{r} \lambda\mu\alpha_{i_s} + \sum_{s=1}^{r} \lambda\mu\beta_{j_s}$$

   for $(i, j, k) \in H_r^n, 1 \leq r < n \leq 12$ where $r = n - 1, n - 3$ and (i, j, k) are ordered triples of integers.

2. Determine the set $H = A + B$ of all possible $\gamma\mu$ for $1 \leq r < n \leq 12$.

Significance of the Study

The state of research done in the area of eigenvalues of sum of Hermitian matrices to the best of the researcher's knowledge and findings is of size T_r^n for $1 \leq r < n \leq 11$. Hence an extension to size H_r^n for $1 \leq r < n \leq 12$ will add up to the existing academic knowledge.

Delimitations

Determining the eigenvalues of sum of Hermitian matrices for $n \leq 12$, the research could have considered the eigenvalues of sum of skew Hermitian matrices. However, for this research emphasis is placed on eigenvalues of sum of Hermitian matrices for $n \leq 12$.

Limitations

The study discussed eigenvalues of sum of Hermitian matrices of size H_r^n for $1 \leq r < n \leq 12$. Thus the study is limited to eigenvalues of sum of Hermitian matrices for $n \leq 12$. Again, we will restrict the research to inequalities of H_r^n for $1 \leq r < n \leq 12$, such that $r = n-1, n-3$.

Organization of the Study

Generally, the research consist of five chapters. Chapter 1 has been cosidered earlier. The remaining chapters have also been defined as follows; Chapter 2 reviews some related literatures on eigenvalues of sum of Hermitian matrices problems. Chapter 3 delivers the research methodology and approach used. Chapter 4 presents the main results of the research. Chapter 5 contains the summary, conclusions and recommendations of the research.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter there is a review of the work done by some researchers on the eigenvalues of sum of Hermitian matrices. As a matter of recognizing the fact that some limited literatures exist on the research topic, the researcher made a decision to consider the various form as of the eigenvalues obtained so far on sum of Hermitian matrices. Despite a critical search at the research on sum of Hermitian matrices, however, it must be noted that eigenvalues of sum of Hermitian matrices are characterized based on eigenvalue inequalities which necessarily hold for $n \times n$ Hermitian matrices (Taylor, 2015). Studies on sum of eigenvalues of Hermitian matrices have been intensive, ranging from physics application to algebraic theorization (Villacampa, Navarro-Gonzalez, Compan-Rosique, & Satorre-Cuerda, 2019). Results are however bound for extension even within the same field of discipline.

Eigenvalue Conjectures

Many efforts have been employed at finding the eigenvalues of sum of Hermitian matrices by researchers (Amir-Moez, 1956; Day, So, & Thompson, 1998; Fulton, 1998; Horn, 1962; Miranda, 2003; Taylor, 2015). In this research, eigenvalues of sum of Hermitian matrices are discussed based on their characteristics such as inequalities bounding the eigenvalues summation of Hermitian matrices and the structure of set of all possible $\gamma\mu$ of $H = A + B$ for $A, B, H \in \text{Herm}(n)$ (Taylor, 2015).

This research is basically eigenvalue problem of sum of Hermitian matrices and before we discuss our research in details, discussion will be made on various conjectures on eigenvalue inequalities and sum of Hermitian matrices as well as what they seek to achieve.

Let A, B be $n \times n$ Hermitian matrices and $H = A + B$. With many effort in dealing with eigenvalues of sum of Hermitian matrices; what are the eigenvalues of H? How will all the possible set of eigenvalues sum $H = A + B$ be determined? There has been many mathematical development resulting from Weyl's question, (Weyl, 1912).

Horn (1962) formulated his conjecture by establishing a number of spectral inequalities for Hermitian matrices A, B and $C = A + B$. The research gave a conditional equality

$$\gamma_1 + \gamma_2 + ... + \gamma_n = \alpha_1 + \alpha_2 + ... + \alpha_n + \beta_1 + \beta_2 + ... + \beta_n$$

and the linear inequalities of the form

$$\gamma_{k_1} + \gamma_{k_2} + ... + \gamma_{k_r} \leq \alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_r} + \beta_{j_1} + \beta_{j_2} + ... + \beta_{j_r}$$

where (i, j, k) are considered increasing sequence of integers. Horn gave several inequalities in completing the set $n = 3$ and $n = 4$. The research conjectured that necessary and sufficient conditions could be given by such inequalities, where subset T_r^n of triples (I, J, K) of cardinality r in $[1, n]$ that occur can be obtained by induction on r. Hence, $I = i_1 < ... < i_r$, $J = j_1 < ... < j_r$, and $K = k_1 < ... < k_r$ such that when $r = 1$, $(I, J, K) \in T_1^n$ wherever $i_1 + j_1 = k_1 + 1$. Also, for $r > 1$, $(I, J, K) \in T_r^n$ wherever

$$\sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + (\frac{r+1}{2})$$

However, the spectrum problem for Hermitian matrices sum is then solved by combining portions in a clever argument that calls for extensive combinatorial difficulties. Horn (1962) conjectured a complete set H_r^n of triples of indices (I, J, K) of cardinality r such that $I = i_1, i_2, ..., i_r$, $J = j_1, j_2, ..., j_r$ and $K = k_1, k_2, ..., k_r$ which are subsets of $1, 2, ..., n$ of $r < n$ by

$$\sum_{i \in I} \lambda \alpha i + \sum_{j \in J} \lambda \beta j \geq \sum_{k \in K} \gamma k$$

. Then $(i, j, k) \in H_r^n$ such that

$$\sum_{x=1}^{s} i_x + \sum_{y=1}^{s} j_y = \sum_{z=1}^{s} k_z + \frac{s(s+1)}{2}$$

for any $s < r$ and for triples of indices $x_1, x_2, ..., x_s$, $y_1, y_2, ..., y_s$ and $z_1, z_2, ..., z_s$ which are subsets of $1, 2, ..., r \in H_s^r$.

In line with this, Horn (1962) proved his conjecture using induction hypothesis and also considering minimax principle where his results suggest that, when $r = 2, (i, j, k) \in S_2^n$ and when $r = 3, (i, j, k) \in S_3^n$. Horn could only prove his conjecture in low dimension because of the interlacing technique's severe combinatorial difficulties, which were compounded by similarly severe combinatorial difficulties in the final state. However, this therefore permitted Horn to only use low dimension to demonstrate his hypothesis for $n \leq 4$.

Thompson and Freede (1971) established a novel family of inequalities that connects the eigenvalues $\alpha_1 \geq ... \geq \alpha_n, \beta_1 \geq ... \geq \beta_n$ and $\gamma_1 \geq ... \geq \gamma_n$ of the Hermitian Liner transformations A, B and $C = A + B$ respectively. The study demonstrated how Amir-Moez (1956) inequalities, may be inferred from their family of inequalities, and hence showed that their inequalities are at least as severe and prevalent as and are often superior to Amir-Moez inequalities. The study further demonstrated by induction on the dimension n, that their inequalities provide a very natural and straight forward generalization of the inequalities and also contain Weyl (1912) inequalities. The study demonstrated by an induction on the dimension n, such that for $n = m$, $i_1 = j_1 = 1 ... i_m + j_m = m$, and hence reduced

$$\sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_{j_s} \geq \sum_{s=1}^{m} \gamma_(i_s + j_s - s)$$

to the trace equality $tr(A) + tr(B) = tr(C)$. Assuming $m < n$, Thompson & Freede established inequalities for Hermitian operators on dimensional space $(n - 1)$.

According to Klyachko (as cited in Fulton, 1998) the problem can be expressed in terms of the eigenvalues of A and B to determine the eigenvalues of the sum $A+B$ of two $n \times n$ Hermitian matrices. In decreasing order the research list the eigenvalues $\lambda_1(H) \geq \lambda_2(H) \geq ... \geq \lambda_n(H)$ of Hermitian matrix H. Write $\lambda(A) : \alpha_1 \geq ... \geq \alpha_n, \lambda(B) : \beta_1 \geq ... \geq \beta_n$ and $\lambda(C) : \gamma_1 \geq ... \geq \gamma_n$; where $C = A + B$. The problem of the research is to describe the triples (I, J, K) that determine inequalities and then demonstrate that the resulting inequalities characterize the possible eigenvalues. The problem was brought on by questions is solid mechanics, where the shapes of ellipsoid are determined by the eigenvalues of symmetric matrices. The research simply show that these inequalities $\gamma_1 \geq \alpha_1 + \beta_1, \gamma_2 \geq \alpha_1 + \beta_2$, and $\gamma_2 \geq \alpha_2 + \beta_1$ are sufficient to describe the possible eigenvalues of the sum $C = A + B$ for $n = 2$. For a complete answer is the case $n = 3$, the research produced another form of inequality $\gamma_2 + \gamma_3 \leq \lambda\alpha_1 + \lambda\alpha_3 + \lambda\beta_1 + \lambda\beta_3$ (Fulton, 1998).

Day *et al.* (1998) reveal a form of structure such that given real elements $(\alpha_1 + \alpha_2 + ... + \alpha_n)$, $(\beta_1 + \beta_2 + ... + \beta_n)$ and $(\gamma_1 + \gamma_2 + ... + \gamma_n)$ with nonincreasing component, necessary and sufficient condition for the existence of Herm (n) matrices A, B and $C = A + B$ such that α, β and γ are respectively components as their sequence of eigenvalues, then the consistent inequalities

$$\sum_{t=1}^{m} \gamma_{h_t} \leq \sum_{t=1}^{m} \alpha_{f_t} + \sum_{t=1}^{m} \beta_{g_t}$$

for $(f, g, h) \in T_m^n$ for $m = [1, n]$ with equality $m = n$. Day $et\ al.$ in a table indicated the space dimension (n), number of terms (k) and number of inequalities (N) of sum of Hermitian matrices for $n \leq 7$. Despite the number of inequalities indicated in the study, Day $et\ al.$ omitted the eigenvalue inequalities to enumerate the sets T_m^n where $m = k$ for $1 \leq k < n \leq 7$.

Miranda (2003), established how the diagonal entries of $A + B$ relate to the eigenvalues of A and B by well-known inequalities for Hermitian matrices A and B. According to Miranda, these inequalities are expanded to more general inequalities if the matrices A and B are perturbed by congruence of $UAU*$ + $VBV*$ where U and V are arbitrary unitary matrices, or if there are sums of more than two matrices. The research examined the extreme circumstances where these inequalities and some generalizations become equal.

In the same light of using inequalities to produce eigenvalues of sum of Hermitian matrices, Taylor (2015) investigate the relationship between Herm (n) matrices A, B eigenvalues and the eigenvalues of their sum $C = A+B$. The research mainly concentrate on inequalities that, for some r less than n, limit sums of r eigenvalues for $C = A + B$ by sums of r eigenvalues for Hermitian matrices A and B. Taylor, in the research also focus on certain inequalities that, from Alfred Horn's hypothesis would totally determine the potential eigenvalues of Hermitian matrices A, B and $C = A + B$. Taylor therefore considered an alternate formulation of Horn's theorem and then proof that when $r = n - 2$, there are necessarily diagonal $r \times r$ Hermitian matrices $A, B, C = A + B$. According to the research, $T_r^n \subset H_r^n$ for all $1 \leq r < n$. Thus, if $(i, j, k) \in T_r^n$, then inequality (I, J, K) holds for all Herm (n) matrices $A, B, C = A + B$. Also if $(\alpha, \beta, \gamma) \in (R^n)^3$ are weakly decreasing and satisfying both the trace equality

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_j$$

and the inequality (I, J, K) for every $(I, J, K) \in T_r^n$ and all $1 \leq r < n$ then there exists $A, B, C = A + B \in$ Herm (n) such that $\lambda(A) = \alpha, \lambda(B) = \beta$ and $\lambda(C) = \gamma$.

Taylor (2015) produced inequalities that generate the eigenvalues of sum of Hermitian matrices A, B and $C = A + B$ when dimension of the matrix and some parameters are given. By the use of prolog program, the research deduced a table to enumerate the sets T_r^n for $1 \leq r < n \leq 11$.

Chapter Summary

In this chapter, a number of forms of eigenvalues inequalities in relation to the sum of Hermitian matrices have been reviewed and discussed, most importantly in relation to Horn's conjecture, existence of minimax principles and some cases induction hypothesis. Some eigenvalue inequalities have been obtained by research and others are still open for solution. this research therefore present a systematic way to generate inequalities bounding the eigenvalues of sum of Hermitian matrices $A, B, H = A + B$ of size H_r^n for $1 \leq r < n \leq 12$, where $r = n - 1, n - 3$ and also determine the eigenvalues of sum of Hermitian matrices for $n \leq 12$.

CHAPTER THREE

RESEARCH METHODS

Introduction

The purpose of the research is to investigate the eigenvalues of sum of Hermitian matrices. This chapter provides information on the research methodology employed in the conduct of this research. The chapter specifically presents a step by step method and procedure used in obtaining eigenvalue inequalities of sum of Hermitian matrices A, B and $H = A + B$. Based on solvability lemma, an algorithm employed by Horn (1962) and Taylor (2015) is modified to construct eigenvalues of sum of Hermitian matrices $H = A + B$ with an extension of size H_r^n for $1 \leq r < n \leq 12$ where $r = n - 1, n - 3$. Thus, firstly a method for generating eigenvalue inequalities is discussed and then a modified algorithm for the solution of eigenvalues of sum of Hermitian matrices.

Method for Generating Possible Eigenvalue Inequalities given the Dimension and Parameters of the Matrix

The research uses an $n \times n$ matrix where $n = 2$ as a base for the development of the method for generating subsequence eigenvalue inequalities. We let A and B be $n \times n$ matrices.

Suppose A and B denote a 2×2 matrices with entries

$$A = \begin{bmatrix} a_{xx} & a_{xy} + i \\ a_{yx} - i & a_{yy} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{xx} & b_{xy} - 2i \\ b_{yx} + 2i & b_{yy} \end{bmatrix}$$

we aim at making both A and B Hermitian. This implies $A = \bar{A}^T$ and $B = \bar{B}^T$, where $\bar{A}^T$ and $\bar{B}^T$ denote the conjugate transpose of A and B respectively.

Thus,

$$A = \begin{bmatrix} a_{xx} & a_{xy}+i \\ a_{yx}-i & a_{yy} \end{bmatrix}$$

implies

$$\bar{A} = \begin{bmatrix} a_{xx} & a_{xy}-i \\ a_{yx}+i & a_{yy} \end{bmatrix}$$

and

$$\bar{A}^T = \begin{bmatrix} a_{xx} & a_{xy}+i \\ a_{yx}-i & a_{yy} \end{bmatrix}$$

which means that $a_{xy} = \bar{a}_{yx}$ and hence $A = \bar{A}^T$. The same process is followed for $B = \bar{B}^T$.

Suppose that $(\lambda\mu\alpha_1, \lambda\mu\alpha_2, ..., \lambda\mu\alpha_n)$ and $(\lambda\mu\beta_1, \lambda\mu\beta_2, ..., \lambda\mu\beta_n)$ are eigenvalues of $n \times n$ Hermitian matrices A and B respectively with entries

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} \end{bmatrix}$$

,

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & . & . & . & b_{1n} \\ b_{21} & b_{22} & b_{23} & . & . & . & b_{2n} \\ b_{31} & b_{32} & b_{33} & . & . & . & b_{3n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ b_{n1} & b_{n2} & b_{n3} & . & . & . & b_{nn} \end{bmatrix}$$

and $(\gamma\mu_1, \gamma\mu_2, ..., \gamma\mu_n)$ be eigenvalues of $H = A + B$, we have $(i, j, k) \in H_r^n$ such that

$$\sum_{s=1}^{r} \gamma\mu_{k_s} \leq \sum_{s=1}^{r} \lambda\mu\alpha_{i_s} + \sum_{s=1}^{r} \lambda\mu\beta_{j_s} \tag{3.1}$$

For $n \times n$ matrices where $n = 2$, there exist matrices $H = A + B$ of triples $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ satisfying the inequlities

$$\left.\begin{aligned} \gamma\mu_1 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_1 \\ \gamma\mu_2 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_2 \\ \gamma\mu_2 &\leq \lambda\mu\alpha_2 + \lambda\mu\beta_1 \end{aligned}\right\} \tag{3.2}$$

Step 1

We determine the matrix dimension (n) and parameter (r) of Hermitian matrix H_r^n such that $r = n - 1, n - 3$.

Step 2

We determine the eigenvalue inequalities depending on the dimension and parameters. We use base on equation (3.2) the following;

Table 1: **Present Eigenvalue Inequality Entries of Matrix Dimension** $n = 2$

$H(\gamma\mu) \leq$	$A(\lambda\mu\alpha)+$	$B(\lambda\mu\beta)$
1	1	1
2	1	2
2	2	1

(Richmond, 2013).

From Table 1, entries representing each of the three rows are such that the difference in the sum of integer entries in H and the sum of integer entries in both A and B is always one. Also, the number of entries in H determines the number of entries in each of A and B. The following tables present the details.

Table 2: **Eigenvalue Sum of Matrix Dimension n and Parameters $r = n - 1$**

n	$r = n-1$	Eigenvalues		Sum H	$A+B$	Diff
2	1	$\gamma\mu_1$	$\lambda\mu\alpha_1 + \lambda\mu\beta_1$	1	2	1
3	2	X_1	X_2	3	6	3
4	3	Y_1	Y_2	6	12	6
5	4	Z_1	Z_2	10	20	10
6	5	V_1	V_2	15	30	15
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
n	$n-1$	U_1	U_2	$\sum_{k=1}^{n}\gamma\mu_k$	$2[\sum_{k=1}^{n}\gamma\mu_k]$	$\sum_{k=1}^{n}\gamma\mu_k$

(Taylor, 2015).

Let,

$$X_1 = \gamma\mu_1 + \gamma\mu_2$$

$$X_2 = \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_2$$

$$Y_1 = X_1 + \gamma\mu_3$$

$$Y_2 = X_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_3$$

$$Z_1 = Y_1 + \gamma\mu_4$$

$$Z_2 = Y_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_4$$

$$V_1 = Z_1 + \gamma\mu_5$$

$$V_2 = Z_2 + \lambda\mu\alpha_5 + \lambda\mu\beta_5$$

$$U_1 = X_1 + ... + \gamma\mu_{n-1}$$

$$U_2 = X_2 + ... + \lambda\mu\alpha_{n-1} + \lambda\mu\beta_{n-1}$$

Table 3: **Eigenvalue Sum of Matrix Dimension n and Parameters $r = n - 3$**

n	$r = n-3$	Eigenvalues		Sum H	$A+B$	Diff
4	1	$\gamma\mu_1$	$\lambda\mu\alpha_1 + \lambda\mu\beta_1$	1	2	1
5	2	P_1	P_2	3	6	3
6	3	Q_1	Q_2	6	12	6
7	4	R_1	R_2	10	20	10
8	5	S_1	S_2	15	30	15
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
n	$n-3$	W_1	W_2	$\sum_{k=1}^{n} \gamma\mu_k$	$2[\sum_{k=1}^{n} \gamma\mu_k]$	$\sum_{k=1}^{n} \gamma\mu_k$

(Taylor, 2015).

Let,

$$P_1 = \gamma\mu_1 + \gamma\mu_2$$

$$P_2 = \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_2$$

$$Q_1 = P_1 + \gamma\mu_3$$

$$Q_2 = P_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_3$$

$$R_1 = Q_1 + \gamma\mu_4$$

$$R_2 = Q_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_4$$

$$S_1 = R_1 + \gamma\mu_5$$

$$S_2 = R_2 + \lambda\mu\alpha_5 + \lambda\mu\beta_5$$

$$W_1 = S_1 + ... + \gamma\mu_{n-3}$$

$$W_2 = S_2 + ... + \lambda\mu\alpha_{n-3} + \lambda\mu\beta_{n-3}$$

Step 3a

We use row five from Table 2 where $n = 6$ and $r = n - 1$ to generate the required number of eigenvalue inequalities by summing the integer entries of $\gamma\mu$ and both $\lambda\mu\alpha$ and $\lambda\mu\beta$ such that

i. the integer entries (i, j, k) in each of the eigenvalues $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ respectively increases from left to right.
Thus $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4, 5$ and $k = 1, 2, 3, 4, 5$.

ii. the sum of the integer entries (k) of $\gamma\mu$ is always 15 less than the sum of entries (i, j) of $\lambda\mu\alpha$ and $\lambda\mu\beta$ respectively.

Thus,

$$\sum \lambda\mu\alpha_{i_s} + \sum \lambda\mu\beta_{j_s} \geq \sum \gamma\mu_{k_s} + 15 \tag{3.3}$$

We suppose that, given the entries $\gamma\mu = 2, 3, 4, 5, 6, \lambda\mu\alpha = 1, 2, 3, 4, 6$ and $\lambda\mu\beta = 1, 3, 4, 5, 6$, we can generate the eigenvalue inequality of the form;

$$\left.\begin{array}{l}\gamma\mu_2 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\alpha_4 + \lambda\mu\alpha_6 \\ +\lambda\mu\beta_1 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_5 + \lambda\mu\beta_6\end{array}\right\} \tag{3.4}$$

which follows from equation (3.3).

Step 3b

We use row two where $n = 5$ and $r = n - 3$ from Table 3 to produce the required number of eigenvalue inequalities by going through step 3ai and

i. the sum of the integer entries (k) of $\gamma\mu$ is always three less than the sum of entries (i, j) of $\lambda\mu\alpha$ and $\lambda\mu\beta$ respectively.

Thus,

$$\sum \lambda\mu\alpha_{i_s} + \sum \lambda\mu\beta_{j_s} \geq \sum \gamma\mu_{k_s} + 3 \quad (3.5)$$

We let $\gamma\mu_k = 3, 4, \lambda\mu\alpha_i = 1, 3$ and $\lambda\mu\beta_j = 2, 3$, we can generate the eigenvalue inequality of the form;

$$\gamma\mu_3 + \gamma\mu_4 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \quad (3.6)$$

which follows from equation (3.5). It must be noted that the entries of $\lambda\mu\alpha$ and $\lambda\mu\beta$ depend strictly on the entries of $\gamma\mu$.

Modified Algorithm to Generate Solution of Sum of Eigenvalues of Hermitian Matrices of size H_r^n for $1 \leq r < n \leq 12$

Suppose that $1 \leq r < n \leq 12$, given the space dimension (n) of the matrices such that $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, we let

$$\lambda n_1 = 2, \lambda n_2 = 3, \lambda n_3 = 4, ..., \lambda n_{11} = 12 \quad (3.7)$$

Based on Theorem 1.3, H_r^n by an algorithm from Taylor (2015) is the following;

i. For H_r^n where $n = 12$ and $r = 1$, the sum of eigenvalues is produced by;

$$H_1^{12} = \begin{bmatrix} \lambda n_1 + 1 \\ \\ \lambda n_1 + \lambda n_2 + 1 \\ \\ \lambda n_1 + \lambda n_2 + \lambda n_3 + 1 \\ \\ \lambda n_1 + \lambda n_2 + \lambda n_3 + \lambda n_4 + 1 \\ . \\ . \\ . \\ \sum_{i=1}^{11} [\lambda n_i] + 1 \end{bmatrix}$$

Thus, $H_r^n = H_1^{12} = \frac{n(n+1)}{2}$, where $n = 12$.

Using a number of mathematical stractegies and algorithms we have the following;

ii. For H_r^n where $n = 12$ and $r = 2$, we let U represent the sum of eigenvalues, di represent the differences in the sum of eigenvalues and R represent rows. Then sum of eigenvalues of H_2^{12} is produced by the following;

Table 4: **Algorithm Showing** $H_r^n = H_2^{12}$

	2	3	4	5	6	7	8	9	10	11	12
2	U_1	U_6	U_{21}	U_{56}	U_{126}	U_{252}	U_{462}	U_{792}	U_{1287}	U_{2002}	U_x
		↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$di_{(1\text{-}10)}R_1$		d_5	d_{15}	d_{35}	d_{70}	d_{126}	d_{210}	d_{330}	d_{495}	d_{715}	$di_{10}R_1$
			↓	↓	↓	↓	↓	↓	↓	↓	↓
$di_{(1\text{-}9)}R_2$			d_{10}	d_{20}	d_{35}	d_{56}	d_{84}	d_{120}	d_{165}	d_{220}	di_9R_2
				↓	↓	↓	↓	↓	↓	↓	↓
$di_{(1\text{-}8)}R_3$				d_{10}	d_{15}	d_{21}	d_{28}	d_{36}	d_{45}	d_{55}	di_8R_3
					↓	↓	↓	↓	↓	↓	↓
$di_{(1\text{-}7)}R_4$					d_5	d_6	d_7	d_8	d_9	d_{10}	di_7R_4

(Day *et al.*, 1998)

Key: ↓ indicates the difference between successor and predecessor.

Let

$$di_{10}R_1 = d[U_x - U_{2002}]$$

$$di_9R_2 = di_{10}R_1 - di_9R_1 = d[(U_x - U_{2002}) - di_9R_1]$$

$$di_8R_3 = di_9R_2 - di_8R_2 = d[((U_x - U_{2002}) - di_9R_1) - di_8R_2]$$

$$di_7R_4 = di_8R_3 - di_7R_3 = d[(((U_x - U_{2002}) - di_9R_1) - di_8R_2) - di_7R_3]$$

From Table 4,

$$di_7R_4 = d[(((U_x - U_{2002}) - di_9R_1) - di_8R_2) - di_7R_3] = d_{11} \quad (3.8)$$

Hence we find its inverse algorithm to generate U_x.

iii. Suppose that we have H_r^n where $n = 12$ and $r = 3$, we let U represent the sum of eigenvalues, dj represent their differences and R represent rows. Then sum of eigenvalues of Hermitian matrix H_3^{12} is as a result of the following;

Table 5: **Algorithm Showing** $H_r^n = H_3^{12}$

	3	4	5	6	7	8	9	10	11	12
3	U_1	U_{10}	U_{56}	U_{228}	U_{751}	U_{2120}	U_{5317}	U_{12140}	U_{25678}	U_x
		↓	↓	↓	↓	↓	↓	↓	↓	↓
$dj_{(1-9)}R_1$		d_9	d_{46}	d_{172}	d_{523}	d_{1369}	d_{3197}	d_{6823}	d_{13538}	dj_9R_1
			↓	↓	↓	↓	↓	↓	↓	↓
$dj_{(1-8)}R_2$			d_{37}	d_{126}	d_{351}	d_{846}	d_{1828}	d_{3626}	d_{6715}	dj_8R_2
				↓	↓	↓	↓	↓	↓	↓
$dj_{(1-7)}R_3$				d_{89}	d_{225}	d_{495}	d_{982}	d_{1798}	d_{3089}	dj_7R_3
					↓	↓	↓	↓	↓	↓
$dj_{(1-6)}R_4$					d_{136}	d_{270}	d_{487}	d_{816}	d_{1291}	dj_6R_4
						↓	↓	↓	↓	↓
$dj_{(1-5)}R_5$						d_{134}	d_{217}	d_{329}	d_{475}	dj_5R_5
							↓	↓	↓	↓
$dj_{(1-4)}R_6$							d_{83}	d_{112}	d_{146}	dj_4R_6

(Day *et al.*, 1998)

Key: ↓ indicates the difference between successor and predecessor.

Let

$$dj_9R_1 = d[U_x - U_{25678}]$$

$$dj_8R_2 = dj_9R_1 - dj_8R_1 = d[(U_x - U_{25678}) - dj_8R_1]$$

$$dj_7R_3 = dj_8R_2 - dj_7R_2 = d[((U_x - U_{25678}) - dj_8R_1) - dj_7R_2]$$

$$dj_6R_4 = dj_7R_3 - dj_6R_3 = d[(((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3]$$

$$dj_5R_5 = dj_6R_4 - dj_5R_4$$

$$= d[((((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3) - dj_5R_4]$$

$$dj_4R_6 = dj_5R_5 - dj_4R_5$$

$$= d[(((((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3) - dj_5R_4) - dj_4R_5]$$

From Table 5, $dj_{(1-3)}R_6$ provides a partern which generate an approximate value for

$$\left.\begin{aligned} dj_4R_6 \quad &= d[(((((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3) - dj_5R_4) \\ &-dj_4R_5] = d_{186} \end{aligned}\right\} \tag{3.9}$$

Hence we find its inverse algorithm which generate an approximate value of U_x.

iv. Suppose that we have H_r^n such that $n = 12$ and $r = 4$, we let U be the sum of eigenvalues, dk be their differences and R represent rows. Then sum of eigenvalues of Hermitian matrix H_4^{12} is through the following;

Table 6: **Algorithm Showing** $H_r^n = H_4^{12}$

	4	5	6	7	8	9	10	11	12
4	U_1	U_{15}	U_{126}	U_{751}	U_{3516}	U_{13704}	U_{46208}	U_{138519}	U_x
$dk_{(1\text{-}8)}R_1$		d_{14}	d_{111}	d_{625}	d_{2765}	d_{10188}	d_{32504}	d_{92311}	dk_8R_1
$dk_{(1\text{-}7)}R_2$			d_{97}	d_{514}	d_{2140}	d_{7423}	d_{22316}	d_{59807}	dk_7R_2
$dk_{(1\text{-}6)}R_3$				d_{417}	d_{1626}	d_{5283}	d_{14893}	d_{37491}	dk_6R_3
$dk_{(1\text{-}5)}R_4$					d_{1209}	d_{3657}	d_{9610}	d_{22598}	dk_5R_4

(Day *et al.*, 1998)

Key: ↓ indicates the difference between successor and predecessor.

Let

$$dk_8R_1 = d[U_x - U_{138519}]$$

$$dk_7R_2 = dk_8R_1 - dk_7R_1 = d[(U_x - U_{138519}) - dk_7R_1]$$

$$dk_6R_3 = dk_7R_2 - dk_6R_2 = d[((U_x - U_{138519}) - dk_7R_1) - dk_6R_2]$$

$$dk_5R_4 = dk_6R_3 - dk_5R_3 = d[(((U_x - U_{138519}) - dk_7R_1) - dk_6R_2) - dk_5R_3]$$

From Table 6, considering the differences in $(dk_2R_4, dk_3R_4$ and $dk_4R_4)$ we assume that,

$$[(dk_2R_4 \times 2) + 2296] = dk_3R_4$$

$$[(dk_3R_4 \times 2) + 3378] = dk_4R_4$$

hence by assumption,

$$[(dk_4R_4 \times 2) + 4460] = dk_5R_4$$

Thus,

$$dk_5R_4 = d[(((U_x - U_{138519}) - dk_7R_1) - dk_6R_2) - dk_5R_3] = d_{49656} \quad (3.10)$$

Hence we find its inverse algorithm which generate an approximate value of U_x.

v. For H_r^n where $n = 12$ and $r = 5$, we let U represent the sum of eigenvalues, dl represent the differences in the sum of eigenvalues and R represent rows. Then sum of eigenvalues of H_5^{12} is generated by the following;

Table 7: **Algorithm Showing** $H_r^n = H_5^{12}$

	5	6	7	8	9	10	11	12
5	U_1	U_{21}	U_{252}	U_{2120}	U_{13704}	U_{71973}	U_{319450}	U_x
$dl_{(1\text{-}7)}R_1$		d_{20}	d_{231}	d_{1868}	d_{11584}	d_{58269}	d_{247477}	dl_7R_1
$dl_{(1\text{-}6)}R_2$			d_{211}	d_{1637}	d_{9716}	d_{46685}	d_{189208}	dl_6R_2
$dl_{(1\text{-}5)}R_3$				d_{1426}	d_{8079}	d_{36969}	d_{142523}	dl_5R_3
$dl_{(1\text{-}4)}R_4$					d_{6653}	d_{28890}	d_{105554}	dl_4R_4

(Day *et al.*, 1998)

Key: $\downarrow$ indicates the difference between successor and predecessor.

Let

$$dl_7R_1 = d[U_x - U_{319450}]$$

$$dl_6R_2 = dl_7R_1 - dl_6R_1 = d[(U_x - U_{319450}) - dl_6R_1]$$

$$dl_5R_3 = dl_6R_2 - dl_5R_2 = d[((U_x - U_{319450}) - dl_6R_1) - dl_5R_2]$$

$$dl_4R_4 = dl_5R_3 - dl_4R_3 = d[(((U_x - U_{319450}) - dl_6R_1) - dl_5R_2) - dl_4R_3]$$

From Table 7, considering the differences in $(dl_1R_4, dl_2R_4$ and $dl_3R_4)$ we employ division concept and assume that,

$$\frac{dl_2R_4}{dl_1R_4} = 4.3424$$

$$\frac{dl_3R_4}{d2_1R_4} = 3.6536$$

Hence we assume that

$$\frac{dl_4R_4}{dl_3R_4} = 2.9648$$

Thus,

$$dl_4R_4 = d[(((U_x - U_{319450}) - dl_6R_1) - dl_5R_2) - dl_4R_3] = d_{312946} \quad (3.11)$$

by assumption. We then find its inverse algorithm which generate an approximate value of U_x.

vi. Suppose that we have H_r^n where $n = 12$ and $r = 6$, we let U represent the sum of eigenvalues, dm represent their differences and R represent rows. Then sum of eigenvalues of Hermitian matrix H_6^{12} is as a result of the following;

Table 8: **Algorithm Showing** $H_r^n = H_6^{12}$

	6	7	8	9	10	11	12
6	U_1	U_{28}	U_{462}	U_{5317}	U_{46208}	U_{319450}	U_x
	↓	↓	↓	↓	↓	↓	
$dm_{(1\text{-}6)}R_1$	d_{27}	d_{434}	d_{4855}	d_{40891}	d_{273242}	dm_6R_1	
	↓	↓	↓	↓	↓		
$dm_{(1\text{-}5)}R_2$	d_{407}	d_{4421}	d_{36036}	d_{232351}	dm_5R_2		

(Day *et al.*, 1998)

Key: ↓ indicates the difference between successor and predecessor.

Let

$$dm_6R_1 = d[U_x - U_{319450}]$$

$$dm_5R_2 = dm_6R_1 - dm_5R_1 = d[(U_x - U_{319450}) - dm_5R_1]$$

From Table 8, considering the differences in $(dm_1R_2, dm_2R_2, dm_3R_2$ and $dm_4R_2)$ we use the concept of division and assume that,

$$\frac{dm_2R_2}{dm_1R_2} = 10.8624$$

$$\frac{dm_3R_2}{dm_2R_2} = 8.1510$$

$$\frac{dm_4R_2}{dm_3R_2} = 6.4477$$

Therefore by assumption,

$$\frac{dm_5R_2}{dm_4R_2} = 5.7525$$

Thus,

$$dm_5R_2 = d[(U_x - U_{319450}) - dm_5R_1] = d_{1336599} \tag{3.12}$$

by assumption. Hence we find its inverse algorithm which generate an approximate value of U_x.

Chapter Summary

In this chapter we assess the inequalities of equation (3.2), (3.3), (3.4), (3.5) and (3.6) to build the possible eigenvalue inequalities of the Hermitian matrices. We then solve to find the relationship bounding the eigenvalues of sum of Hermitian matrices. This relationship together with other numerical algorithms is then considered to formulate the structure of eigenvalues of sum Hermitian matrices.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

The purpose of this research is to establish the solution of eigenvalues of sum of Hermitian matrices for $n \leq 12$. Thus our specific objectives were to;

1. Formulate all possible inequalities;

$$\sum_{s=1}^{r} \gamma\mu_{k_s} \leq \sum_{s=1}^{r} \lambda\mu\alpha_{i_s} + \sum_{s=1}^{r} \lambda\mu\beta_{j_s}$$

   for $(i, j, k) \in H_r^n, 1 \leq r < n \leq 12$ where $r = n - 1, n - 3$ and (i, j, k) are ordered triples of integers.

2. Determine the set H_r^n of all possible $\gamma\mu$ for $1 \leq r < n \leq 12$, where $H = A + B$.

In this chapter, our results from the research are presented and discussed in relation to the research problem below;

1. Given $n_i \times n_j$ Hermitian matrices A and B with corresponding eigenvalues $(\lambda\mu\alpha_1, \lambda\mu\alpha_2, ..., \lambda\mu\alpha_n)$ and $(\lambda\mu\beta_1, \lambda\mu\beta_2, ..., \lambda\mu\beta_n)$ and $H \in$ Herm (n) with eigenvalues $(\gamma\mu_1, \gamma\mu_2, ..., \gamma\mu_n)$ where $H = A + B$, we generate possible inequalities of size H_r^n for $1 \leq r < n \leq 12$.

2. Given $n_i \times n_j$ Hermitian matrices A, B and $H = A + B$, we find the set of all possible $\gamma\mu$ of H for $1 \leq r < n \leq 12$.

Results and Discussion

In order to find answer to the above research question, we begin with $n_i \times n_j$ matrice. Suppose that $n = 2$, we have

$$\begin{bmatrix} \lambda\mu\alpha_1 & 0 \\ 0 & \lambda\mu\alpha_2 \end{bmatrix} + \begin{bmatrix} u & w \\ \bar{w} & v \end{bmatrix} = \begin{bmatrix} (\lambda\mu\alpha_1 + u) & w \\ \bar{w} & (\lambda\mu\alpha_2 + v) \end{bmatrix}$$

Then $\lambda\mu\beta = \frac{(u+v)\pm\sqrt{(u-v)^2+|w|^2}}{2}$

Hence $\gamma\mu$ implies $\frac{(\lambda\mu\alpha_1+\lambda\mu\alpha_2)+(u+v)\pm\sqrt{(\lambda\mu\alpha_1-\lambda\mu\alpha_2+u-v)|w|^2}}{2}$

Thus, when $n = 2$, there exist Hermitian matrices $H = A + B$ of triples $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ satifying

$$\left.\begin{aligned} &\gamma\mu_1 \leq \lambda\mu\alpha_1 + \lambda\mu\beta_1 \\ &\gamma\mu_2 \leq \lambda\mu\alpha_1 + \lambda\mu\beta_2 \\ &\gamma\mu_2 \leq \lambda\mu\alpha_2 + \lambda\mu\beta_1 \\ &\gamma\mu_1 + \gamma\mu_2 = \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_2 \end{aligned}\right\} \quad (4.1)$$

(Richmond, 2013).

Horn problem

Let A, B and H be $n_i \times n_j$ Hermitian matrices where $H = A + B$ and we let $\lambda\mu\alpha_i$, $\lambda\mu\beta_i$ and $\gamma\mu_i$ for $i = (1, 2, ..., n)$ be their respective eigenvalues such that

$$\lambda\mu\alpha_1 \geq \lambda\mu\alpha_2 \geq ... \geq \lambda\mu\alpha_n$$

$$\lambda\mu\beta_1 \geq \lambda\mu\beta_2 \geq ... \geq \lambda\mu\beta_n$$

$$\gamma\mu_1 \geq \gamma\mu_2 \geq ... \geq \gamma\mu_n$$

(Altunbulak, 2002).

According to Podkopaeva (2012) we have a simple relation of trace

$$tr(A) + tr(B) = tr(H)$$

. Hence,

$$\sum_{i=1}^{n} \lambda\mu\alpha_i + \sum_{j=1}^{n} \lambda\mu\beta_j = \sum_{k=1}^{n} \gamma\mu_k \tag{4.2}$$

form the relation with equality and for all $r < n$, we obtain a collection of $(I, J, K) \subset [n] := 1, 2, ..., n$ of size r hence,

$$\sum_{i=I} \lambda\mu\alpha_i + \sum_{j=J} \lambda\mu\beta_j \geq \sum_{k=K} \gamma\mu_k \tag{4.3}$$

It follows from equation (4.3) that the maximum eigenvalue of the summand of two Hermitian matrices H can not be greater than their individual maximal eigenvalues.

Lemma 4.1

Let B be an $n_i \times n_j$ Hermitian matrix with eigenvalues $(\lambda\mu\beta_1, \lambda\mu\beta_2, ..., \lambda\mu\beta_n)$ Then we have ;

$$\lambda\mu\beta_1 = \max_{v \cdot v = 1} v \cdot Bv$$

and

$$\lambda\mu\beta_n = \min_{v \cdot v = 1} v \cdot Bv$$

for all v such that $\lambda\mu\beta_1 v \cdot v \geq v \cdot Bv \geq \lambda\mu\beta_n v \cdot v$ (Podkopaeva, 2012).

Proof: Let U be a unitary matrix where $UDU\cdot = B$ and $D = \lambda\mu\beta_1, \lambda\mu\beta_2,$

$\lambda\mu\beta_n$. Then we have;

$$v \cdot Bv = v \cdot (UDU) \cdot (v) = (U \cdot v) \cdot D(U \cdot v) = D(U \cdot v) \cdot (U \cdot v)$$

$$= \sum_{i=1}^{n} \lambda\mu\beta_i |[(U \cdot v) \cdot (U \cdot v)]_i| = \sum_{i=1}^{n} \lambda\mu\beta_i |(U \cdot v)_i|^2$$

Also, having $\lambda\mu\beta_i(v \cdot v) \geq v \cdot Bv \geq \lambda\mu\beta_n(v \cdot v)$ implies

$$\sum_{i=1}^{n} \lambda\mu\beta_i |(U \cdot v)_i|^2 \geq v \cdot (UDU) \cdot v \geq \sum_{i=1}^{n} \lambda\mu\beta_n |(U \cdot v)_i|^2$$

which satisfy the inequalities.

Example 4.1

Given $n_i \times n_j$ matrices with $n = 4$ and $r = 2$ where $(I, J, K) = (1, 3), (2, 3), (2, 4)$ provides the inequality

$$\lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \geq \gamma\mu_2 + \gamma\mu_4$$

Example 4.2

Given $n_i \times n_j$ matrices with $n = 5$ and $r = 3$ where $(I, J, K) = (1, 2, 4), (1, 3, 5), (1, 4, 5)$ gives the eigenvalue inequality

$$\lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_1 + \lambda\mu\beta_3 + \lambda\mu\beta_5 \geq \gamma\mu_1 + \gamma\mu_4 + \gamma\mu_5$$

such that $(I, J, K) = (\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$.

Horn conjecture for the case $r = 4$

Theorem 4.2

Suppose that $(\alpha f, \beta g, \gamma h)$ are ordered set of integers satisfying

$$\left.\begin{aligned} 1 \leq \textstyle\sum_{s=1}^{4} \alpha f_s \leq n \\ 1 \leq \textstyle\sum_{s=1}^{4} \beta g_s \leq n \\ 1 \leq \textstyle\sum_{s=1}^{4} \gamma h_s \leq n \end{aligned}\right\} \tag{4.4}$$

$$\alpha f_1 + \beta g_1 \leq \gamma h_1 + 1 \tag{4.5}$$

$$\left.\begin{aligned} \alpha f_1 + \beta g_2 \\ \alpha f_2 + \beta g_1 \end{aligned}\right\} \leq \gamma h_2 + 1 \tag{4.6}$$

$$\left.\begin{aligned} \alpha f_1 + \beta g_3 \\ \alpha f_2 + \beta g_2 \\ \alpha f_3 + \beta g_1 \end{aligned}\right\} \leq \gamma h_3 + 1 \tag{4.7}$$

$$\left.\begin{aligned} \alpha f_1 + \beta g_4 \\ \alpha f_2 + \beta g_3 \\ \alpha f_3 + \beta g_2 \\ \alpha f_4 + \beta g_1 \end{aligned}\right\} \leq \gamma h_4 + 1 \tag{4.8}$$

$$\alpha f_1 + \alpha f_2 + \beta g_1 + \beta g_2 \leq \gamma h_1 + \gamma h_2 + 3 \tag{4.9}$$

$$\left.\begin{array}{l}\alpha f_1 + \alpha f_2 + \beta g_1 + \beta g_3 \\ \alpha f_1 + \alpha f_3 + \beta g_1 + \beta g_2\end{array}\right\} \leq \gamma h_1 + \gamma h_3 + 3 \tag{4.10}$$

$$\left.\begin{array}{l}\alpha f_1 + \alpha f_2 + \beta g_2 + \beta g_3 \\ \alpha f_2 + \alpha f_3 + \beta g_1 + \beta g_2 \\ \alpha f_1 + \alpha f_3 + \beta g_1 + \beta g_3\end{array}\right\} \leq \gamma h_2 + \gamma h_3 + 3 \tag{4.11}$$

$$\left.\begin{array}{l}\alpha f_1 + \alpha f_2 + \beta g_3 + \beta g_4 \\ \alpha f_2 + \alpha f_3 + \beta g_2 + \beta g_3 \\ \alpha f_3 + \alpha f_4 + \beta g_1 + \beta g_2 \\ \alpha f_1 + \alpha f_4 + \beta g_1 + \beta g_4\end{array}\right\} \leq \gamma h_2 + \gamma h_4 + 4 \tag{4.12}$$

$$\alpha f_1 + \alpha f_2 + \alpha f_3 + \beta g_1 + \beta g_2 + \beta g_3 \leq \gamma h_1 + \gamma h_2 + \gamma h_3 + 6 \tag{4.13}$$

$$\left.\begin{array}{l}\alpha f_1 + \alpha f_2 + \alpha f_3 + \beta g_2 + \beta g_3 + \beta g_4 \\ \alpha f_1 + \alpha f_2 + \alpha f_4 + \beta g_1 + \beta g_3 + \beta g_4 \\ \alpha f_1 + \alpha f_3 + \alpha f_4 + \beta g_1 + \beta g_2 + \beta g_4 \\ \alpha f_2 + \alpha f_3 + \alpha f_4 + \beta g_1 + \beta g_2 + \beta g_3\end{array}\right\} \leq \gamma h_2 + \gamma h_3 + \gamma h_4 + 6 \tag{4.14}$$

$$\sum_{s=1}^{4} \alpha f_s + \sum_{s=1}^{4} \beta g_s \leq \sum_{s=1}^{4} \gamma h_s + 10 \tag{4.15}$$

then $(\alpha f, \beta g, \gamma h) \in H_4^n$.

Proof: We move in line with the proof of ordered triples of integers (i, j, k), see Horn (1962). Asumming $n = \gamma h_4$ and by induction on n, suppose that $n = 4$ the theorem follows from equation (1.1). If the theorem holds for all $n < N$ and $N > 4$, when $n = 4$,

$$\left.\begin{aligned} &\alpha f_q = \beta g_q = \gamma h_q = 1 \\ &\alpha f_{q+1} = \beta g_{q+1} = \gamma h_{q+1} = 2 \\ &\alpha f_{q+2} = \beta g_{q+2} = \gamma h_{q+2} = 3 \\ &\alpha f_{q+3} = \beta g_{q+3} = \gamma h_{q+3} = 4 \\ &q = 1 \end{aligned}\right\} \tag{4.16}$$

We assume

$$\left.\begin{aligned} &\alpha f_1 = \beta g_1 = \gamma h_1 = 1 \\ &\alpha f_2 = \beta g_2 = 2 \end{aligned}\right\} \tag{4.17}$$

Thus,

$$\left.\begin{aligned} &\alpha f_1, \alpha f_2 - 1, \alpha f_3 - 1, \alpha f_4 - 1 \\ &\beta g_1, \beta g_2 - 1, \beta g_3 - 1, \beta g_4 - 1 \\ &\gamma h_1, \gamma h_2 - 1, \gamma h_3 - 1, \gamma h_4 - 1 \end{aligned}\right\} \tag{4.18}$$

guarantee equation (4.4)-(4.14) and if

$$\alpha f_2 + \beta g_4 \geq \gamma h_4 + 2 \tag{4.19}$$

then by induction hypothesis and Horns theorem (5), with $x = 2, y = 4$ and $z = 1$ such that $(x, y, z) = (u, v, w)$ respectively, result the theorem. Also equation (4.19) condition guarantee equation (4.18). Example; let $\gamma h_1 - 1 \geq 1$, since $(\gamma h_1 = 1, ..., \gamma h_n = n)$, then by equation (4.12) and (4.17), $\alpha f_2 + \beta g_4 \leq \gamma h_4 + 2$ which does not move in line with equation (4.19). Again, the two middle inequalities of equation (4.8) in combination with (4.19) and $\alpha f_4 = \beta g_4 = \gamma h_4$ which result from equation (4.8) guaranree $\alpha f_3 < \alpha f_4 - 1$ and $\beta g_3 < \beta g_4 - 1$. Hence by assumption;

$$\left.\begin{aligned} \alpha f_2 + \beta g_4 \\ \alpha f_4 + \beta g_2 \end{aligned}\right\} \leq \gamma h_4 + 2 \tag{4.20}$$

Formulating the set of possible eigenvalue inequalities

We move to the problem of forming and generating the set of all possible eigenvalue inequalities defined in the overview. Suppose H is a set of points $\gamma\mu$ defined by $\gamma\mu_1 \geq \gamma\mu_2 \geq ... \geq \gamma\mu_n$, and $\gamma\mu_{k_1} + \gamma\mu_{k_2} + ... + \gamma\mu_{k_r} \leq \lambda\mu\alpha_{i_1} + \lambda\mu\alpha_{i_2} + ... + \lambda\mu\alpha_{i_r} + \lambda\mu\beta_{j_1} + \lambda\mu\beta_{j_2} + ... + \lambda\mu\beta_{j_r}$ where $(i, j, k) \in H_r^n$; $1 \leq r < n$ such that $r = n - 1, n - 3$, and $n \leq 12$.

Theorem 4.3 *Given the eigenvalues $(\gamma\mu, \lambda\mu\alpha, \lambda\mu\beta)$ of Hermitian matrices (H, A, B) respectively, the eigenvalue inequalities of matrix dimension $n \leq 12$ has a solution provided that $(\gamma\mu, \lambda\mu\alpha, \lambda\mu\beta) \in H_r^n$ and $r = n - 1$ arbitrary parameters are priscribed.*

Proof:

1. Let A, B and $H = A + B$ be $n \times n$ Hermitian matrices where $n = 5$ with eigenvalues $\lambda\mu\alpha, \lambda\mu\beta$ and $\gamma\mu$ respectively. The set H_r^n for $1 \leq r < n \leq 5$ such that $r = n - 1$ yield the following eigenvalue inequalities below:

$$
\left.\begin{aligned}
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_4 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\beta_1+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5 \le \lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_4+\lambda\mu\beta_5
\end{aligned}\right\} \quad (4.21)
$$

$$\left.\begin{aligned}
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_4+\gamma\mu_5\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_5\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5\leq\lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\alpha_4+\lambda\mu\alpha_5\\
&+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_5
\end{aligned}\right\}\quad(4.22)$$

From equation (4.21) and (4.22) the set H_4^5 for all possible eigenvalue inequalities has 15 elements which involve the sum of the three eigenvalues $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ for Hermitian matrices $A, B, H = A + B$ respectively such that $A, B \in$ Herm(5). The set H_1^5 and H_4^5 provide 15 eigenvalue inequalities each and there exist 56 eigenvalue inequalities each for both H_2^5 and H_3^5 while H_1^5 provides one equality. Thus, the set H_r^5 provides a total eigenvalue inequalities of $\sum_r^4 |H_{r=1}^5| = 142$ which holds for all $A, B, H = A + B \in$ Herm(5).

2. Suppose A, B and $H = A + B$ are Herm(n) matrices where $n = 6$ with eigenvalues $\lambda\mu\alpha, \lambda\mu\beta$ and $\gamma\mu$ respectively. The set H_r^n for $1 \leq r < n \leq 6$ such that $r = n - 1$ gives the following eigenvalue inequalities;

$$\left.\begin{aligned}
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_6\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_5+\lambda\mu\beta_6\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_4+\gamma_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_4+\lambda\mu\beta_5+\lambda\mu\beta_6\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5+\lambda\mu\beta_6\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5+\lambda\mu\beta_6\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_4+\lambda\mu\alpha_6+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_3+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_5+\lambda\mu\alpha_6+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_2+\gamma\mu_4+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_4\\
&+\lambda\mu\alpha_5+\lambda\mu\alpha_6+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_1+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\alpha_4\\
&+\lambda\mu\alpha_5+\lambda\mu\alpha_6+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_3+\lambda\mu\beta_4+\lambda\mu\beta_5\\
&\gamma\mu_2+\gamma\mu_3+\gamma\mu_4+\gamma\mu_5+\gamma\mu_6\leq\lambda\mu\alpha_1+\lambda\mu\alpha_2+\lambda\mu\alpha_3\\
&+\lambda\mu\alpha_5+\lambda\mu\alpha_6+\lambda\mu\beta_1+\lambda\mu\beta_2+\lambda\mu\beta_4+\lambda\mu\beta_5+\lambda\mu\beta_6
\end{aligned}\right\}\quad(4.23)$$

$$
\left.
\begin{aligned}
&\gamma\mu_2 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_2 + \lambda\mu\alpha\mu_3 + \lambda\mu\alpha_4 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_5 \\
&\gamma\mu_1 + \gamma\mu_2 + \gamma\mu_3 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_4 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_6 \\
&\gamma\mu_1 + \gamma\mu_2 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_4 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_5 + \lambda\mu\beta_6 \\
&\gamma\mu_1 + \gamma\mu_2 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_6 \\
&\gamma\mu_1 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_4 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_4 + \lambda\mu\beta_5 + \lambda\mu\beta_6 \\
&\gamma\mu_1 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_5 + \lambda\mu\beta_6 \\
&\gamma\mu_1 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_4 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_6 \\
&\gamma\mu_2 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_3 \\
&+\lambda\mu\alpha_4 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_5 + \lambda\mu\beta_6 \\
&\gamma\mu_2 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\alpha_4 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_5 + \lambda\mu\beta_6 \\
&\gamma\mu_2 + \gamma\mu_3 + \gamma\mu_4 + \gamma\mu_5 + \gamma\mu_6 \leq \lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\alpha_4 \\
&+\lambda\mu\alpha_5 + \lambda\mu\alpha_6 + \lambda\mu\beta_1 + \lambda\mu\beta_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 + \lambda\mu\beta_6
\end{aligned}
\right\} \quad (4.24)
$$

From equation (4.23) and (4.24) the set H_5^6 for all possible eigenvalue inequalities has 21 elements which involve the sum of the eigenvalues $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ for Hermitian matrices $A, B, H = A + B$ respectively such that $A, B \in$ Herm(6). The set H_1^6 and H_5^6 provide 21 eigenvalue inequalities each. H_2^6 and H_4^6 give 126 eigenvalue inequalities each. H_3^6 provides 228 eigenvalue inequalities while H_6^6 provides one equality. Hence the set H_r^6 provides a total eigenvalue inequalities of $\sum_{r=1}^{5}|H_r^6| = 522$ which holds for all $A, B, H = A + B \in$ Herm(6). Hence Theorem 4.3 has a solution.

Theorem 4.4 *Given the eigenvalues* $(\gamma\mu, \lambda\mu\alpha, \lambda\mu\beta)$ *of Hermitian matrices* (H, A, B) *respectively, the eigenvalue inequalities of matrix dimension* $n \leq 12$ *has a solution provided that* $(\gamma\mu, \lambda\mu\alpha, \lambda\mu\beta) \in H_r^n$ *and* $r = n - 3$ *arbitrary parameters are priscribed.*

Proof:

1. Let A, B and $H = A + B$ be $n \times n$ Hermitian matrices where $n = 4$ with eigenvalues $\lambda\mu\alpha, \lambda\mu\beta$ and $\gamma\mu$ respectively. The set H_r^n for $1 \leq r < n \leq 4$ such that $r = n - 3$ yield the following eigenvalue inequalities;

$$\left.\begin{aligned}
\gamma\mu_1 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_1 \\
\gamma\mu_2 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_2 \\
\gamma\mu_3 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_3 \\
\gamma\mu_4 &\leq \lambda\mu\alpha_1 + \lambda\mu\beta_4 \\
\gamma\mu_2 &\leq \lambda\mu\alpha_2 + \lambda\mu\beta_1 \\
\gamma\mu_3 &\leq \lambda\mu\alpha_2 + \lambda\mu\beta_2 \\
\gamma\mu_4 &\leq \lambda\mu\alpha_2 + \lambda\mu\beta_3 \\
\gamma\mu_3 &\leq \lambda\mu\alpha_3 + \lambda\mu\beta_1 \\
\gamma\mu_4 &\leq \lambda\mu\alpha_3 + \lambda\mu\beta_2 \\
\gamma\mu_4 &\leq \lambda\mu\alpha_4 + \lambda\mu\beta_1
\end{aligned}\right\} \tag{4.25}$$

From equation (4.25) the set H_1^4 for all possible inequalities gives 10 eigenvalue inequalities involving the sum of eigenvalues $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$

for Hermitian matrices $A, B, H = A + B$ respectively such that $A, B \in$ Herm(4). The set H_1^4 and H_3^4 give 10 eigenvalue inequalities each. H_2^4 gives 21 eigenvalue inequalities while H_4^4 gives one equality. Hence the set H_r^4 provides a total eigenvalue inequalities of $\sum_{r=1}^{3} |H_r^4| = 41$ which holds for all $A, B, H = A + B \in$ Herm(4).

2. Suppose A, B and $H = A + B$ are Herm(n) matrices where $n = 5$ with eigenvalues $\lambda\mu\alpha, \lambda\mu\beta$ and $\gamma\mu$ respectively. The set H_r^n for $1 \leq r < n \leq 5$ such that $r = n - 3$ provides the following eigenvalue inequalities;

$$\left.\begin{aligned}
\gamma\mu_1 + \gamma\mu_2 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_2 \\
\gamma\mu_1 + \gamma\mu_3 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_3 \\
\gamma\mu_1 + \gamma\mu_4 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_1 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_5 \\
\gamma\mu_2 + \gamma\mu_3 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_2 + \gamma\mu_4 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_1 + \lambda\mu\beta_5 \\
\gamma\mu_2 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_2 + \lambda\mu\beta_5 \\
\gamma\mu_3 + \gamma\mu_4 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_3 + \lambda\mu\beta_4 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_3 + \lambda\mu\beta_5 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_5 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_2 + \lambda\mu\beta_4 + \lambda\mu\beta_5 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\beta_3 + \lambda\mu\beta_5 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_5
\end{aligned}\right\} \quad (4.26)$$

$$\left.\begin{aligned}
\gamma\mu_1+\gamma\mu_3 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_1+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_1+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_1+\lambda\mu\beta_4\\
\gamma\mu_2+\gamma\mu_3 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_2+\gamma\mu_4 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_2+\gamma\mu_5 &\le \lambda\mu\alpha_3+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_3+\gamma\mu_4 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_3+\gamma\mu_5 &\le \lambda\mu\alpha_3+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_3+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_3+\lambda\mu\beta_4\\
\gamma\mu_4+\gamma\mu_5 &\le \lambda\mu\alpha_4+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_4+\gamma\mu_5 &\le \lambda\mu\alpha_3+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_3\\
\gamma\mu_1+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_1+\lambda\mu\beta_3\\
\gamma\mu_1+\gamma\mu_5 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_2\\
\gamma\mu_2+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_2+\lambda\mu\beta_3\\
\gamma\mu_2+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_4\\
\gamma\mu_3+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_3+\lambda\mu\beta_2+\lambda\mu\beta_4\\
\gamma\mu_3+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_5\\
\gamma\mu_3+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_5+\lambda\mu\beta_2+\lambda\mu\beta_3\\
\gamma\mu_4+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_4+\lambda\mu\beta_3+\lambda\mu\beta_4\\
\gamma\mu_4+\gamma\mu_5 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_5+\lambda\mu\beta_2+\lambda\mu\beta_4\\
\gamma\mu_1+\gamma\mu_5 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_3+\lambda\mu\beta_1+\lambda\mu\beta_3\\
\gamma\mu_2+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_3\\
\gamma\mu_2+\gamma\mu_5 &\le \lambda\mu\alpha_2+\lambda\mu\alpha_4+\lambda\mu\beta_1+\lambda\mu\beta_3\\
\gamma\mu_3+\gamma\mu_4 &\le \lambda\mu\alpha_1+\lambda\mu\alpha_5+\lambda\mu\beta_1+\lambda\mu\beta_3
\end{aligned}\right\} \tag{4.27}$$

$$\left.\begin{aligned}
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_4 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_5 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_5 + \lambda\mu\beta_1 + \lambda\mu\beta_5 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_5 \\
\gamma\mu_2 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_3 + \lambda\mu\beta_1 + \lambda\mu\beta_5 \\
\gamma\mu_3 + \gamma\mu_4 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_5 + \lambda\mu\beta_1 + \lambda\mu\beta_3 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_1 + \lambda\mu\beta_5 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_3 + \lambda\mu\beta_4 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_4 \\
\gamma\mu_2 + \gamma\mu_5 &\leq \lambda\mu\alpha_1 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_3 + \lambda\mu\alpha_4 + \lambda\mu\beta_1 + \lambda\mu\beta_3 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_3 + \lambda\mu\beta_2 + \lambda\mu\beta_4 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_3 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_5 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_2 + \lambda\mu\alpha_5 + \lambda\mu\beta_2 + \lambda\mu\beta_3 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_3 + \lambda\mu\alpha_4 + \lambda\mu\beta_1 + \lambda\mu\beta_4 \\
\gamma\mu_4 + \gamma\mu_5 &\leq \lambda\mu\alpha_3 + \lambda\mu\alpha_4 + \lambda\mu\beta_2 + \lambda\mu\beta_3
\end{aligned}\right\} \quad (4.28)$$

From equation (4.26), (4.27) and (4.28) the set H_2^5 for all possible inequalities has 56 eigenvalue inequalities which hold for sum of eigenvalues $(\lambda\mu\alpha, \lambda\mu\beta, \gamma\mu)$ for Hermitian matrices $A, B, H = A+B$ respectively where $A, B \in$ Herm(5). Hence Theorem 4.4 has a solution.

Determining the set H_r^n for all possible $\gamma\mu$ for $1 \leq r < n \leq 12$

Theorem 4.5 *For any $1 \leq r < n$ such that $r = 1$, we have*
$H_r^n = \sum_{i=1}^{n-1}[\lambda n_i] + 1.$

Proof: Let $H_r^n = H_1^{12}$ we have;

$$\sum_{i=1}^{11}[\lambda n_i] + 1$$

$$= [\lambda n_1 + \lambda n_2 + ... + \lambda n_{11}] + 1$$

where $\lambda n_1 = 2, \lambda n_2 = 3, ..., \lambda n_{11} = 12$ by equation (3.7)

Thus, $[2 + 3 + ... + 11] + 1 = 78$ which satisfy Theorem 4.5

i. For H_2^{12} we have from equation (3.8)

$$di_7R_4 = d[(((U_x - U_{2002}) - di_9R_1) - di_8R_2) - di_7R_3] = d_{11}$$

$$di_8R_3 = di_7R_4 + di_7R_3 = d[(((U_x - U_{2002}) - di_9R_1) - di_8R_2)] = d_{66}$$

$$di_9R_2 = di_8R_3 + di_8R_2 = d[((U_x - U_{2002}) - di_9R_1)] = d_{286}$$

$$di_{10}R_1 = di_9R_2 + di_9R_1 = d[U_x - U_{2002}] = d_{1001}$$

$$U_x = di_{10}R_1 + U_{2002} = U_{3003}$$

ii. For H_3^{12} we have from equation (3.9)

$$dj_4R_6 = d[(((((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3) - dj_5R_4) - dj_4R_5]$$

$$= d_{186}$$

$$dj_5R_5 = dj_4R_6 + dj_4R_5 = d[((((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3)$$

$$-dj_5R_4] = d_{661}$$

$$dj_6R_4 = dj_5R_5 + dj_5R_4 = d[(((U_x - U_{25678}) - dj_8R_1) - dj_7R_2) - dj_6R_3]$$

$$= d_{1952}$$

$$dj_7R_3 = dj_6R_4 + dj_6R_3 = d[((U_x - U_{25678}) - dj_8R_1) - dj_7R_2] = d_{5041}$$

$$dj_8R_2 = dj_7R_3 + dj_7R_2 = d[(U_x - U_{25678}) - dj_8R_1] = d_{11756}$$

$$dj_9R_1 = dj_8R_2 + dj_8R_1 = d[U_x - U_{25678}] = d_{661}$$

$$U_x = dj_9R_1 + U_{25678} = U_{50972}$$

iii. For H_4^{12} we have from equation (3.10)

$$dk_5R_4 = d[(((U_x - U_{138519}) - dk_7R_1) - dk_6R_2) - dk_5R_3] = d_{49656}$$

$$dk_6R_3 = dk_5R_4 + dk_5R_3 = d[((U_x - U_{138519}) - dk_7R_1) - dk_6R_2] = d_{87147}$$

$$dk_7R_2 = dk_6R_3 + dk_6R_2 = d[(U_x - U_{138519}) - dk_7R_1] = d_{146954}$$

$$dk_8R_1 = dk_7R_2 + dk_7R_1 = d(U_x - U_{138519}) = d_{239265}$$

$$U_x = dk_8R_1 + U_{138519} = U_{377784}$$

iv. For H_5^{12} we have from equation (3.11)

$$dl_4R_4 = d[(((U_x - U_{319450}) - dl_6R_1) - dl_5R_2) - dl_4R_3] = d_{312946}$$

$$dl_5R_3 = dl_4R_4 + dl_4R_3 = d[((U_x - U_{319450}) - dl_6R_1) - dl_5R_2] = d_{455469}$$

$$dl_6R_2 = dl_5R_3 + dl_5R_2 = d[(U_x - U_{319450}) - dl_6R_1] = d_{644677}$$

$$dl_7R_1 = dl_6R_2 + dl_6R_1 = d[U_x - U_{319450}] = d_{892154}$$

$$U_x = dl_7R_1 + U_{319450} = U_{1211604}$$

v. For H_6^{12} we have from equation (3.12)

$$dm_5R_2 = d[(U_x - U_{319450}) - dm_5R_1] = d_{1336599}$$

$$dm_6R_1 = dm_5R_2 + dm_5R_1 = d[U_x - U_{319450}] = d_{1609841}$$

$$U_x = dm_6R_1 + U_{319450} = U_{1929291}$$

Table 9 presents the size H_r^n for $1 \leq r < n \leq 12$. Thus the set H_r^{12} produces an approximated total eigenvalue inequalities of $\sum_{r=1}^{11} |H_r^{11}| = 5216173$ which holds for all Hermitian matrices A, B and $H = A + B \in$ Herm(12). It must be noted that the last natural number for each Herm(n) matrix in Table 9 is the trace equality.

Table 9: **Sum of Eigenvalues of H_r^n for $n \leq 12$ such that $H_r^n = H_{n-r}^n$**

H_r^n						n					
r	2	3	4	5	6	7	8	9	10	11	12
1	3	6	10	15	21	28	36	45	55	66	78
2	1	6	21	56	126	252	462	792	1287	2002	3003
3	0	1	10	56	228	751	2120	5317	12140	25678	50972*
4	0	0	1	15	126	751	3516	30704	46208	138519	377784*
5	0	0	0	1	21	252	2120	13704	71973	319450	1211604*
6	0	0	0	0	1	28	462	5317	46208	319450	1929291*
7	0	0	0	0	0	1	36	792	12140	138519	1211604
8	0	0	0	0	0	0	1	45	1287	25678	377784
9	0	0	0	0	0	0	0	1	55	2002	50972
10	0	0	0	0	0	0	0	0	1	66	3003
11	0	0	0	0	0	0	0	0	0	1	78

(Taylor, 2015).

Note: * represent an approximate value.

Chapter Summary

This chapter established a solution of eigenvalue inequalities of sum of Hermitian matrices for $n \leq 12$ given the parameters $r = n - 1$, $r = n - 3$. Using a number of algorithms, exact values and some approximated values were obtained for all the possible set of eigenvalue sum $H = A + B$ for $n \leq 12$.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter provides a brief summary, conclusions and recommendations of the research. The purpose of this research was to establish the solution of eigenvalues of sum of Hermitian matrices for $n \leq 12$ using a number of numerical algorithms.

Summary

The main aim of the study was to construct eigenvalues of sum of Hermitian matrices for the set $\gamma\mu$ of $H = A + B$ for $1 \leq r < n \leq 12$. Firstly, some already existing literatures in relation to eigenvalues of sum of Hermitian matrices were reviewed. Relating to the literatures by Taylor (2015), the study provided a number of numerical algorithm to include eigenvalue inequalities for $r = (n-1, n-3)$, providing numerical examples of eigenvalues of sum of Hermitian matrices for $n \leq 12$.

Conclusions

Key findings

1. The research modified numerical algorithm to formulate eigenvalue inequalities for $n_i \times n_j$ Hermitian matrices with parameters $(r = n - 1)$.

2. The research modified numerical algorithm to formulate eigenvalue inequalities for $n_i \times n_j$ Hermitian matrices with parameters $(r = n - 3)$.

3. Lastly, the research generated the solution showing real values for some H_r^{12} and approximated values for the other H_r^{12} of eigenvalues of sum of

Hermitian matrices of size H_r^n for $1 \leq r < n \leq 12$, given that $H_r^n = H_{n-r}^n$.

Recommendations

In relation to the reserch findings, the following are recommended for attention.

1. The use of mathematical softwares and programs should be applied in solving eigenvalues of sum of Hermitian matrices to prevent difficulties in obtaining solutions.

2. Solutions of approximated values of eigenvalues of sum of Hermitian matrices should be reconsidered before it is applied in the field of physics and most specifically quantum mechanics.

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