UNIVERSITY OF CAPE COAST

MATHEMATICAL ANALYSIS OF MALARIA TRANSMISSION MODEL WITH INFECTIVE IMMIGRANTS

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature ..... Date .....

Name: Fred Ameko

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature ..... Date .....

Name: Dr. Samuel Mindakifoe Naandam

ABSTRACT

In this thesis, we developed an *SEIR - SEI* model of malaria transmission with the inclusion of susceptible, exposed and infected immigrants. Analysis of the model were carried out to find the equilibrium points and their stabilities. We have discovered that our model has no disease-free and hence no basic reproduction number $\mathcal{R}_0$ due to the influx of exposed and infected immigrants. However, when the proportions of exposed and infected immigrants approaches zero, disease-free status will be attained whenever $\mathcal{R}_0 < 1$. The unique endemic equilibrium point for which there are exposed and infected immigrants is both locally and globally stable. Numerical simulations were performed to know the effect of exposed and infected immigrants and the results from our simulations showed that exposed and infected immigrants entering the population rendered the basic reproduction number $\mathcal{R}_0$ irrelevant and can not be used to determine the extinction and the prevalence of malaria. Sensitivity analysis was carried out on the parameters that the basic reproduction number $\mathcal{R}_0$ depend on. The result from the sensitivity analysis revealed that the most sensitive parameter is the mosquito biting rate v. We recommended that immigrants should be screened at our borders and be sure of malaria free before allowing them to enter the population to ensure the health and well-being of everyone in the community.

KEY WORDS

Basic reproduction number

Disease-free

Exposed immigrants

Infected immigrants

Simulation

Sensitivity analysis

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DEDICATION

To my mother, Patience Kuapah and my father, Samuel Xevi

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LIST OF ABBREVIATIONS

DFE	Disease Free Equilibrium
EE_1	Endemic Equilibrium Point when there are no Exposed and Infected Immigrants
EE_2	Endemic Equilibrium when there are Exposed and Infected Immigrants
SI	Susceptible, Infected
CDC	Center for Disease Control and Prevention
WHO	World Health Organization
ITNs	Insecticide-Treated Nets
SIS	Susceptible, Infected, Susceptible
SEI	Susceptible, Exposed, Infected
SEIR	Susceptible, Exposed, Infected, Recovery
IPTi	Intermittent Preventive Treatment of infants
SIR	Susceptible Infected Recovered
IPTp	Intermittent Preventive Treatment of pregnant women
MDa	Mass Drug administration
ACT	Artemisinin-Based Combination Therapy
ODEs	Ordinary Diferential Equations
DDT	Dichlorodiphenyltrichloroethane

CHAPTER ONE

INTRODUCTION

Mathematical modelling, an essential research tool has been employed over the years to study the transmission dynamics of infectious diseases. Having a comprehensive insight of how these infectious diseases are transmitted could aid in developing the appropriate tools to mitigate the transmission of these infectious diseases.

Background to the Study

Malaria is perceived as both epidemic and endemic disease. History has it that there have been controversies among researchers concerning the cause of malaria. There were two conjectures with respect to malaria transmission that is bad air and insect vectors (Hempelmann & Krafts, 2013). Sir Patrick Manson discovered in 1878 that a parasite that causes human infection is capable of infecting mosquito (CDC, 2015). Malaria parasite was discovered on 20th October 1880 by Dr Alphonse Laveran at Military hospital in the Constantine, Algeria (Garnham, 1988). In 1895, Sir Ronald Ross embarked on a journey in pursuit of proving the conjectures of Dr. Alphonse Laveran and his contemporary Sir Patrick Manson that mosquitoes were responsible for the spread of malaria. On 20th August 1897 in Secunderbad, Sir Ronald Ross also found the malaria parasite after he dissected the stomach tissues of an anopheles mosquito that fed four days previously on malaria patient and he proceeded to prove the role of anopheles mosquitoes in the transmission of malaria parasites in humans (CDC, 2015).

Malaria, a lethal disease caused by the parasite of the plasmodium family which is rife in Sub-Sahara Africa and seen as a penury-induced disease (WHO, 2019). According to WHO (2021), an estimated 241 million cases of malaria were recorded globally in the year 2020 out of which Africa recorded

228 million cases against 213 million cases of malaria in Africa in the year 2019. Comparatively, there was a surge in malaria cases in Africa from 2019 to 2020, which was attributed to the arrival of COVID-19 pandemic. The death toll of malaria in Africa increased from 534000 to 602000 in the year 2019 and 2020 respectively (WHO, 2021). In 2021 world malaria report, it was estimated that 6.7 and 5.9 million cases of malaria were presumed and confirmed in 2019 and 2020 respctively. There were 336 deaths in 2019 and 308 deaths in 2020 recorded in Ghana.

Malaria parasite is generally transmitted when the malaria parasite enter the bloodstream of a susceptible human, after they (susceptible human) are been bitten by female anopheles mosquito that is infected. Suscpetible mosquitoes become infected when they bite an infected human. Since malaria parasite is found in red blood cells of an infected person, malaria can be transmitted from human to human via blood transfusion, organ transplant or sharing of needles or syringes contaminated with blood (CDC, 2022). According to Otieno (2016), pregnant women that have malaria may also transmit the malaria parasite vertically to her unborn child before or during birth (congenital malaria). There are five species of malaria parasite that infect human and cause illness: Plasmodium falciparum, Plasmodium malariae, Plasmodium vivax, Plasmodium ovale and Plasmodium knowlesi. Plasmodium falciparum malaria is a life-threatening human parasite which accounted for 80% of all recorded malaria cases globally and 90% of death is rife in the tropical areas of Africa and South East Asia (Mia, Begum, Er, Abiden & Pereira, 2011). The first symptoms of malaria are fever, headache and chills usually appeared 10-15 days after infectd female mosquito deposit the parasite into human and may be clement and hard to recognize as malaria. If left untreated, plasmodium falciparum malaria can progress to severe illness which can lead to death within 24 hours.

Controlling mosquito population plays a crucial role in decreasing human-female mosquito interaction which will intend decrease the spread of malaria.

According to WHO (2021), insecticide-treated nets (ITNs) and indoor residual spraying (IRS) are important tools to be employed in reducing mosquito population. Adulticide and larvicide can be applied to reduce the size of female mosquito population (Nordin, Ahmad & Ahmad, 2015). Effort in planetary malaria control is impeded by anopheles mosquito resistance to insecticides. In 2021 world malaria report, 78 countries reported mosquito's resistance to at least one out of four commonly-used insecticide classes in the period $2010 - 2019$ whiles 29 countries reported mosquito's resistance to all main insecticide classes (WHO, 2021).

Preventive chemotherapy can be used to thwart malaria infections and their implications. It consists of chemoprophylaxis, intermittent preventive treatment of infants (IPTi) and pregnant women (IPTp), seasonal malaria chemoprevention (SMC) and mass drug administration (MDA). These safe and economical strategies are geared towards malaria control activities, including vector control measures, prompt diagnosis of suspected malaria, and treatment of confirmed cases with antimalarial medicines (WHO, 2021).

Vaccination in an effective way of preventing the transmission of malaria. World Health Organization target for malaria vaccine is at least 75% efficacy. In October 2021, global advisory bodies for immunization convoked to review the RTS,S/AS01 malaria vaccine among children living in regions with moderate to high plasmodium falciparum malaria transmission. This vaccine was piloted in three countries in Sub-Sahara Africa that is Ghana, Kenya, and Malawi of which more than six hundred and fifty thousand children benefited in the exercise. The vaccine has been observed to decrease malaria drastically, and deadly severe malaria among young children so in view of that, World Health Organization recommended wider use of this vaccine (WHO, 2021). Ball, Knock, and O' Neil (as cited in Koutou, Sangare & Traore, 2020) have observed that albeit malaria vaccine could be imperfect, using it together with other control strategies could aid in decreasing the rifeness of malaria monumentally.

Early diagnosis and treatment of malaria does not only mitigate the disease but it also thwarts death and transmission rate drastically. According to WHO (2021), parasite-based diagnostic testing should be used to affirm malaria cases which would enable health care workers to promptly differentiate between malarial and non-malarial fevers for apt treatment. Artemisinin-based combination therapy (ACT) is the best treatment specifically for P. falciparum malaria (WHO, 2021). The main motive of treatment is to guarantee zero plasmodium parasites to prevent a simple case of malaria from advancing to serious illness leading to death.

WHO (2021) underscored that antimalarial drug resistance is an impediment to planetary malaria control efforts in the Greater Mekong subregion. There were report from Africa indicating drug-resistant to malaria. Strategies are being put in place by World Health Organization to ameliorate drug resistance in Africa.

Malaria elimination is the primary objective of health care providers. However, this task is not a walk on the park as the case of malaria increases astronomically each year. In order to allow malaria elimination see the light of the day, protracted measures to prevent re-establishment of transmission and relapse of the disease is required. In 2020, twenty-six countries in the world reported less than 100 cases of malaria. Also, countries like China and El Salvador were certified by World Health Organization in 2021 as malaria-free and European Region has been malaria-free since 2015 which is a confirmation that malaria can be eliminated (WHO, 2021).

Malaria surveillance enable health care providers to figure out which areas are most affected and does not only aids countries to track change in disease patterns but it also assists them to plan effective health interventions and examine the effect of their malaria restrain interventions

Statement of the Problem

Malaria is a lethal infectious disease affecting nearly half of the world's population. The WHO African Region accounted-for 95% of malaria cases and 96% of malaria deaths in 2020. Children below five years accounted-for 80% of all the malaria deaths in the WHO African Region (WHO, 2021).

Regrettably, Ghana is among the first 10 highest malaria burdened countries in the world (WHO, 2021). In 2021, a total of 5.7 million cases of malaria were confirmed in Ghana, a modest increase of 2020 estimated number of 5.1 million cases. The number of admission due to malaria were increased from 308,358 in 2020 to 391,052 in 2021, albeit, there is a drop in inpatients deaths from 312 to 275 in 2020 and 2021 respectively (Annoh, 2022). These figures should make us restless, especially because malaria is a preventable and treatable disease. Ghana is doing everything possible to join the queue of malaria-free countries, however, our efforts failed to see the light of the day.

Due to comparatively low level of border control in Ghana which makes it virtually possible for immigrants to infiltrate as a result of the porous nature of our borders, it is therefore extremely important that we conduct more research in order to fully understand the impacts of infective immigrants in the transmission of malaria using mathematical modelling as a tool.

Purpose of the Study

The purpose of this thesis is to formulate a deterministic epidemic model of malaria transmission where influx of infective immigrants into the human population is allowed.

Research Objectives

General Objectives

The overall objective of this thesis is to formulate an $SEIR-SEI$ model for malaria transmission with infective immigrants to understand the role play by infective immigrants in the transmission of malaria.

Specific Objectives

This thesis has the following specific objectives:

- To formulate a mathematical model of malaria with the inclusion of infective immigrants into the human population.
- To determine the equilibrium states of the model.
- To determine the basic reproduction number, $\Re_0$.
- To determine both local and global stability of the model .
- To perform sensitivity analysis to understand the parameters that influence the model dynamics .
- To perform numerical simulations to know the full extent of the effect of infective immigrants on malaria transmission .

Significance of the Study

- An essential research tool that can assist us to study and understand the transmission dynamics of malaria is deterministic mathematical modelling.
- There are several mathematical models on malaria transmission, nevertheless, much work has not been done extensively taking into account the effect of infective immigrants on the transmission dynamics of malaria.

- Therefore, it is imperative that we come with a mathematical model that allow influx of infective immigrants, in order to inform our policy makers on the best strategies that will effectively combat the spread of malaria.

Delimitation

This thesis is limited to finding out the crucial role played by exposed and infected immigrants in the transmission dynamics of malaria

Limitation

This thesis has a limitation of correct estimation of parameter values because we depend primarily on values from literature and some assumed values for our numerical simulations.

Definition of Terms

In this section, we state some fundamental defintions needed to comprehend the model.

Definition 1.1

A differential equation of the form

$$\frac{dy}{dt} = f(y), \tag{1.1}$$

where f does not depend explicitly on t is called an autonomous differential equation: otherwise it is nonautonomous

Definition 1.2

A system of first-order ordinary differential equations of the form

$$\frac{dx(t)}{dt} = f(t, x(t)), \tag{1.2}$$

is independent of t or $f(t, x(t)) = f(x(t))$ for $x \in \mathbb{R}^n$, $f \in \mathbb{R}^n$, $t \in \mathbb{R}$ is called an autonomous system.

Definition 1.3

In differential equation (1.2), if a point x^c is such that

$$f(x^c) = 0, \qquad x^c \in \mathbb{R}^n \tag{1.3}$$

then x^c is called an equilibrium point (critical point, fixed point , steady-state)

Definition 1.4

An equilibrium point x^c is said to be locally stable if for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\|x^0 - x^c\| < \delta \Rightarrow \|x(t) - x^c\|, \qquad \forall t > 0 \tag{1.4}$$

Instinctively, an equilibrium point $x^c = (x_1^c, x_2^c, ..., x_n^c) \in \mathbb{R}^n$, of the autonomous system (1.2) is called stable, if the initial point x^0 is close to x^c, then the trajectory x(t) will remain close to x^c for future time for all $t \geq 0$

Definition 1.5

An equilibrium point x^c is locally asymptotically stable it it is stable and in addition there exists $r(t_0) \geq 0$ such that

$$\|x^0 - x^c\| \geq r(t_0) \Rightarrow \lim_{t \to \infty} \|x(t) - x^c\| = 0 \tag{1.5}$$

Organization of the Study

This thesis is divided into five chapters. Chapter one of this work describes the background to the study, statement of the problem, research objectives and

the significance of the problem. Chapter two deals with the review of some related literature on mathematical models of malaria with the inclusion of infected immigrants. In chapter three, we formulate the model and investigate its stabilities. Chapter four we carry out numerical simulations and sensitivity analysis of our model. In chapter five, we present summary, conclusions recommendations.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this section, we re-examine related work on deterministic mathematical models of malaria with infective immigrants. Deterministic mathematical model gives us a clear picture of the transmission dynamics of infectious disease.

Mathematical Models on Infectious Disease

Malaria is a long in the tooth infectious disease that researchers have tried to comprehend its transmission dynamics for many years due to its burden on human population globally. The breakthrough happened when the mosquito involvement in the transmission cycle was uncovered by Grassi and Ross in 1897 (Mandal, Sarkar & Sinha, 2011). Deterministic mathematical model for malaria transmission is attributed to Ross (1911). Ross was the first person to publish a paper on simple mathematical model that provided a clear picture of interactive factors and their role in the eradication of malaria disease. In his work, he used the SIS model for human population and SI model for mosquito population with standard incidence and constant population. According to Ross (1911), decreasing malaria transmission does not require extinction of mosquito population, in lieu, reducing mosquito population below a certain threshold is enough to control malaria transmission. Scores of work on malaria transmission was investigated after Ross (1911). For instance, Macdonald et al. (1957) reiterates the significance of mathematical epidemiology and extended Ross (1911) model by adding the exposed compartment to the mosquito population. The human population is the same as in the Ross model thus SIS model while the mosquito population is modified to SEI model. This model gave a clear understanding of malaria cycle and underscored that the survival of adult mosquito

is the weakest link in the cycle. This led to massive malaria eradication campaign by the World Health Organization (WHO) by concentrating on the use of DDT as an insecticide to eliminate mosquito in Africa (Macdonald et al., 1957; Mandal, Sarkar & Sinha, 2011).

Mojeeb, Osman and Isaac (2017) did a work on $SEIR$ model followed by $SEIR-SEI$ model of malaria transmission. They assumed that permanent immunity is conferred on recovered individuals. Their analysis indicated that both models are locally asymptotically stable whenever the associated basic reproduction numbers are less than unity and unstable, when they are greater than unity. They went further by saying that in order to control malaria, the rate of infection between humans and mosquitoes must be decreased, also reducing the interaction between mosquitoes and humans and the use of malaria drugs, insecticides, and treated bed nets would decrease mosquito population which will in turn keep human population stable.

Several models on malaria were developed after that. For example, Newman, Parise, Barber and Steketee (2004) reported that nearly 1500 malaria cases occur each year in the United States, of which 60% are among United States travellers (imported malaria cases). This phenomenon is as a result of immigrants from malaria endemic regions act as a source of malaria when they move to malaria free zone that has uninfected mosquitoes.

Also, Tumwiine, Luboobi and Mugisha (2005) did a work on a host-vector mathematical model for malaria with infective immigrants. Analysis of their model indicated that the basic reproduction number does not exist because there is no disease-free equilibrium point due to the influx of infective immigrants.

Mandal and Sarkar (2011) asserted that the reason why various strategies to eradicate infectious disease fail to see the light of the day is the disregard for the mobility pattern of the host (human), and this confirmed the recent surge in malaria incidence not only in endemic areas but also in areas where malaria had been eliminated.

Also, Budhwar and Daniel (2017) analysed the stability of $SEIR - SI$ model for malaria with infective immigrants. They found out in their analysis that both the disease-free and endemic equilibria are stable locally and globally. The apparent drawback of their work is that, they have not done any numerical simulations and sensitivity analysis to comprehend the full extent of the impact of infective immigrants on the spread of malaria in the population.

Furthermore, Sigdel and McCluskey (2014) studied global stability of an SEI model of infectious disease with immigration into all the three compartments. The result from their study indicated that there was no basic reproduction number because disease-free steady state does not exist due to the influx of infective humans. Further analysis of their model reveals the existence of endemic steady state which was globally asymptotically stable. They concluded that elimination of disease becomes virtually unattainable if there is an influx of infected immigrants in the population.

Wedajo, Bole and Koya (2018), published a paper on $SIR - SI$ mathematical model of malaria with the inclusion of infected immigrants. Their numerical analysis indicated that preventing the influx of infected immigrants have a strong impact on the malaria disease control.

Chapter Summary

In this chapter, we reviewed some related works on mathematical models of malaria taken into account the inclusion of infected immigrants into the human population. Many of the research on the mathematical models of malaria do not allow the influx of exposed immigrants and even those that include exposed immigrants, they have not done any numerical simulations to comprehend the effect of those immigrants in the malaria transmission. In this study, we developed a $SEIR - SEI$ model of malaria with the inclusion of exposed and infected immigrants into the human population.

CHAPTER THREE

RESEARCH METHODS

Introduction

In this chapter, we formulate a mathematical model for human-female mosquito transmission of malaria using compartmental approach with mass action as incidence rate. We consider four and three compartments in the human and female mosquito population respectively. In the human population, there is an influx of immigrants A where proportions ω, ψ and $(1-\omega-\psi)$ are exposed, infected and uninfected with malaria respectively such that ($0 \leq \omega + \psi < 1$). We consider two cases in the analysis of our model. In the first case where we assume there are no immigrants ($A = 0$), we compute equilibrium points, the basic reproduction number $\Re_0$ and investigate both local and global stability of disease-free and endemic equilibria. In the second case of our model, where there are influx of immigrants ($A > 0$), where proportions of the imigrants ω, ψ and $(1-\omega-\psi)$ are exposed, infected and uninfected with malaria respectively such that ($0 \leq \omega + \psi < 1$), we compute the equilibrium point and investigate its stability.

According to Wedajo et al. (2018), it is without a shred of doubt that immigrants play a vital role in the transmission of malaria so in view of that, we formulate a deterministic model for human-female mosquito transmission of malaria incorporating infective immigrants in the human population. We make the assumption that there is a transfer of individuals from the susceptible compartment to exposed compartment when there is an interaction between susceptible humans and infected female mosquitoes given that there is a transmission in the process of the interaction.

Mathematical Backgrounds

In this section, basic definitions and theorems needed to comprehend the model are reviewed.

Definition 3.1

An equilibrium point (steady-state, fixed-point or critical point) of a differential equation (1.2), is a constant solution $x^c = (x_1^c, x_2^c, ..., x_n^c) \in \mathbb{R}^n$, satisfying

$$f(x^c) = 0 \tag{3.1}$$

Example 3.1

The logistic differential equation,

$$\frac{dx}{dt} = wx\left(1 - \frac{x}{L}\right), \qquad w, L > 0, \tag{3.2}$$

has two points of equilibrium: $x_1^c = 0$, and $x_2^c = L$

Example 3.2

The SEI epidemic model

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta IS - \mu S \\ \frac{dE}{dt} &= \beta IS - (\mu + \epsilon)E \\ \frac{dI}{dt} &= \epsilon E - \mu I \end{aligned} \tag{3.3}$$

has two equilibrium points namely

- a disease-free equilibrium point: $p_1 = (\frac{\Lambda}{\mu}, 0, 0)$, and

- an endemic equilibrium point $P_2 = (S_2^*, E_2^*, I_2^*)$, where

$$S_2^* = \frac{(\mu+\epsilon)\mu}{\beta\epsilon}$$
$$E_2^* = \frac{\mu(\Lambda\beta\epsilon - \mu^2(\mu+\epsilon)}{\beta\mu(\mu+\epsilon)\epsilon}$$
$$I_2^* = \frac{\Lambda\beta\epsilon - \mu^2(\mu+\epsilon)}{\beta\mu(\mu+\epsilon)}$$

The next generation matrix and the basic reproduction number

In epidemiology, the next-generation matrix is a method used to derive the basic reproduction number, for a compartmental model of the spread of infectious diseases. This method is given by (Dekmann, Heesterbeek & Metz, 1990; Van den Driessche & Watmough, 2002). Many of today's most important emerging infectious diseases are multi-host infections by their very nature. As a result, they require a slightly more complex formalism for investigating epidemic thresholds, etc. The basic tool for examining epidemic thresholds in complex, structured models is the so-called next generation matrix.

Consider a population of individuals (or species) subdivided into n compartments, of which m are infected. Let x_i represent the proportion of the population in the ith compartment and let the vector of the proportions in all the compartments be x. In order to compute $\Re_0$, it is important to distinguish new infections from all other changes in the population. Let

- $\mathcal{F}_i(x)$ be the rate of appearance of new infections in compartment i,

- $V_i^+(x)$ be the rate of transfer of individuals into compartment i by all other means, and

- $V_i^-(x)$ be the rate of transfer of individuals out of compartment i.

It is assumed that each function is continuously differentiable at least twice in each variable.The disease transmission model consists of nonnegative initial conditions together with the following system of equations:

$$\dot{x_i} = f_i(x) = \mathcal{F}_i(x) - \mathcal{V}_i(x), i = 1, ..., n \tag{3.4}$$

where $\mathcal{V}_i = \mathcal{V}_i^- - \mathcal{V}_i^+$. We define the matrices,

$$F = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(Q^0)\right], \qquad V = \left[\frac{\partial \mathcal{V}_i}{\partial x_j}(Q^0)\right],$$

where Q^0 denotes the DFE with indices $i, j = 1, ..., m$. The entries of the matrix

$$G = FV^{-1}$$

gives the rate at which infected individuals of state j generate new infections of type i. The matrix G is called the next generation matrix (Diekmann et al., 1990). R_0 is the dominant eigenvalue of G. That is

$$\Re_0 = \rho(G) = \rho\left(FV^{-1}\right). \tag{3.5}$$

For example , considering the epidemic model Equation (3.3) which has a disease-free quilibrium point $p_1 = (\frac{\Lambda}{\mu}, 0, 0)$.

To compute $\Re_0$ for the epidemic model Equation (3.3), we note the two disease states of the epidemic model which are E and I. The vectors V_i and F_i are given respectively as

$$\mathcal{F} = \begin{bmatrix} \beta IS \\ 0 \end{bmatrix} \quad and \quad \mathcal{V}_i^- - \mathcal{V}_i^+ = \mathcal{V} = \begin{bmatrix} (\mu + \epsilon)E \\ \mu i - \epsilon E \end{bmatrix}.$$

The matrices $\boldsymbol{F}$ and $\boldsymbol{V}$ are defined respectively as

$$\boldsymbol{F} = \begin{bmatrix} 0 & \beta S \\ 0 & 0 \end{bmatrix} \quad and \quad \boldsymbol{V} = \begin{bmatrix} \mu + \epsilon & 0 \\ -\epsilon & \mu \end{bmatrix}.$$

Now,

$$\boldsymbol{V}^{-1} = \begin{bmatrix} \frac{1}{\mu+\epsilon} & 0 \\ \frac{\epsilon}{\mu(\mu+\epsilon)} & \frac{1}{\mu} \end{bmatrix}.$$

The matrix $\boldsymbol{G}$ is now given given by

$$\boldsymbol{G} = \boldsymbol{F}\boldsymbol{V}^{-1} = \begin{bmatrix} \frac{\beta\epsilon}{\mu(\mu+\epsilon)} & \frac{\beta}{\mu} \\ 0 & 0 \end{bmatrix}.$$

The basic reproduction number is given by the spectral radius of $\boldsymbol{G}$, denoted by $\rho(\boldsymbol{G})$. That is ,

$$\Re_0 = \rho(\boldsymbol{G}) = \frac{\beta\epsilon}{\mu(\mu+\epsilon)}.$$

Global Stability analysis

The Lyapunov direct method is generally used to study the stability properties of an equilibrium point of systems of non-linear ordinary differential equation globally. There is no general techniques for constructing Lyapunov functions for ODEs.

Theorem 3.1

Let $\mathbb{Q} \in \mathbb{R}^n$ be a domain of origin. Let us consider the equation,

$$X' = f(x), \tag{3.6}$$

on $[0, \infty) \times Q$ with $f(0) = 0$ so that $x_* = 0$ is an equilibrium point of Equation (3.8). Assume that V is a Lyapunov function.

- If $v'(x) \leq 0$, then $x_* = 0$ globally stable .

- If $V'(x) < 0$, $x \neq 0$ (or $-V'(x)$ is positive definite), then $x_* = 0$ is globally asymptotically stable .

- If $V'(x) > 0$, $x \neq 0$, then $x_* = 0$ is unstable.

Below are some common Lyapunov candidates functions.

- Logarithemic Lyapunov Function:

  $\boldsymbol{V}(y_1, y_2, ..., y_m) = \sum_{i=1}^{m} q_i(y_i - y_i^* \ln \frac{y_i}{y_i^*})$,

- Common Quadratic Lyapunov Function:

  $\boldsymbol{V}(y_1, y_2, ..., y_m) = \sum_{m}^{i=1} \frac{q_i}{2}(y_i - y^*)^2$,

- Composite Quadratic Lyapunov Function:

  $\boldsymbol{V}(y_1, y_2, ..., y_m) = \frac{q}{2}\left[\sum_{m}^{i=1}(y_i - y^*)\right]^2$

Sensitivity analysis

The sensitivity analysis enables the researcher to ascertain the effect of the parameters in the model on the independent variable, $\Re_0$. For example the researcher may want to know if increasing a particular parameter will result in increase or decrease in the dependent variable .

The sensitivity index on $\Re_0$ is given by

$$\prec_{\rho}^{\Re_0} = \frac{\partial \Re_0}{\partial \rho} \cdot \frac{\rho}{\Re_0}, \tag{3.7}$$

where $\Re_0$ is the basic reproductive number (independent variable) and ρ is the parameter of interest.

Example 3.3

Let us consider the SEI model in system Equation (3.3) and compute the sensitivity index of β with $\mu = 0.0113$ and $\epsilon = 0.1$ and interprete the result.

The basic reproduction number for Equation (3.3) is given by

$$\Re_0 = \frac{\beta\epsilon}{\mu(\mu+\epsilon)}. \tag{3.8}$$

The sensitivity index of β on $\Re_0$ is given by,

$$\prec_{\beta}^{\Re_0} = \frac{\partial\Re_0}{\partial\beta} \cdot \frac{\beta}{\Re_0}. \tag{3.9}$$

Differentiating (3.8) partially with respect to β, we have

$$\frac{\partial\Re_0}{\partial\beta} = \frac{\epsilon}{\mu(\mu+\epsilon)}. \tag{3.10}$$

Substituting Equation (3.8) and Equation (3.10) into Equation (3.9), we obtain

$$\prec_{\beta}^{\Re_0} = \frac{\epsilon}{\mu(\mu+\epsilon)} \cdot \frac{\beta\mu(\mu+\epsilon)}{\beta\epsilon}. \tag{3.11}$$

Simplifying Equation (3.11), we obtain

$$\prec_{\beta}^{\Re_0} = +1. \tag{3.12}$$

The plus (+) sign of the right hand side of Equation (3.12) indicates there is a direct relationship between β and $\Re_0$. The 1 means that, a unit increase in β will result in a unit increase in $\Re_0$.

Model formulation

In our model, we consider two populations that is human population ($N_h(t)$) and female-mosquito population ($N_m(t)$). We compartmentalized human population into Susceptible (S_h), Exposed (E_h), Infected (I_h) and Recovered (R_h) at a given time. We illustrate total human population mathematically as

$$N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t).$$

Influx into the susceptible compartment is from two sources that is natural birth at the rate of Λ_h and immigration rate $(1 - \omega - \psi)A$. There is a movement of individuals from the susceptible compartment to the exposed compartment at a bitting rate ν and the rate of transmission β_h. Susceptible human become exposed at the rate of $\nu\beta_h I_m$. Furthermore, in the exposed compartment, there is an influx of immigrants at the rate ω which is a proportion of the total number of immigrants A coming into the human population at a given time. Individuals move from the exposed compartment to the infected compartment at the rate α_h which is a proportion of the total number of exposed individuals E_h. Also, in the infected compartment, there is an influx of immigrants at the rate of ψ which is a proportion of the total number of immigrants A. Infected individuals may die of the disease at a rate δ which is a proportion of the total number of infected individuals I_h. Infected individuals recover at the rate of ρ which is a proportion of I_h. Individuals in all the compartment can die naturally at the rate μ_h which are proportions of the individuals' respective status at a given time.

In a similar fashion, we compartmentalized female-mosquito population into susceptible (S_m), exposed (E_m), and infected (I_m) mosquitoes. The female-mosquito's total population is formulated mathematically as

$$N_m(t) = S_m(t) + E_m(t) + I_m(t).$$

Female-mosquitoes are rucruited into the susceptible compartment by natural birth rate Λ_m. Susceptible mosquitoes become infected when they have an interaction with infected human at the bitting rate ν and the rate of transmission β_m. Susceptible female mosquitoes become infected at the rate of $\nu\beta_m I_h$. After the susceptible female-mosquito has an interaction with infected human, they will progress to the exposed compartment. Female mosquitoes in the exposed compartment move to the infected compartment at the rate α_m which is a proportion of the total number of exposed mosquitoes. Also, female-mosquitoes in the infected compartment remain infected until they die naturally (Wan & Cui, 2009). Female-mosquitoes in all the three compartment die naturally at the rate μ_m which are proportions of female-mosquitoes' respective status. Our model excludes male mosquitoes since they do not take part in the transmission process.

State variables and parameters description

Explicit description of the state variabls and parameters of the model are given in Table 1 and 2 below respectively.

Table 1: State Variables and their Description

State Variables	Description
S_h	Susceptible human
E_h	Exposed human
I_h	Infected human
R_h	Rocovered human
S_m	Susceptible female mosquitoes
E_m	Exposed female mosquitoes
I_m	Infected female mosquitoes

Source: Mojeeb, Osman and Isaac (2017)

Table 2: Parameters Description for the Model

Parameters	Description
β_h	Rate of transmission from infectious female mosquitoes to susceptible human
β_m	Rate of transmission from infectious human to susceptible female mosquitoes
Λ_h	Natural birth rate for human
Λ_m	Natural birth rate for female mosquitoes
μ_h	Natural death rate for human
μ_m	Natural death rate for female mosquito
ν	Biting rate of female mosquitoes
ω	proportion of exposed human immigrants
ψ	proportion of infected immigrants
α_h	Human progression rate from exposed to infected
α_m	mosquito progression rate from exposed to infected
δ	Human disease-induced death rate
ρ	Human recovery rate
A	Total number of immigrants

Source: Mojeeb et al. (2017)

Based on the state variables and parameters description in Table 1 and 2 above respectively and the assumption made, we present $SEIR - SEI$ model of malaria transmission in Figure 1

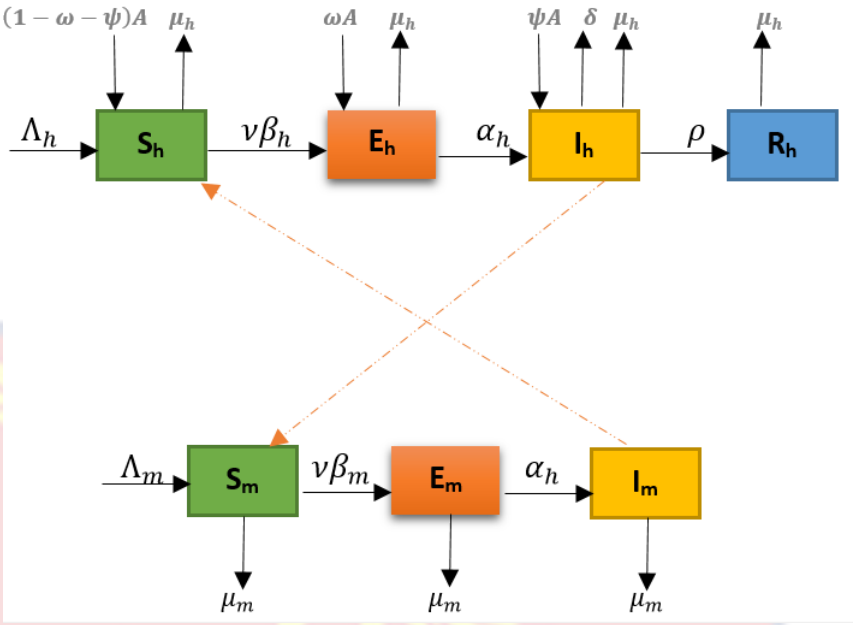

Figure 1: Compartmental Model for Human-Female Mosquito Transmission of Malaria with Infective Immigrants.

Equations for the model

From Figure 1, we obtained system of seven (7) non-linear differential equations.

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + (1-\omega-\psi)A - \nu\beta_h S_h I_m - \mu_h S_h \\
\frac{dE_h}{dt} &= \omega A + \nu\beta_h S_h I_m - (\mu_h + \alpha_h)E_h \\
\frac{dI_h}{dt} &= \psi A + \alpha_h E_h - (\mu_h + \delta + \rho)I_h \\
\frac{dR_h}{dt} &= \rho I_h - \mu_h R_h \\
\frac{dS_m}{dt} &= \Lambda_m + \nu\beta_m S_m I_h - \mu_m S_m \\
\frac{dE_m}{dt} &= \nu\beta_m S_m I_h - (\mu_m + \alpha_m)E_m \\
\frac{dI_m}{dt} &= \alpha_m E_m - \mu_m I_m
\end{aligned}\right\} \tag{3.13}$$

Reduced model

Since the first three of system Equation (3.13) are independent of R_h,

$$N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t)$$

and we can obtain R_h from this equation that is

$$R_h(t) = N_h(t) - (E_h(t) + E_h(t) + I_h(t))$$

We then concentrate on the reduced system because they have the same dynamical bahaviour as system Equation (3.13). For this reason, the model system of the reduced form excluding the R_h is given as

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + (1-\omega-\psi)A - \nu\beta_h S_h I_m - \mu_h S_h \\
\frac{dE_h}{dt} &= \omega A + \nu\beta_h S_h I_m - (\mu_h + \alpha_h)E_h \\
\frac{dI_h}{dt} &= \psi A + \alpha_h E_h - (\mu_h + \delta + \rho)I_h \\
\frac{dS_m}{dt} &= \Lambda_m - \nu\beta_m S_m I_h - \mu_m S_m \\
\frac{dE_m}{dt} &= \nu\beta_h S_m I_h - (\mu_m + \alpha_m)E_m \\
\frac{dI_m}{dt} &= \alpha_m E_m - \mu_m I_m
\end{aligned}\right\}. \tag{3.14}$$

Basic Properties of the Malaria Model

Here, we illustrate the positivity and boundedness of solutions of system Equation (3.14).

Positivity of solutions

For the malaria model transmission to be mathematically and epidemiologically correct, it is imperative to show that all state vaiables in the model system are non-negative at all time. That is to say, model system Equation (3.14) with non-negative initial condition will give a non-negative solution at all time.

Lemma 3.2

Suppose that the initial conditions

$$\{(S_h(0), S_m(0)) > 0, (E_h(0), I_h(0), E_m(0), I_m(0)) \geq 0\} \in \Omega, \tag{3.15}$$

then the solution set

$$\{S_h(t), S_m(t), E_h(t), I_h(t), E_m(t), I_m(t)\},$$

for model Equation (3.14) is

$$\{(S_h(0), S_m(0)) > 0, (E_h(0), I_h(0), E_m(0), I_m(0)) \geq 0\} \in \Omega$$

for $t > 0$.

Proof 3.1

We assume that

in considering the first equation in model system Equation (3.14), we have

$$\frac{dS_h}{dt} = \Lambda_h + (1 - \omega - \psi)A - \nu\beta_h S_h I_m - \mu_h S_h.$$

It follows that

$$\frac{dS_h}{dt} \geq -\mu_h S_h.$$

By separation of variables, we have

$$\frac{dS_h}{S_h} \geq -\mu_h dt.$$

Integrating both sides,

$$\int \frac{dS_h}{S_h} \geq -\int \mu_h dt,$$

we obtain

$$\ln |S_h| \geq -\mu_h + c_o.$$

Taking antilog on both sides, we obtain

$$S_h(t) \geq e^{-\mu_h t + c_o} = e^{-\mu_h t} e^{c_o}$$

$$\geq K_1 e^{-\mu_h t},$$

where $K_1 = e^{c_o} > 0$. Taking the initial condition at t = 0, we have $S_h(0) \geq K_1$. This implies

$$S_h(t) \geq S_h(0) e^{-\mu_h t} \geq 0.$$

For the second equation of model Equation (3.14), we have

$$\frac{dE_h}{dt} = \omega A + \nu \beta_h I_m - (\mu_h + \alpha_h) E_h.$$

It follows that,

$$\frac{dE_h}{dt} \geq -(\mu_h + \alpha_h) E_h,$$

By separation of variables, we obtain

$$\frac{dE_h}{E_h} \geq -(\mu_h + \alpha_h) dt.$$

Integrating both sides,

$$\int \frac{dE_h}{E_h} \geq -\int (\mu_h + \alpha) dt.$$

The result is,

$$\ln |E_h| \geq -(\mu_h + \alpha_h) t + c_1.$$

Taking antilog on both sides, we have

$$E_h(t) \geq e^{-(\mu_h+\alpha_h)t+c_1} = e^{-(\mu_h+\alpha_h)}e^{c_1}$$

$$\geq K_2 e^{-(\mu_h+\alpha_h)t},$$

where $K_2 = e^{c_1} > 0$. At t = 0, $E_h(0) \geq K_2$.

This implies,

$$E_h(t) \geq E_h(0)e^{(-\mu_h+\alpha_h)} \geq 0.$$

For the third equation of model Equation (3.14), we have

$$\frac{dI_h}{dt} = \psi A + \alpha_h E_h - (\mu_h + \delta + \rho)I_h.$$

It follows that,

$$\frac{dI_h}{dt} \geq -(\mu_h + \delta + \rho)I_h.$$

This gives,

$$\frac{dI_h}{I_h} \geq -(\mu_h + \delta + \rho)dt.$$

Integrating both sides,

$$\int \frac{dI_h}{I_h} \geq -\int (\mu_h + \delta + \rho)dt.$$

The result is,

$$\ln |I_h| \geq -(\mu_h + \delta + \rho) + c_2.$$

Taking antilog on both sides, we have

$$I_h(t) \geq e^{-(\mu_h+\delta+\rho)+c_2} = e^{-(\mu_h+\delta+\rho)}e^{c_2}$$

$$\geq K_3 e^{-(\mu_h+\delta+\rho)},$$

where $K_3 = e^{c_2} > 0$. At t = 0, $I_h(0) \geq K_3$,

This implies,

$$I_h t \geq I_h(0)e^{-(\mu_h+\delta+\rho)t} \geq 0.$$

For the fourth equation of model Equation (3.14), we have

$$\frac{dS_m}{dt} = \Lambda_m + \nu\beta_m S_m I_h - \mu_m S_m.$$

It follows that,

$$\frac{dS_m}{dt} \geq -\mu_m S_m.$$

This gives

$$\frac{dS_m}{S_m} \geq -\mu_m dt.$$

Integrating both sides,

$$\int \frac{dS_m}{S_m} \geq -\int \mu_m dt.$$

The result is,

$$\ln|S_m| \geq -\mu_m t + c_3.$$

Taking antilog on both sides, we have

$$S_m(t) \geq e^{-\mu_m t+c_3} = e^{-\mu_m t}e^{c_3}$$

$$\geq K_4 e^{-\mu_m t},$$

where $K_4 = e^{c_3} > 0$. At t = 0, $S_m(0) \geq K_4$.

This implies,

$$S_m(t) \geq S_m(0)e^{-\mu_m t} \geq 0.$$

For the fifth equation of model Equation (3.14), we have

$$\frac{dE_m}{dt} = \nu\beta_h S_m I_h - (\mu_m + \alpha_m)E_m.$$

It follows that,

$$\frac{dE_m}{dt} \geq -(\mu_m + \alpha_m)E_m.$$

This gives,

$$\frac{dE_m}{E_m} \geq -(\mu_m + \alpha_m)dt.$$

Integrating both sides give,

$$\int \frac{dE_m}{E_m} \geq -\int (\mu_m + \alpha_m)dt.$$

The result is,

$$\ln|E_m| \geq -(\mu_m + \alpha_m)t + c_4.$$

Taking antilog on both sides, we have

$$E_m(t) \geq e^{-(\mu_m+\alpha_m)t+c_4}$$

$$\geq K_5 e^{-(\mu_m+\alpha_m)t},$$

where $K_5 = e^{c_4} > 0$. At t = 0, $E_m(0) \geq K_5$.

This implies,

$$E_m(t) \geq E_m(0)e^{-(\mu_m+\alpha_m)t} \geq 0.$$

For the sixth equation of model Equation (3.14), we have

$$\frac{dI_m}{dt} = \alpha_m E_m - \mu_m I_m.$$

It follows that,

$$\frac{dI_m}{dt} \geq -\mu_m I_m.$$

This gives,

$$\frac{dI_m}{I_m} \geq -\mu_m dt.$$

Integrating both sides,

$$\int \frac{dI_m}{I_m} \geq -\int \mu_m dt.$$

The result is,

$$\ln |I_m| \geq -\mu_m t + c_5.$$

Taking antilog on both sides, we have

$$I_m(t) \geq e^{-\mu_m t + c_5} = e^{-\mu_m} e^{c_5}$$

$$\geq K_6 e^{-\mu_m t},$$

Where $K_6 = e^{c_5} > 0$. At t = 0, $I_m(0) \geq K_6$.

This implies,

$$I_m(t) \geq I_m(0) e^{-\mu_m t} \geq 0.$$

Boundedness of solution

Considering the human population of model system Equation (3.14), given initial conditions

$$S_h(0) > 0, (E_h(0), I_h(0)) \geq 0.$$

The total human population is given by

$$N_h(t) = S_h(t) + E_h(t) + I_h(t).$$

The rate at which the total human population is changing over time is given by

$$\frac{dN_h(t)}{dt} = \frac{dS_h(t)}{dt} + \frac{dE_h(t)}{dt} + \frac{dI_h(t)}{dt}. \tag{3.16}$$

Substituting model system Equation (3.14) into Equation (3.16), we obtain

$$\begin{aligned} \frac{dN_h(t)}{dt} &= \Lambda_h + A - \mu_h S_h - \mu_h E_h - \mu_h I_h - (\delta + \rho) I_h, \\ &= \Lambda_h + A - \mu_h (S_h + E_h + I_h) - (\delta + \rho) I_h, \\ &= \Lambda_h + A - \mu_h N_h - (\delta + \rho) I_h. \end{aligned} \tag{3.17}$$

Lemma 3.3

We have the result below for the boundedness of model Equation (3.14). The feasible region of the human population is defined by

$$\Omega_h = \left\{ (S_h, E_h, I_h) \in \mathbb{R}^3_+ : S_h + E_h + I_h \leq \frac{\Lambda_h + A}{\mu_h}, S_h > 0, (E_h, I_h) \geq 0 \right\}.$$

From Equation (3.17), we have

$$\frac{dN_h(t)}{dt} \geq \Lambda_h + A - \mu_h N_h,$$

$$\frac{dN_h(t)}{dt} + \mu_h N_h \geq \Lambda_h + A. \tag{3.18}$$

Using integrating factor $e^{\mu_h t}$, the computation for solving Equation (3.18) is given as

$$e^{\mu_h t} \frac{dN_h(t)}{dt} + e^{\mu_h t} \mu_h N_h \geq e^{\mu_h t} (\Lambda_h + A)$$

$$\frac{d(e^{\mu_h t} N_h)}{dt} \leq e^{\mu_h t} (\Lambda_h + A)$$

$$d(e^{\mu_h} N_h) \leq e^{\mu_h}(\Lambda_h + A)dt$$

$$\int d(e^{\mu_h} N_h) \leq \int e^{\mu_h}(\Lambda_h + A)dt$$

$$e^{\mu_h t} N_h(t) \leq \frac{(\Lambda_h + A)}{\mu_h} e^{\mu_h t} + c_8,$$

where $c_8 > 0$ is the constant of integration

$$N_h(t) \leq \frac{\Lambda_h + A}{\mu_h} + e^{-\mu_h t} c_8.$$

At $t = 0$, $N_h(0) \leq \frac{\Lambda_h + A}{\mu_h} + c_8$, where

$c_8 = N_h(0) - \frac{\Lambda_h + A}{\mu_h}$.

$$N_h(t) \leq \frac{\Lambda_h + A}{\mu_h} + e^{-\mu_h t}\left(N_h(0) - \frac{\Lambda_h + A}{\mu_h}\right). \tag{3.19}$$

Taking limit of Equation (3.19) as $t \to \infty$, we have

$$\lim_{t\to\infty}\left(N_h(t) \leq \frac{\Lambda_h + A}{\mu_h} + e^{-\mu_h t}(N_h(0) - \frac{\Lambda_h + A}{\mu_h})\right),$$

$$\lim_{t\to\infty} N_h(t) \leq \lim_{t\to\infty} \frac{\Lambda_h + A}{\mu_h} + \lim_{t\to\infty} e^{-\mu_h t}\left(N_h(0) - \frac{\Lambda_h + A}{\mu_h}\right),$$

which is simplifies to

$$\lim_{t\to\infty} N_h(t) \leq \frac{\Lambda_h + A}{\mu_h}. \tag{3.20}$$

Hence, the human population is bounded above by the carrying capacity $\frac{\Lambda_h + A}{\mu_h}$ and its feasible set is given by

$$\Omega_h := \left\{(S_h, E_h, I_h) \in \mathbb{R}^3_+ : S_h + E_h + I_h \leq \frac{\Lambda_h + A}{\mu_h}, S_h > 0, (E_h, I_h) \geq 0\right\}.$$

Also, considering the mosquito population of model Equation (3.14), given

initial conditions

$$S_m(0) > 0, (E_m(0), I_m(0)) \geq 0.$$

The total mosquito population is given by

$$N_m(t) = S_m(t) + E_m(t) + I_m(t).$$

The rate at which the total mosquito population is changing over time is given by

$$\frac{dN_m(t)}{dt} = \frac{dS_m(t)}{dt} + \frac{dE_m(t)}{dt} + \frac{dI_m(t)}{dt}. \tag{3.21}$$

Substituting model system Equation (3.14) into (3.21), we obtain

$$\begin{aligned} \frac{dN_m(t)}{dt} &= \Lambda_m - \mu_m S_m - \mu_m E_m - \mu_m I_m, \\ &= \Lambda_m - \mu_m (S_m + E_m + I_m), \\ &= \Lambda_m - \mu_m N_m. \end{aligned} \tag{3.22}$$

Lemma 3.4

We have the result for the boundedness of model Equation (3.14), the feasible region for the female-mosquito population is define by

$$\Omega_m = \left\{ (S_m, E_m, I_m) \in \mathbb{R}^3_+ : S_m + E_m + I_m \leq \frac{\Lambda_m}{\mu_m}, S_m > 0, (E_m, I_m) \geq 0 \right\}.$$

From Equation (3.21), we have

$$\frac{dN_m(t)}{dt} \leq \Lambda_m - \mu_m N_m,$$

$$\frac{dN_m(t)}{dt} + \mu_m N_m \leq \Lambda_m. \tag{3.23}$$

Using integration factor $e^{\mu_m t}$, the computation for solving Equation (3.23) is given by

$$\begin{aligned} e^{\mu_m t}\frac{dN_m(t)}{dt} + e^{\mu_m t}\mu_m N_m &\leq e^{\mu_m t}\Lambda_m \\ \frac{d}{dt}(e^{\mu_m t}\mu_m N_m) &\leq e^{\mu_m t}\Lambda_m \\ d(e^{\mu_m t}N_m) &\leq e^{\mu_m t}\Lambda_m \\ \int d(e^{\mu_m t}N_m) &\leq \int e^{\mu_m t}\Lambda_m dt \\ e^{\mu_m t}N_m(t) &\leq \frac{\Lambda_m}{\mu_m}e^{\mu_m t} + c_9, \end{aligned}$$

where c_9 is the constant of integration

$$N_m(t) \leq \frac{\Lambda_m}{\mu_m} + e^{-\mu_m t}c_9$$

At $t = 0$, $N_m(0) \leq \frac{\Lambda_m}{\mu_m} + c_9$, where

$c_9 = N_m(0) - \frac{\Lambda_m}{\mu_m}$

$$N_m(t) \leq \frac{\Lambda_m}{\mu_m} + e^{-\mu_m t}\left(N_m(0) - \frac{\Lambda_m}{\mu_m}\right). \tag{3.24}$$

Taking limit of Equation (3.24) as $t \to \infty$, we have

$$\begin{aligned} \lim_{t\to\infty}\left(N_m(t) \leq \frac{\Lambda_m}{\mu_m} + e^{-\mu_h t}(N_m(0) - \frac{\Lambda_m}{\mu_m})\right), \\ \lim_{t\to\infty}(N_m(t) \leq \lim_{t\to\infty}\frac{\Lambda_m}{\mu_m} + \lim_{t\to\infty}e^{-\mu_h t}\left(N_m(0) - \frac{\Lambda_m}{\mu_m}\right), \end{aligned}$$

which is simplifies to

$$\lim_{t\to\infty}(N_m(t) \leq \frac{\Lambda_m}{\mu_m}.$$

Hence, the female-mosquito population is bounded above by the carrying ca-

pacity $\frac{\Lambda_m}{\mu_m}$ and its feasible set is given by

$$\Omega_m := \left\{ (S_m, E_m, I_m) \in \mathbb{R}^3_+ : S_m + E_m + I_m \leq \frac{\Lambda_m}{\mu_m}, S_m > 0, (E_m, I_m) \geq 0 \right\}.$$

Model Steady State for the First Case: $A = 0$

The system Equation (3.14) has two equilibria that is disease-free equilibrium point (DFE) and endemic equilibrium point (EE_1) when we there are no exposed and infected immigrants $(\omega = \psi = 0)$.Equating the right hand side of system Equation (3.14) to zero, we have,

$$\left.\begin{aligned} \Lambda_h - \nu\beta_h S_h I_m - \mu_h S_h &= 0 \\ \nu\beta_h S_h I_m - (\mu_h + \alpha_h) E_h &= 0 \\ \alpha_h E_h - (\mu_h + \delta + \rho) I_h &= 0 \\ \Lambda_m - \nu\beta_m S_m I_h - \mu_m S_m &= 0 \\ \nu\beta_m S_m I_h - (\mu_m + \alpha_m) E_m &= 0 \\ \alpha_m E_m - \mu_m I_m &= 0 \end{aligned}\right\}. \tag{3.25}$$

Disease-Free Equlibrium (DFE)

At the disease-free equilibrium, we assume that there is no malaria in the population. Therefore, at the disease-free equilibrium, we have $E_h^0 = I_h^0 = E_m^0 = I_m^0 = 0$.

Substituting $E_h^0 = I_h^0 = E_m^0 = I_m^0 = 0$ into the first and second equations of the system in Equation (3.25), we have

$$\Lambda_h - \mu_h S_h^0 = 0, \tag{3.26}$$

and

$$\Lambda_m - \mu_m S_m^0 = 0. \tag{3.27}$$

Making S_h^0 the subject of Equation (3.26), we obtain

$$S_h^0 = \frac{\Lambda_h}{\mu_h}.$$

Similarly, making S_m^0 the subject of Equation (3.27), we obtain

$$S_m^0 = \frac{\Lambda_m}{\mu_m}.$$

Therefore , the disease- free equilibrium point Q^0 with the axis is given by

$$Q^0 = (S_h^0, E_h^0, I_h^0, S_m^0, E_m^0, I_m^0),$$

which is defined by

$$Q^0 = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, \frac{\Lambda_m}{\mu_m}, 0, 0\right).$$

The Basic Reproduction Number

In this section, we employed next generation matrix appraoch to compute the basic reproduction number, $\Re_0$. The basic reproduction, $\Re_0$ plays an eminent role in epidemiological theory for infectious diseases. The basic reproduction number, $\Re_0$ in this context is nothing more than the expected number of humans who would be infected after one generation of the parasite by a singular infectious human who is introduced into a susceptible population. What is essential about the $\Re_0$ is that, it measures how swiftly a disease can spread in its incipient stage and can foretell whether an infectious disease will die out or will become endemic in a population whenever $\Re_0 < 1$ and $\Re_0 > 1$ respectively. It can also be used to establish stability of steady states of disease models.

The system Equation (3.14) has four infected states that is E_h, I_h, E_m and I_m given by

$$\left.\begin{aligned}
\frac{dE_h}{dt} &= \omega A + \nu\beta_h S_h I_m - (\mu_h + \alpha_h)E_h \\
\frac{dI_h}{dt} &= \psi A + \alpha_h E_h - (\mu_h + \delta + \rho)I_h \\
\frac{dE_m}{dt} &= \nu\beta_h S_m I_h - (\mu_m + \alpha_m)E_m \\
\frac{dI_m}{dt} &= \alpha_m E_m - \mu_m I_m
\end{aligned}\right\}. \tag{3.28}$$

and two uninfected states, S_h, S_m. At the disease free state $E_h^0 = I_h^0 = E_m^0 = E_m^0 = A = 0$ and $S_h^0 = \frac{\Lambda_h}{\mu_h}$, $S_m^0 = \frac{\Lambda_m}{\mu_m}$.

The vectors F_i and V_i are given by

$$F_i = \begin{bmatrix} \nu\beta_h S_h I_m \\ 0 \\ \nu\beta_m S_m I_h \\ 0 \end{bmatrix}, \tag{3.29}$$

and

$$V_i = \begin{bmatrix} (\mu_h + \alpha_h)E_h \\ (\mu_h + \delta + \rho)I_h - \alpha_h E_h \\ (\mu_m + \alpha_m)E_m \\ \mu_m I_m + \alpha_m E_m \end{bmatrix}, \tag{3.30}$$

respectively. The partial derivative of Equation (3.29) with respect to E_h, I_h, E_m and I_m is given by

$$F = \begin{bmatrix} 0 & 0 & 0 & \nu\beta_h S_h \\ 0 & 0 & 0 & 0 \\ 0 & \nu\beta_m S_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.31}$$

Evaluating Equation (3.31) at the disease-free equilibrium point (Q^0) and the

Jacobian matrix of F_i is given by

$$F = \begin{bmatrix} 0 & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\nu\beta_m\Lambda_m}{\mu_m} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the partial derivative of Equation (3.30) with respect to E_h, I_h, E_m,and I_m is given by

$$V = \begin{bmatrix} (\mu_h + \alpha_h) & 0 & 0 & 0 \\ -\alpha_h & (\mu_h + \delta + \rho) & 0 & 0 \\ 0 & 0 & (\mu_m + \alpha_m) & 0 \\ 0 & 0 & -\alpha_m & \mu_m \end{bmatrix}. \tag{3.32}$$

Evaluating Equation (3.32) at the disease-free equilibrium point (Q^0) and the Jacobian matrix of V_i is given by

$$V = \begin{bmatrix} (\mu_h + \alpha_h) & 0 & 0 & 0 \\ -\alpha_h & (\mu_h + \delta + \rho) & 0 & 0 \\ 0 & 0 & (\mu_m + \alpha_m) & 0 \\ 0 & 0 & -\alpha_m & \mu_m \end{bmatrix}.$$

The inverse of the matrix V is given by

$$V^{-1} = \begin{bmatrix} \frac{1}{(\mu_h+\alpha_h)} & 0 & 0 & 0 \\ \frac{\alpha_h}{(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} & \frac{1}{(\mu_h+\delta+\rho)} & 0 & 0 \\ 0 & 0 & \frac{1}{(\mu_m+\alpha_m)} & 0 \\ 0 & 0 & \frac{\alpha_m}{\mu_m(\mu_m+\alpha_m)} & \frac{1}{\mu_m} \end{bmatrix}.$$

The Next Generation Matrix is given by

$$G = FV^{-1},$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{\nu\beta_h\Lambda}{\mu_h} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\nu\beta_m\Lambda_m}{\mu_m} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\mu_h+\alpha_h)} & 0 & 0 & 0 \\ \frac{\alpha_h}{(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} & \frac{1}{(\mu_h+\delta+\rho)} & 0 & 0 \\ 0 & 0 & \frac{1}{(\mu_m+\alpha_m)} & 0 \\ 0 & 0 & \frac{\alpha_m}{\mu_m(\mu_m+\alpha_m)} & \frac{1}{\mu_m} \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & \frac{\nu\beta_h\Lambda_h\alpha_m}{\mu_h\mu_m(\mu_m+\alpha_m)} & \frac{\nu\beta_h\Lambda_h\alpha_m}{\mu_h\mu_m} \\ 0 & 0 & 0 & 0 \\ \frac{\nu\beta_m\Lambda_m\alpha_h}{\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} & \frac{\nu\beta_m\Lambda_m}{\mu_m(\mu_h+\delta+\rho)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values of G are given by

$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = -\sqrt{\frac{\nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}}$ and $\lambda_4 = \sqrt{\frac{\nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}}$.

The basic reproduction number $\Re_0$ is the spectral radius of the next generation matrix G,

$$\Re_0 = \sqrt{\frac{\nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}}. \tag{3.33}$$

Equation (3.33) can also be expressed as

$$\Re_0^2 = \frac{\nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}. \tag{3.34}$$

Endemic Equilibrium (EE_1) **when** $A = 0$

In this section, we determine the endemic equilibrium point by solving the system in Equation (3.25) simulteneously for all the state variables. The endemic equilibrium points are the steady-state solutions where the malaria cannot be eliminated but remains in the total population. At the endemic equilibrium, the following equations are satisfied:

$$\left.\begin{aligned}
\Lambda_h - \nu\beta_h S_h^* I_m^* - \mu_h S_h^* &= 0 \\
\nu\beta_h S_h^* I_m^* - (\mu_h + \alpha_h) E_h^* &= 0 \\
\alpha_h E_h^* - (\mu_h + \delta + \rho) I_h^* &= 0 \\
\Lambda_m - \nu\beta_m S_m^* I_h^* - \mu_m S_m^* &= 0 \\
\nu\beta_m S_m^* I_h^* - (\mu_m + \alpha_m) E_m^* &= 0 \\
\alpha_m E_m^* - \mu_m I_m^* &= 0
\end{aligned}\right\}. \tag{3.35}$$

Making S_m^* the subject of the fourth equation in Equation (3.35), we have

$$S_m^* = \frac{\Lambda_m}{\nu\beta_m I_h^* + \mu_m}. \tag{3.36}$$

Substituting Equation (3.36) into the fifth equation in Equation (3.35), we have

$$\frac{\nu\beta_m \Lambda_m I_h^*}{\nu\beta_m I_h^* + \mu_m} - (\mu_m + \alpha_m) E_m^* = 0. \tag{3.37}$$

Making E_m^* the the subject of Equation (3.37) , we have

$$E_m^* = \frac{\nu\beta_m \Lambda_m I_h^*}{(\mu_m + \alpha_m)(\nu\beta_m I_h^* + \mu_m)}. \tag{3.38}$$

Also, substituting E_m^* into the sixth equation in Equation (3.35), we have

$$\frac{\nu\beta_m \Lambda_m \alpha_m I_h^*}{(\mu_m + \alpha_m)(\nu\beta_m I_h^* + \mu_m)} - \mu_m I_m^* = 0. \tag{3.39}$$

Making I_m^* the subject of Equation (3.39), give us

$$I_m^* = \frac{\nu\beta_m \Lambda_m \alpha_m I_h^*}{\mu_m(\mu_m + \alpha_m)(\nu\beta_m I_h^* + \mu_m)}. \tag{3.40}$$

Again, making E_h^* the subject of equation three in Equation (3.35), we have

$$E_h^* = \frac{1}{\alpha_h}\left[(\mu_h + \delta + \rho) I_h^*\right]. \tag{3.41}$$

Also, adding the first and the second equations in Equation (3.35) together, give us

$$\Lambda_h - \mu_h S_h^* - (\mu_h + \alpha_h)E_h^* = 0. \tag{3.42}$$

Substituting Equation (3.41) into Equation (3.42) give us

$$\Lambda_h - \mu_h S_h^* - \frac{(\mu_h + \alpha_h)(\mu_h + \delta + \rho)I_h^*}{\alpha_h} = 0. \tag{3.43}$$

Making S_h^* the subject of Equation (3.43), we have

$$S_h^* = \frac{1}{\alpha_h \mu_h}\left[\alpha_h \Lambda_h - (\mu_h + \alpha_h)(\mu_h + \delta + \rho)I_h^*\right]. \tag{3.44}$$

Similarly, substituting Equation (3.40) and Equation (3.41) into the second equation in Equation (3.25), give us

$$\frac{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m S_h^* I_h^*}{\mu_m(\mu_m + \alpha_m)(\nu\beta_m I_h^* + \mu_m)} - \frac{(\mu_h + \alpha_h)(\mu_h + \delta + \rho)I_h^*}{\alpha_h} = 0. \tag{3.45}$$

Making S_h^* the subject of Equation (3.45), we have

$$S_h^* = \frac{\mu_m(\mu_m + \alpha_m)(\mu_h + \alpha_h)(\mu_h + \delta + \rho)(\nu\beta_m I_h^* + \mu_m)}{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m \alpha_h}. \tag{3.46}$$

From Equation (3.44) and Equation (3.46), we have

$$\frac{\alpha_h \Lambda_h - q_1 I_h^*}{\mu_h} = \frac{q_1 \mu_m(\mu_m + \alpha_m)(\nu\beta_m I_h^* + \mu_m)}{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m}, \tag{3.47}$$

where $q_1 = (\mu_h + \alpha_h)(\mu_h + \delta + \rho)$.

Simplifying and rearranging Equation (3.47) , we obtain

$$q_1 \nu \beta_m M I_h^* - K = 0. \tag{3.48}$$

Replacing the value of q_1 with $(\mu_h + \alpha_h)(\mu_h + \delta + \rho)$ in Equation (3.48), we

have

$$\nu\beta_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)MI_h^*-K=0. \tag{3.49}$$

Making I_h^* the subject of Equation (3.49), we have

$$I_h^*=\frac{K}{\nu\beta_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M}. \tag{3.50}$$

where

$$\left.\begin{aligned} K &= \nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h-\mu_h\mu_m(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho \\ M &= \nu\beta_h\Lambda_m\alpha_m+\mu_h\mu_m(\mu_m+\alpha_m) \end{aligned}\right\} \tag{3.51}$$

Substituting Equation (3.50) into the Equation (3.36), we have

$$S_m^*=\frac{\Lambda_m}{\frac{\nu\beta_m K}{\nu\beta_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho M}+\mu_m}. \tag{3.52}$$

Simplifying Equation (3.52), we obtain

$$S_m^*=\frac{\Lambda_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M}{K+\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M}.$$

Similarly, substituting Equation (3.50) into Equation (3.41), we obtain

$$E_h^*=\frac{(\mu_h+\delta+\rho)K}{\nu\beta_m\alpha_h(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M}. \tag{3.53}$$

Simplifying Equation (3.53), we have

$$E_h^*=\frac{K}{\nu\beta_m\alpha_h(\mu_h+\alpha_h)M}.$$

The rest of the points are obtained in a similar fashion. The endemic equilibrium point Q^* is given by

$$Q^* = (S_h^*, E_h^*, I_h^*, S_m^*, E_m^*, I_m^*),$$

where

$$\left.\begin{aligned} S_h^* &= \frac{\mu_m(\mu_m+\alpha_m)[K+\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M]}{\nu^2\beta_h\beta_m\Lambda_m\alpha_m\alpha_h M} \\ E_h^* &= \frac{K}{\nu\beta_m\alpha_h(\mu_h+\alpha_h)M} \\ I_h^* &= \frac{K}{\nu\beta_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M} \\ S_m^* &= \frac{\Lambda_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M}{K+\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M} \\ E_m^* &= \frac{\Lambda_m K}{(\mu_m+\alpha_m)[K+\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M]} \\ I_m^* &= \frac{\Lambda_m\alpha_m K}{\mu_m(\mu_m+\alpha_m)[K+\mu_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho)M]} \end{aligned}\right\}. \tag{3.54}$$

We can express the endemic equilibium points (Q^*) in terms of the basic reproduction number $(\Re_0)$. From Equation (3.51),

$$K = \nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h - \mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho). \tag{3.55}$$

Factorizing Equation (3.55), we have

$$K = \mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\left[\frac{\nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\delta+\rho)} - 1\right]. \tag{3.56}$$

Substituting Equation (3.34) into Equation (3.56), we obtained

$$K = \mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)[\Re_0^2 - 1]. \tag{3.57}$$

Substituting Equation (3.57) into the third equation in Equation (3.54) and simplifying further, we have

$$I_h^* = \frac{\mu_h\mu_m^2(\mu_m+\alpha_m)[\Re_0^2-1]}{\nu\beta_m M}.$$

Silimilarly, substititing Equation (3.57) into the second equation in Equation

(3.54) and further simplification give us

$$E_h^* = \frac{\mu_h\mu_m(\mu_m+\alpha_m)(\mu_h+\delta+\rho)[\Re_0^2-1]}{\nu\beta_m\alpha_h M}.$$

We obtain the rest of the points in a similar fashion. The endemic equililibrium state in terms of $\Re_0$ are provided in Equation (3.58)

$$\begin{aligned}
S_h^* &= \frac{\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)[\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]}{\nu^2\beta_h\beta_m\Lambda_m\alpha_m\alpha_h M},\\
E_h^* &= \frac{\mu_h\mu_m(\mu_m+\alpha_m)(\mu_h+\delta+\rho)[\Re_0^2-1]}{\nu\beta_m\alpha_h M},\\
I_h^* &= \frac{\mu_h\mu_m^2(\mu_m+\alpha_m)[\Re_0^2-1]}{\nu\beta_m M},\\
S_m^* &= \frac{\Lambda_m M}{\mu_m[\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]},\\
E_m^* &= \frac{\Lambda_m\mu_h\mu_m(\mu_m+\alpha_m)[\Re_0^2-1]}{\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]},\\
I_m^* &= \frac{\Lambda_m\alpha_m\mu_h[\Re_0^2-1]}{\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]}.
\end{aligned} \tag{3.58}$$

Local Stability Analysis at the Disease Free Equilibrium Point (DFE), Q^0

Now, we investigate the local stability of the disease-free equilibrium point (DFE) when we assume that there are no immigrants that come into the human population, that is $A = 0$ using the theorem below.

Theorem 3.5

The disease-free equilibrium point (Q^0) which is locally asymptotically stable if $\Re_0^2 < 1$ and unstable if $\Re_0^2 > 1$.

Proof 3.2

The Jacobian matrix of system Equation (3.14) is given by

$$J = \begin{bmatrix} -b_1 - \mu_h & 0 & 0 & 0 & 0 & -\nu\beta_h S_h \\ b_1 & b_2 & 0 & 0 & 0 & \nu\beta_h S_h \\ 0 & \alpha_h & b_3 & 0 & 0 & 0 \\ 0 & 0 & -\nu\beta_m S_m & -b_4 - \mu_m & 0 & 0 \\ 0 & 0 & \nu\beta_m S_m & b_4 & -(\mu_m + \alpha_m) & 0 \\ 0 & 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix},$$

where $b_1 = \nu\beta_h I_m$, $b_2 = -(\mu_h + \alpha_h)$, $b_3 = -(\mu_h + \delta + \rho)$ and $b_4 = \nu\beta_m I_h$.

Evaluating the above matrix at DFE gives $J(Q^0)$ as

$$J(Q^0) = \begin{bmatrix} -\mu_h & 0 & 0 & 0 & 0 & -\frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & b_2 & 0 & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & \alpha_h & b_3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\nu\beta_m\Lambda_m}{\mu_m} & -\mu_m & 0 & 0 \\ 0 & 0 & \frac{\nu\beta_m\Lambda_m}{\mu_m} & 0 & -(\mu_m + \alpha_m) & 0 \\ 0 & 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}.$$

From columns 1 and 4, we get two of the eigenvlaues $\lambda_1 = -\mu_h$ and $\lambda_2 = -\mu_m$ which are all negatives. we have the remaining matrix as

$$J_r = \begin{bmatrix} -(\mu_h + \alpha_h) & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ \alpha_h & -(\mu_h + \delta + \rho) & 0 & 0 \\ 0 & \frac{\nu\beta_m\Lambda_m}{\mu_m} & -(\mu_m + \alpha_m) & 0 \\ 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.59}$$

We perform an elementary row-transformation for Equation (3.59) to enable us get the main diagonal as the eigenvalues. We obtain the following matrix

$$\frac{\alpha_h}{(\mu_h+\alpha_h)}R_1+R_2\rightarrow R_2,$$

$$\begin{bmatrix} -(\mu_h+\alpha_h) & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & -(\mu_h+\delta+\rho) & 0 & \frac{\nu\beta_h\Lambda_h\alpha_h}{\mu_h(\mu_h+\alpha_h)} \\ 0 & \frac{\nu\beta_m\Lambda_m}{\mu_m} & -(\mu_m+\alpha_m) & 0 \\ 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.60}$$

Also,

$$\frac{\nu\beta_m\Lambda_m}{\mu_m(\mu_h+\delta+\rho)}R_2+R_3\rightarrow R_3,$$

$$\begin{bmatrix} -(\mu_h+\alpha_h) & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & -(\mu_h+\delta+\rho) & 0 & \frac{\nu\beta_h\Lambda_h\alpha_h}{\mu_h(\mu_h+\alpha_h)} \\ 0 & 0 & -(\mu_m+\alpha_m) & \frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h}{\mu_m\mu_h(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} \\ 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.61}$$

Similarly,

$$\frac{\alpha_m}{(\mu_m+\alpha_m)}R_3+R_4\rightarrow R_4,$$

$$\begin{bmatrix} -(\mu_h+\alpha_h) & 0 & 0 & \frac{\nu\beta_h\Lambda_h}{\mu_h} \\ 0 & -(\mu_h+\delta+\rho) & 0 & \frac{\nu\beta_h\Lambda_h\alpha_h}{\mu_h(\mu_h+\alpha_h)} \\ 0 & 0 & -(\mu_m+\alpha_m) & \frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h}{\mu_m\mu_h(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} \\ 0 & 0 & 0 & D \end{bmatrix}, \tag{3.62}$$

where

$$D = -\mu_m + \frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h\alpha_m}{\mu_m\mu_h(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}.$$

The eigen values of Equation (3.62) are

$\lambda_3 = -(\mu_h+\alpha_h) < 0$, $\lambda_4 = -(\mu_h+\delta+\rho) < 0$, $\lambda_5 = -(\mu_m+\alpha_m) < 0$ and $\lambda_6 = D$

For stability, $\lambda_6 < 0$. Since $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda 5_1$.

Using the value of D, we have

$$-\mu_m + \frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h\alpha_m}{\mu_h\mu_m(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} < 0. \tag{3.63}$$

We can expressed Equation (3.63) as

$$\frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h\alpha_m}{\mu_m\mu_h(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} < \mu_m. \tag{3.64}$$

Dividing Equation (3.64) by μ_m, we have

$$\frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h\alpha_m}{\mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)} < 1. \tag{3.65}$$

The right hand side of Equation (3.65) is the same as $\Re_0^2$ in Equation (3.34) so expressing Equation (3.65) in terms of the basic reproduction number, $\Re_0$, we have

$$\Re_0^2 < 1\,,$$
$$\Longrightarrow \Re_0 < 1.$$

This shows that the disease-free equilibrium point is locally asymptotically stable if $\Re_0 < 1$.

Local Stability Analysis at the Endemic Equilibrium Point (EE_1), Q^*

Again, we investigate the local stability of the endemic equilibrium point with $A = 0$ using the theorem below.

Theorem 3.6

The endemic equilibrium (Q^*) with $A = 0$ is locally asymptotically stable if $\Re_0^2 > 1$ and unstable if $\Re_0^2 < 1$.

We will investigate the local stability of the EE_1 by writing the infected compartments for the EE_1 in terms of $\Re_0^2$. The infected compartments of Q^* in terms of $\Re_0^2$ in Equation (3.58) is given by

$$\left.\begin{aligned} E_h^* &= \frac{\mu_h\mu_m(\mu_m+\alpha_m)(\mu_h+\delta+\rho)[\Re_0^2-1]}{\nu\beta_m\alpha_h M} \\ I_h^* &= \frac{\mu_h\mu_m^2(\mu_m+\alpha_m)[\Re_0^2-1]}{\nu\beta_m M} \\ E_m^* &= \frac{\Lambda_m\mu_h\mu_m(\mu_m+\alpha_m)[\Re_0^2-1]}{\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]} \\ I_m^* &= \frac{\Lambda_m\alpha_m\mu_h[\Re_0^2-1]}{\mu_h\mu_m(\mu_m+\alpha_m)\Re_0^2+\nu\beta_h\Lambda_m\alpha_m]} \end{aligned}\right\}. \tag{3.66}$$

Since human and female-mosquito populations cannot assume a negative value, it is crystal clear that E_h^*, I_h^*, E_m^* and I_m^* are positive whenever $\Re_0^2 > 1$. In other words, the only way the endemic equilibrium EE_1 in system Equation (3.66) can exist is when $\Re_0^2 > 1$.

Global Stability Analysis of the Disease Free Equilibrium (DFE), Q^0

In this section, we will examine the global stability of the DFE in the feasible region Ω. When the DFE is globally asymptotically stable, no matter the size of the initial population , the disease will not persist in the population. We will use Lyapunov method to study the global stabilty of the DFE.

Theorem 3.7

If $\Re_0^2 \geq 1$, then the disease free equilibrium (DFE) is globally asymptotically stable in the feasible region Ω.

Proof 3.3

Let us consider the Lyapunov function below

$$V(t) = E_h(t) + q_1 I_h(t) + q_2 \nu \beta_h \Lambda_h E_m(t) + q_3 I_m(t), \tag{3.67}$$

where $q_1 > 0$, $q_2 > 0$ and $q_1 > 0$.

The derivative of Equation (3.67) along system Equation (3.14) gives the expression

$$V'(t) = E'_h(t) + q_1 I'_h(t) + q_2 \nu \beta_h \Lambda_h E'_m(t) + q_3 I'_m(t), \tag{3.68}$$

which $'$ denotes the derivate with respect to time t. Substituting system Equation (3.28) into Equation (3.68) with $A = 0$, we have

$$\begin{aligned} V'(t) &= \nu\beta_h S_h^0 I_m^0 - (\mu_h + \alpha_h)E_h^0 + q_1[\alpha_h E_h^0 - (\mu_h + \delta + \rho)I_h^0] \\ &\quad + q_2 \nu \beta_h \Lambda_h \alpha_h [\nu \beta_m S_m^0 I_h^0 - (\mu_m + \alpha_m)E_m^0] \\ &\quad + q_3[\alpha_m E_m^0 - \mu_m I_m^0], \\ &= \nu\beta_h S_h^0 I_m^0 - (\mu_h + \alpha_h)E_h^0 + q_1\alpha_h E_h^0 - q_1(\mu_h + \delta + \rho)I_h^0 \\ &\quad + q_2 \nu^2 \beta_m \beta_h \Lambda_h \alpha_h S_m^0 I_h^0 - q_2 \nu \beta_h \Lambda_h \alpha_h (\mu_m + \alpha_m) E_m^0 \\ &\quad + q_3 \alpha_m E_m^0 - q_3 \mu_m I_m^0. \end{aligned} \tag{3.69}$$

Choosing q_1, q_2 and q_3 respectively as

$\frac{(\mu_h+\alpha_h)}{\alpha_h}$, $\frac{\alpha_h}{\alpha_h \mu_m (\mu_m+\mu_m)}$ and $\frac{\nu\beta_h\Lambda_h}{\mu_h\mu_m}$,

and substituting it into Equation (3.69) and simplifying further, we have

$$\begin{aligned} V'(t) = \nu\beta_h S_h^0 I_m^0 &- \frac{(\mu_h + \alpha_h)(\mu_h + \delta + \rho)I_h^0}{\alpha_h} \\ &+ \frac{\alpha_m \nu^2 \beta_m \beta_h \Lambda_h \alpha_h}{\mu_h \mu_m (\mu_m + \alpha_m)} - \frac{\nu \beta_h \Lambda_h I_m^0}{\mu_h}. \end{aligned} \tag{3.70}$$

Note that

$$S_h^0 = \frac{\Lambda_h}{\mu_h}, S_m^0 = \frac{\Lambda_m}{\mu_m}. \tag{3.71}$$

Substituting Equation (3.71) into Equation (3.70), we have

$$\begin{aligned} V'(t) = \frac{\nu\beta_h\Lambda_h I_m^0}{\mu_h} - \frac{(\mu_h+\alpha_h)(\mu_h+\delta+\rho)I_h^0}{\alpha_h} \\ + \frac{\alpha_m\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_h}{\mu_h\mu_m^2(\mu_m+\alpha_m)} - \frac{\nu\beta_h\Lambda_h I_m^0}{\mu_h}. \end{aligned} \tag{3.72}$$

Simplifying Equation (3.72), we have

$$V'(t) = \frac{\nu^2\beta_m\beta_h\Lambda_h\Lambda_m\alpha_m I_h^0}{\mu_h\mu_m^2(\mu_m+\alpha_m)} - \frac{(\mu_h+\alpha_h)(\mu_h+\delta+\rho)I_h^0}{\alpha_h}. \tag{3.73}$$

Epressing Equation (3.73) in terms of $\Re_0$, we have

$$V'(t) = \frac{(\mu_h+\alpha_h)(\mu_h+\delta+\rho)}{\alpha_h}[\Re_0^2 - 1]I_h^0. \tag{3.74}$$

Hence,

$$\begin{cases} V'(t) = 0 & \text{if} \quad \Re_0^2 = 1 \quad \text{or} \quad I_h^0 = 0 \\ V'(t) < 0 & \text{if} \quad \Re_0^2 < 1. \end{cases} \tag{3.75}$$

Therefore, the disease-free equilibrium is globally asymptotically stable in the feasible region Ω whenever $\Re_0^2 \leq 1$.

Global Stability of Endemic Equilibrium Point (EE_1), Q^*

In this section, we will examine the global stability of endemic equilibrium Q^*.

Theorem 3.8

If $A = 0$ and $\Re_0 > 1$, the endemic equilibrium Q^* of system Equation (3.14) is globally asymptotically stable in the feasible region Ω .

Proof 3.4

Let us consider the Lyapunov function below

$$\begin{aligned} L_2(t) = a_1(S_h - S_h^* \ln S_h) + a_2(E_h - E_h^* \ln E_h) + \\ a_3(I_h - I_h^* \ln I_h) + a_4(S_m - S_m^* \ln S_m) + \\ a_5(E_m - E_m^* \ln E_m) + a_6(I_m - I_m^* \ln I_m). \end{aligned} \tag{3.76}$$

Differentiating Equation (3.76) along system Equation (3.14) gives the expression below

$$\begin{aligned} L_2'(t) = a_1 S_h'(1 - \frac{S_h^*}{S_h}) + a_2 E_h'(1 - \frac{E_h^*}{E_h}) + \\ a_3 I_h'(1 - \frac{I_h^*}{I_h}) + a_4 S_m'(1 - \frac{S_m^*}{S_m}) + \\ a_5 E_m'(1 - \frac{E_m^*}{S_m}) + a_6 I_m'(1 - \frac{I_m^*}{I_m}). \end{aligned} \tag{3.77}$$

Substituting system Equation (3.14) into Equation (3.77) gives us

$$\begin{aligned} L_2'(t) = a_1 \left(\Lambda_h + A - \nu\beta_h S_h I_m - \mu_h\right) \left[1 - \frac{S_h^*}{S_h}\right] \\ + a_2 \left(\nu\beta_h I_m - (\mu_h + \alpha_h) E_h\right) \left[1 - \frac{E_h^*}{E_h}\right] \\ + a_3 \left(\alpha_h E_h - (\mu_h + \delta + \rho) I_h\right) \left[1 - \frac{I_h^*}{I_h}\right] \\ + a_4 \left(\Lambda_m + \nu\beta_m S_m I_h - \mu_m S_m\right) \left[1 - \frac{S_m^*}{S_m}\right] \\ + a_5 \left(\nu\beta_h S_m I_h - (\mu_m + \alpha_m) E_m\right) \left[1 - \frac{E_m^*}{S_m}\right] \\ + a_6 \left(\alpha_m E_m - \mu_m I_m\right) \left[1 - \frac{I_m^*}{I_m}\right]. \end{aligned} \tag{3.78}$$

Equation (3.78) can be expressed as

$$
\begin{aligned}
L_2'(t) = {} & a_1\left(\nu\beta_h S_h^* I_m^* + \mu_h S_h^* - \nu\beta_h S_h I_m - \mu_h S_h\right)\left[1 - \frac{S_h^*}{S_h}\right] \\
& + a_2\left(\nu\beta_h S_h I_m\right)\left[1 - \frac{E_h^*}{E_h}\right] \\
& + a_3\left(\alpha_h E_h\right)\left[1 - \frac{I_h^*}{I_h}\right] \\
& + a_4\left(\nu\beta_m S_m^* I_h^* + \mu_m S_m^* - \nu\beta_m S_m I_h - \mu_m S_m\right)\left[1 - \frac{S_m^*}{S_m}\right] \\
& + a_5\left(\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m^* I_h^*}{E_m^*} E_m\right)\left[1 - \frac{E_m^*}{S_m}\right] \\
& + a_6\left(\alpha_m E_m - \frac{\alpha_m E_m^*}{I_m^*} I_m\right)\left[1 - \frac{I_m^*}{I_m}\right].
\end{aligned} \tag{3.79}
$$

$$
\begin{aligned}
L_2'(t) = {} & a_1\Bigg[\mu_h S_h^* - \mu_h \frac{S_h^{*2}}{S_h} - \mu_h S_h + \mu_h S_h^* + \nu\beta_h S_h^* I_m^* \\
& - \frac{\nu\beta_h S_h^{*2} I_m^*}{S_h} - \nu\beta_h S_h I_m + \nu\beta_h S_h^* I_m\Bigg] + a_2\Bigg[\nu\beta_h S_h I_m - \frac{\nu\beta_h I_m E_h^*}{E_h} \\
& - \frac{\nu\beta_h S_h^* I_m^* E_h}{E_h^*} + \nu\beta_h S_h^* I_m^*\Bigg] + a_3\Bigg[\alpha_h E_h - \frac{\alpha_h E_h I_h^*}{I_h} \\
& - \frac{\alpha_h E_h I_h}{I_h^*} + \alpha_h E_h^*\Bigg] + a_4\Bigg[\nu\beta_m S_m^* I_h^{**} + \mu_m S_m^* \\
& - \frac{\nu\beta_m S_m^{*2} I_h^*}{S_m} - \frac{\mu_m S^{*2}}{S_m} - \nu\beta_m S_m I_h + \nu\beta_m S_m^* I_h \\
& - \mu_m S_m + \mu_m S_m^*\Bigg] + a_5\Bigg[\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m I_h E_m^*}{E_m} \\
& - \frac{\nu\beta_m S_m^* I_h^* E_m}{E_m^*} + \nu\beta_m S_m^* I_h^*\Bigg] + a_6\Bigg[\alpha_m E_m - \\
& - \frac{\alpha_m Em I_m^*}{I_m} - \frac{\alpha_m E_m^* I_m}{I_m^*} + \alpha_m E_m^*\Bigg].
\end{aligned} \tag{3.80}
$$

$$
\begin{aligned}
L_2'(t) = a_1 \Bigg[& 2\mu_h S_h^* - \mu_h S_h - \mu_h \frac{S_h^{*2}}{S_h} + \nu\beta_h S_h^* I_m^* \\
& - \frac{\nu\beta_h S_h^{*2} I_m^*}{S_h} - \nu\beta_h S_h I_m + \nu\beta_h S_h^* I_m \Bigg] + a_2 \Bigg[\nu\beta_h S_h I_m - \frac{\nu\beta_h I_m E_h^*}{E_h} \\
& - \frac{\nu\beta_h S_h^* I_m^* E_h}{E_h^*} + \nu\beta_h S_h^* I_m^* \Bigg] + a_3 \Bigg[\alpha_h E_h - \frac{\alpha_h E_h I_h^*}{I_h} - \frac{\psi A I_h}{I_h^*} \\
& - \frac{\alpha_h E_h I_h}{I_h^*} + \alpha_h E_h^* \Bigg] + a_4 \Bigg[2\mu_m S_m^* + \nu\beta_m S_m^* I_h^* \\
& - \frac{\nu\beta_m S_m^{*2} I_h^*}{S_m} - \frac{\mu_m S^{*2}}{S_m} - \nu\beta_m S_m I_h + \nu\beta_m S_m^* I_h \\
& - \mu_m S_m \Bigg] + a_5 \Bigg[\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m I_h E_m^*}{E_m} \\
& - \frac{\nu\beta_m S_m^* I_h^* E_m}{E_m^*} + \nu\beta_m S_m^* I_h^* \Bigg] + a_6 \Bigg[\alpha_m E_m - \\
& - \frac{\alpha_m Em I_m^*}{I_m} - \frac{\alpha_m E_m^* I_m}{I_m^*} + \alpha_m E_m^* \Bigg].
\end{aligned}
\tag{3.81}
$$

$$
\begin{aligned}
L_2'(t) = a_1 & \left[\mu_h S_h^* \left(2 - \frac{S_h}{S_h^*} - \frac{S_h^*}{S_h} \right) + \nu\beta_h S_h^* I_h^* \left(1 - \frac{S_h^*}{S_h} + \frac{I_m}{I_m^*} - \frac{s_h I_m}{S_h^* I_m^*} \right) \right] \\
& + a_2 \left[\nu\beta_h S_h^* I_m^* \left(1 - \frac{S_h I_m E_h^*}{S_h^* I_m^* E_h} + \frac{S_h I_m}{S_h^* I_m^*} - \frac{E_h}{E_h^*} \right) \right] \\
& + a_3 \left[\alpha_h E_h^* \left(1 - \frac{E_h}{E_h^*} + \frac{E_h I_h^*}{E_h^* I_m} - \frac{I_h}{I_h^*} \right) \right] \\
& + a_4 \left[\mu_m S_m^* \left(2 - \frac{S_m}{S_m^*} - \frac{S_m^{**}}{S_m} \right) + \nu\beta_m S_m^* I_m^{**} \left(1 - \frac{S_m^*}{S_m} + \frac{I_h}{I_h^{**}} - \right. \right. \\
& \left. \left. \frac{s_m I_h}{S_m^* I_h^*} \right) \right] + a_5 \left[\nu\beta_m S_m^* I_h^* \left(1 - \frac{S_m I_h E_m^*}{S_m^* I_h^* E_m} + \frac{S_m I_h}{S_m^* I_h^*} - \frac{E_m}{E_m^*} \right) \right] \\
& + a_6 \left[\alpha_m E_m^* \left(1 - \frac{E_m}{E_m^*} + \frac{E_m I_m^*}{E_m^* I_m} - \frac{I_m}{I_m^*} \right) \right].
\end{aligned}
\tag{3.82}
$$

We choose the constants a_1 , a_2 , a_3 , a_4 , a_5 and a_6 respectively as

$$1\ ,\ 1,\ \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*}\ ,\ \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*}\ ,\ \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*} \quad \text{and} \quad \frac{\nu\beta_h S_h^* I_m^*}{\alpha_m E_h^*}.$$

Sustituting a_1 , a_2 , a_3 , a_4 , a_5 and a_6 into Equation (3.82) gives

$$\begin{aligned}
L_2'(t) = &\left[\mu_h S_h^*\left(2 - \frac{S_h}{S_h^*} - \frac{S_h^*}{S_h}\right) + \nu\beta_h S_h^* I_h^*\left(1 - \frac{S_h^*}{S_h} + \frac{I_m}{I_m^*} - \frac{s_h I_m}{S_h^* I_m^*}\right)\right] \\
&+ \left[\nu\beta_h S_h^* I_m^*\left(1 - \frac{S_h I_m E_h^*}{S_h^* I_m^* E_h} + \frac{S_h I_m}{S_h^* I_m^*} - \frac{E_h}{E_h^*}\right)\right] \\
&+ \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*}\left[\alpha_h E_h^{**}\left(1 - \frac{E_h}{E_h^*} + \frac{E_h I_h^*}{E_h^* I_m} - \frac{I_h}{I_h^{**}}\right)\right] \\
&+ \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*}\left[\mu_m S_m^*\left(2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m}\right) + \nu\beta_m S_m^* I_m^*\left(1 - \frac{S_m^*}{S_m}\right.\right. \\
&\left.\left.+ \frac{I_h}{I_h^*} - \frac{s_m I_h}{S_m^* I_h^*}\right)\right] + \frac{\beta_h S_h^* I_m^*}{\beta_m S_m^* I_h^*}\left[\nu\beta_m S_m^* I_h^*\left(1 - \frac{S_m I_h E_m^{**}}{S_m^* I_h^* E_m}\right.\right. \\
&\left.\left.+ \frac{S_m I_h}{S_m^* I_h^*} - \frac{E_m}{E_m^*}\right)\right] + \frac{\nu\beta_h S_h^* I_m^*}{\alpha_m E_h^*}\left[\alpha_m E_m^*\left(1 - \frac{E_m}{E_m^*} + \frac{E_m I_m^*}{E_m^* I_m} - \frac{I_m}{I_m^*}\right)\right].
\end{aligned} \tag{3.83}$$

Simplifying Equation (3.83) further gives us

$$\begin{aligned}
L_2'(t) = \mu_h S_h^*\left[2 - \frac{S_h}{S_h^*} - \frac{S_h^*}{S_h}\right] &+ \nu\beta_h S_h^* I_m^*\left[6 - \frac{S_h^*}{S_h}\right. \\
&- \frac{S_m^*}{S_m} - \frac{E_h I_h^*}{E_h^* I_h} - \frac{S_h E_h^* I_m}{S_h^* E_h I_m^*} - \frac{S_m E_m^* I_h}{S_m^* E_m I_h^*} \\
&\left. - \frac{E_m I_m^*}{E_m^* I_m}\right] + \frac{\mu_m\beta_h S_h^* I_m^*}{\beta_m I_h^*}\left[2 - \frac{S_m^*}{S_m} - \frac{S_m}{S_m^*}\right].
\end{aligned} \tag{3.84}$$

From Equation (3.84), it is clear that

$$\begin{aligned}
2 - \tfrac{S_h}{S_h^*} - \tfrac{S_h^*}{S_h} &\leq 0, \\
6 - \tfrac{S_h^*}{S_h} - \tfrac{S_m^*}{S_m} - \tfrac{E_h I_h^*}{E_h^* I_h} - \tfrac{S_h E_h^* I_m}{S_h^* E_h I_m^*} - \tfrac{S_m E_m^* I_h}{S_m^* E_m I_h^{**}} - \tfrac{E_m I_m^*}{E_m^* I_m} &\leq 0, \\
2 - \tfrac{E_h^*}{E_h} - \tfrac{E_h}{E_h^*} &\leq 0, \\
2 - \tfrac{I_h^*}{I_h} - \tfrac{1_h}{I_h^*} &\leq 0, \\
2 - \tfrac{S_m^*}{S_m} - \tfrac{S_m}{S_m^*} &\leq 0.
\end{aligned}$$

Hence,

$$L_2'(t) \leq 0 \text{ for } \Re_0 > 1.$$

All the parameters in Equation (3.84) are nonnegative, so $L_2'(t) = 0$ if $S_h = S_h^*, E_h = E_h^*, I_h = I_h^*, S_m = S_m^*, E_m = E_m^*$ and $I_m = I_m^*$.

Therefore, the endemic equilibrium $Q^* = (S_h^*, E_h^*, I_h^*, S_m^*, E_m^*, I_m^*)$ are globally asymptotically stable whenever $\Re_0 > 1$. What Theorem (3.8) means epidemiologically is that malaria will persist whenever $\Re_0 > 1$, regardless of the number of infected individuals at the initial stage of the population.

Model Steady State for the Second Case: ($A > 0$ **and** $0 < \omega < 1$**,** $0 < \psi < 1$**)**

In this section, we determine the endmic equilibrium point, (EE_2) when there are proportions of exposed and infected immigrants ($0 < \omega < 1$ and $0 < \psi < 1$). At the equilibrium points, state variables are static with respect to time. To compute the equilibrium points, we equate the right-hand side of system Equation (3.14) to zero and solve the resulting system simulteneously. Equating the right hand side of system Equation (3.14) to zero, we have,

$$\left.\begin{aligned}
\Lambda_h + (1-\omega-\psi)A - \nu\beta_h S_h^{**} I_m^{**} - \mu_h S_h^{**} &= 0 \\
\omega A + \nu\beta_h S_h^{**} I_m^{**} - (\mu_h + \alpha_h) E_h^{**} &= 0 \\
\psi A + \alpha_h E_h^{**} - (\mu_h + \delta + \rho) I_h^{**} &= 0 \\
\Lambda_m - \nu\beta_m S_m^{**} I_h^{**} - \mu_m S_m^{**} &= 0 \\
\nu\beta_h S_m^{**} I_h^{**} - (\mu_m + \alpha_m) E_m^{**} &= 0 \\
\alpha_m E_m^{**} - \mu_m I_m^{**} &= 0
\end{aligned}\right\}. \tag{3.85}$$

Making S_m^{**} the subject from the fourth equation in Equation (3.85), we have

$$S_m^{**} = \frac{\Lambda_m}{\nu\beta_m I_h^{**} + \mu_m}. \tag{3.86}$$

Substituting Equation (3.86) into the fifth equation in Equation (3.85), we have

$$\frac{\nu\beta_m\Lambda_m I_h^{**}}{\nu\beta_m I_h^{**} + \mu_m} - (\mu_m + \alpha_m)E_m^{**} = 0 \tag{3.87}$$

Making E_m^{**} the the subject of Equation (3.87) , we have

$$E_m^{**} = \frac{\nu\beta_m\Lambda_m I_h^{**}}{(\mu_m + \alpha_m)(\nu\beta_m I_h^{**} + \mu_m)}. \tag{3.88}$$

Also, substituting Equation (3.88) into the sixth equation in Equation (3.85), we have

$$\frac{\nu\beta_m\Lambda_m\alpha_m I_h^{**}}{(\mu_m + \alpha_m)(\nu\beta_m I_h^{**} + \mu_m)} - \mu_m I_m^{**} = 0. \tag{3.89}$$

Making I_m^{**} the subject of Equation (3.89), give us

$$I_m^{**} = \frac{\nu\beta_m\Lambda_m\alpha_m I_h^{**}}{\mu_m(\mu_m + \alpha_m)(\nu\beta_m I_h^{**} + \mu_m)}. \tag{3.90}$$

Again, making E_h^{**} the subject of the third equation in Equation (3.85), we have

$$E_h^{**} = \frac{1}{\alpha_h}\left[(\mu_h + \delta + \rho)I_h^{**} - \psi A\right]. \tag{3.91}$$

Also, adding the first and the second equation in Equation (3.85) together, give us

$$\Lambda_h + (1 - \psi)A - \mu_h S_h^{**} - (\mu_h + \alpha_h)E_h^{**} = 0. \tag{3.92}$$

Substituting Equation (3.91) into Equation (3.92) give us

$$\Lambda_h + (1 - \psi)A - \mu_h S_h^{**} - (\mu_h + \alpha_h)\left[\frac{q_2 I_h^{**} - \psi A}{\alpha_h}\right] = 0, \tag{3.93}$$

where, $q_2 = \mu_h + \delta + \rho$.

Making S_h^{**} the subject in Equation (3.93), we have

$$S_h^{**} = \frac{1}{\alpha_h \mu_h}\left[\alpha_h(\Lambda_h + A) + \mu_h \psi A - (\mu_h + \alpha_h) q_2 I_h^{**}\right]. \tag{3.94}$$

Similarly, substituting Equation (3.90) and Equation (3.91) into the second equation in Equation (3.85), give us

$$\omega A + \frac{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m S_h^{**} I_h^{**}}{\mu_m(\mu_m + \alpha_m)(\nu \beta_m I_h^{**} + \mu_m)} - (\mu_h + \alpha_h)\left(\frac{q_2 I_h^{**} - \psi A}{\alpha_h}\right) = 0. \tag{3.95}$$

Making S_h^{**} the subject in Equation (3.95), we have

$$S_h^{**} = \frac{q_3(q_2 I_h^{**} - \psi A)(\nu \beta_m I_h^{**} + \mu_m) - \omega A \alpha_h \mu_m(\mu_m + \alpha_m)(\nu \beta_m I_h^{**} + \mu_m)}{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m \alpha_h I_h^{**}}, \tag{3.96}$$

$$\text{where, } q_3 = \mu_m(\mu_m + \alpha_m)(\mu_h + \alpha_h).$$

From Equations (3.94) and (3.96), we have

$$\frac{\alpha_h(\Lambda_h + A) + \mu_h \psi A - (\mu_h + \alpha_h) q_2 I_h^{**}}{\mu_h} = \frac{N_1}{\nu^2 \beta_h \beta_m \Lambda_m \alpha_m I_h^{**}}, \tag{3.97}$$

where,

$$N_1 = q_3(q_2 I_h - \psi A)(\nu \beta_m I_h^{**} + \mu_m) - \omega A \alpha_h \mu_m(\mu_m + \alpha_m)(\nu \beta_m I_h^{**} + \mu_m).$$

Substituting the values of q_2 and q_3 into Equation (3.47) and simplifying further, we have

$$XM I_h^{**2} - [K + Z_1(\psi M + \alpha_h Y(\psi + \omega))]\, I_h^{**} - [Z_2 Y(\psi \mu_h + \alpha_h(\omega + \psi))] = 0, \tag{3.98}$$

where

$$\left.\begin{aligned} K &= \nu^2\beta_h\Lambda_h\beta_m\Lambda_m\alpha_m\alpha_h - \mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho) \\ M &= \nu\beta_h\Lambda_m\alpha_m + \mu_h\mu_m(\mu_m+\alpha_m) \\ X &= \nu\beta_m(\mu_h+\alpha_h)(\mu_h+\delta+\rho) \\ Y &= \mu_m(\mu_m+\alpha_m) \\ Z_1 &= \nu\beta_m\mu_h A \\ Z_2 &= \mu_h\mu_m A \end{aligned}\right\} \quad (3.99)$$

Equation (3.98) has two roots. One negative , which is given by

$$I_h^{**} = \frac{[K + Z_1(\psi M + \alpha_h Y[\psi+\omega])] - B}{2XM}, \quad (3.100)$$

(which is meaningless epidemiologically) and one positive. The positive root is given by

$$I_h^{**} = \frac{[K + Z_1(\psi M + \alpha_h Y[\psi+\omega])] + B}{2XM}, \quad (3.101)$$

where

$$B = \sqrt{[K + Z_1(\psi M + \alpha_h X(\psi+\omega))]^2 + 4MXYZ_2[\psi\mu_h + \alpha_h(\omega+\psi)]}.$$

The unique endemic equilibrium point Q^{**} are determined from the Equations (3.86), (3.88), (3.90), (3.91) and (3.94) with the coordinates

$$Q^{**} = (S_h^{**}, E_h^{**}, I_h^{**}, S_m^{**}, E_m^{**}, I_m^{**}),$$

where

$$\left.\begin{aligned} S_h^{**} &= \frac{\alpha_h(\Lambda_h+A)+\mu_h\psi A-(\mu_h+\alpha_h)(\mu_h+\delta+\rho)I_h^{**}}{\alpha_h\mu_h} \\ E_h^{**} &= \frac{(\mu_h+\delta+\rho)I_h^{**}-\psi A}{\alpha_h} \\ I_h^{**} &= \frac{[K+Z_1(\psi M+\alpha_h Y[\psi+\omega])]+B}{2XM} \\ S_m^{**} &= \frac{\Lambda_m}{\mu_m+\nu\beta_m I_h^{**}} \\ E_m^{**} &= \frac{\nu\beta_m\Lambda_m I_h^{**}}{(\mu_m+\alpha_m)(\mu_m+\nu\beta_m I_h^{**})} \\ I_m^{**} &= \frac{\nu\beta_m\Lambda_m\alpha_m I_h^{**}}{\mu_m(\mu_m+\alpha_m)(\mu_m+\nu\beta_m I_h^{**})} \end{aligned}\right\}. \tag{3.102}$$

From Equation (3.101), we notice that as $(\omega, \psi) \to 0$, we obtain

$$\lim_{(\omega,\psi)\to 0} I_h^{**} = \frac{K+\sqrt{K^2}}{2XM} = \frac{K+|K|}{2XM}. \tag{3.103}$$

We can express K in terms of $\Re_0$ which is given by

$$K = \mu_h\mu_m^2(\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)[\Re_0^2-1]. \tag{3.104}$$

From Equation (3.103), if $\Re_0 < 1$, $K < 0$, so in view of that, Equation (3.103) becomes

$$\lim_{(\omega,\psi)\to 0} I_h^{**} = 0. \tag{3.105}$$

Equation (3.105) implies that there will be no malaria in the population if $K < 0$ and as $(\omega, \psi) \to 0$.

Meanwhile, from Equation (3.103), if $\Re_0 > 1$, $K > 0$, then Equation (3.103) becomes

$$\lim_{(\omega,\psi)\to 0} I_h^{**} = \frac{K}{XM}. \tag{3.106}$$

Equation (3.106) means that malaria will be endemic in the population if $\Re_0 > 1$ and as $(\omega, \psi) \to 0$.

However, if we take ω and ψ to be sufficiently small, Equation (3.101) becomes

$$I_h^{**} = \frac{K + \sqrt{K^2 + 4Z_2YMX(\psi\mu_h + \alpha_h(\omega + \psi))}}{2XM}. \tag{3.107}$$

We can rewrite Equation (3.107) as

$$2MXI_h^{**} = K + |K|\left(1 + \frac{4Z_2MXY(\psi\mu_h + \alpha_h(\omega + \psi))}{K^2}\right)^{\frac{1}{2}}. \tag{3.108}$$

Let us consider the binomial approximation in Equation (3.109)

$$(1 + x)^n = 1 + nx. \tag{3.109}$$

Using the Equation (3.109) to expand Equation (3.108), we have

$$2MXI_h^{**} = K + |K|\left(1 + \frac{2Z_2MXY(\psi\mu_h + \alpha_h(\omega + \psi))}{K^2}\right). \tag{3.110}$$

Exapnding and simplifying Equation (3.110), we have

$$I_h^{**} = \frac{K + |K|}{2MX} + \frac{Z_2Y(\psi\mu_h + \alpha_h(\omega + \psi))}{K}. \tag{3.111}$$

From Equation (3.111), it is apparent that if $\Re_0 < 1$, then $K < 0$, we have

$$I_h^{**} \approx \frac{Z_2Y(\psi\mu_h + \alpha_h(\omega + \psi))}{|K|}. \tag{3.112}$$

On the other hand, from Equation (3.111) if $\Re_0 > 1$, then $K > 0$, we obtain

$$I_h^{**} \approx \frac{K}{MX} + \frac{Z_2Y(\psi\mu_h + \alpha_h(\omega + \psi))}{K}. \tag{3.113}$$

Also, if we add the second and the third eqations in Equation (3.14) together and set $E_h = I_h = 0$, we have

$$\left(\frac{dE_h}{dt} + \frac{dI_h}{dt}\right)_{E_h=I_h=0} = A(\omega + \psi), \tag{3.114}$$

Where, $0 < \omega + \psi < 1$.

From Equtions (3.112) and (3.113), it can be seen that there is no threshold effect if the values of ω and ψ are sufficiently small. That is the coventional way of determining the extinction and the spread of diseases when the $\Re_0 < 1$ and $\Re_0 > 1$ respectively is irrelevant when there are exposed and infected immigrants. Similarly, from Equation (3.114), we observed that when there is no disease in the population, the entering of the exposed and infected immigrants into the population will lead to the persistence of malaria in the population. Our model system Equation (3.14) does not have a disease free if $0 < \omega < 1$ and $0 < \psi < 1$ and subsequently, no basic reproduction number $(\Re_0)$. Nevertheless, we noticed from Equations (3.105) and (3.106) that there is a threshold effect as ω and ψ goes to zero.

Local Stability Analysis at the Unique Endemic Equilibrium Point (EE_2), Q^{**}

Also, we investigaate the local stability of the unique endemic equilibrium (EE_2) when there are constant inflow of exposed and infected immigrants into the population, that is $0 < \omega < 1$ and $0 < \psi < 1$ using the theorem below.

Theorem 3.9

The unique endemic equilibrium point (Q^{**}) which is locally asymptotically stable if $0 < \omega < 1$ and $0 < \psi < 1$ and unstable otherwise.

Proof 3.5

Linearization of the system Equuation (3.14) about a unique endemic equilibrium Q^{**} gives the Jacobian matrix:

$$J_1 = \begin{bmatrix} -d_1 & 0 & 0 & 0 & 0 & -d_9 \\ d_2 & -d_3 & 0 & 0 & 0 & d_9 \\ 0 & \alpha_h & -d_4 & 0 & 0 & 0 \\ 0 & 0 & -d_5 & -d_6 & 0 & 0 \\ 0 & 0 & d_5 & d_7 & -d_8 & 0 \\ 0 & 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.115}$$

Where

$d_1 = \nu\beta_h I_m^{**} + \mu_h$, $d_2 = \nu\beta_h I_m^{**}$, $d_3 = \mu_h + \alpha_h$, $d_4 = \mu_h + \delta + \rho$, $d_5 = \nu\beta_m S_m^{**}$, $d_6 = \nu\beta_m I_h^{**} + \mu_m$, $d_7 = \nu\beta_m I_h^{**}$, $d_8 = \mu_m + \alpha_m$ and $d_9 = \nu\beta_h S_h^{**}$.

To determine the local stability of the unique endemic equilibrium Q^{**}, we perform an elementary row-transformation for the matrix Equation (3.115) to enable us get the main diagonal as the eigenvalues. We obtain the following matrix:

$$\frac{d_2}{d_1}R_1 + R_2 \to R_2,$$

$$J_2 = \begin{bmatrix} -d_1 & 0 & 0 & 0 & 0 & -d_9 \\ 0 & -d_3 & 0 & 0 & 0 & \frac{d_9(d_1-d_2)}{d_1} \\ 0 & \alpha_h & -d_4 & 0 & 0 & 0 \\ 0 & 0 & -d_5 & -d_6 & 0 & 0 \\ 0 & 0 & d_5 & d_7 & -d_8 & 0 \\ 0 & 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.116}$$

From column 1 in Equation (3.116), we have one of the eigen value to be $\lambda_1 = -d_1 < 0$, the remaining matrix becomes

$$J_3 = \begin{bmatrix} -d_3 & 0 & 0 & 0 & \frac{d_9(d_1-d_2)}{d_1} \\ \alpha_h & -d_4 & 0 & 0 & 0 \\ 0 & -d_5 & -d_6 & 0 & 0 \\ 0 & d_5 & d_7 & -d_8 & 0 \\ 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.117}$$

From Equation (3.117), we have

$$\frac{\alpha_h}{d_3}R_1 + R_2 \to R_2,$$

$$J_4 = \begin{bmatrix} -d_3 & 0 & 0 & 0 & \frac{d_9(d_1-d_2)}{d_1} \\ 0 & -d_4 & 0 & 0 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ 0 & -d_5 & -d_6 & 0 & 0 \\ 0 & d_5 & d_7 & -d_8 & 0 \\ 0 & 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.118}$$

From column 1 in Equation (3.118), the eigenvalue is $\lambda_2 = -d_3 < 0$. the remaining matrix is given in Equation (3.119):

$$J_5 = \begin{bmatrix} -d_4 & 0 & 0 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ -d_5 & -d_6 & 0 & 0 \\ d_5 & d_7 & -d_8 & 0 \\ 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.119}$$

Also, from Equation (3.119), we have

$$\frac{d_7}{d_6}R_2 + R_3 \to R_3,$$

$$J_6 = \begin{bmatrix} -d_4 & 0 & 0 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ -d_5 & -d_6 & 0 & 0 \\ \frac{d_5(d_6-d_7)}{d_6} & 0 & -d_8 & 0 \\ 0 & 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.120}$$

Again, from column 2 in Equation (3.120), we have an eigenvalue of $\lambda_3 = -d_6 < 0$. The remaining matrix is given in Equation (3.121):

$$J_7 = \begin{bmatrix} -d_4 & 0 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ \frac{d_5(d_6-d_7)}{d_6} & -d_8 & 0 \\ 0 & \alpha_m & -\mu_m \end{bmatrix}. \tag{3.121}$$

Simlilarly, from Equation (3.121) we have

$$\frac{\alpha_m}{d_8} R_2 + R_3 \to R_3,$$

$$J_8 = \begin{bmatrix} -d_4 & 0 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ \frac{d_5(d_6-d_7)}{d_6} & -d_8 & 0 \\ \frac{\alpha_m d_5(d_6-d_7)}{d_6 d_8} & 0 & -\mu_m \end{bmatrix}. \tag{3.122}$$

Also, from column 2 in Equation (3.122), we have the eigenvalue to be $\lambda_4 = -d_8 < 0$. The remaining matrix is given in Equation (3.123):

$$J_9 = \begin{bmatrix} -d_4 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ \frac{\alpha_m d_5(d_6-d_7)}{d_6 d_8} & -\mu_m \end{bmatrix}. \tag{3.123}$$

Also, from Equation (3.123), we have

$$\frac{\alpha_m d_5(d_6-d_7)}{d_4 d_6 d_8} R_1 + R_2 \to R_2,$$

$$J_{10} = \begin{bmatrix} -d_4 & \frac{\alpha_h d_9(d_1-d_2)}{d_1 d_3} \\ 0 & -\mu_m + \frac{\alpha_m \alpha_h d_5 d_9(d_1-d_2)(d_6-d_7)}{d_1 d_3 d_4 d_6 d_8} \end{bmatrix}. \tag{3.124}$$

From Equation (3.124), the eigenvalues are $\lambda_5 = -d_4 < 0$ and

$\lambda_6 = -\mu_m + \frac{\alpha_m \alpha_h d_5 d_9 (d_1 - d_2)(d_6 - d_7)}{d_1 d_3 d_4 d_6 d_8}$,

$\lambda_6 < 0$, if and only if

$$\mu_m d_1 d_3 d_4 d_6 d_8 - \alpha_m \alpha_h d_5 d_9 (d_1 - d_2)(d_6 - d_7) > 0. \tag{3.125}$$

Using the values of d_1, d_2, d_6 and d_7, we can rewrite Equation (3.125) as

$$\mu_m d_1 d_3 d_4 d_6 d_8 - \mu_m \mu_h \alpha_m \alpha_h d_5 d_9 > 0. \tag{3.126}$$

Factoring out μ_m and divide through by μ_m, Equation (3.126) becomes

$$d_1 d_3 d_4 d_6 d_8 - \mu_h \alpha_m \alpha_h d_5 d_9 > 0. \tag{3.127}$$

From Equaation (3.127), it is legit to say that: $d_1 d_3 d_4 d_6 d_8 > \mu_h \alpha_m \alpha_h d_5 d_9$

Hence,

$$\lambda_6 < 0.$$

Since all the eigenvalues have negative real part, we therefore conclude that the unique endemic equilibrium (Q^{**}) is locally asymptotically stable.

Global Stability of Endemic Equilibrium Point Q^{**}

We have already shown that our model has no disease-free equilibrium point when there are exposed and infected immigrants. This implies that irrespective of the value of $\Re_0$, malaria will persist at the endemic equilibrium level if there are exposed and infected immigrants in the population . In this section, we will examine the global stability of our endemic equilibrium point Q^{**}, using Lyapunov Indirect Method.

Theorem 3.10

If $0 < \omega < 1$ and $0 < \psi < 1$, then the unique endemic equilibrium Q^{**} of system Equation (3.14) is globally asymptotically stable in the feasible region Ω.

Proof 3.6

Let us consider the Lyapunov function below

$$\begin{aligned} L_3(t) = a_1(S_h - S_h^{**}\ln S_h) + a_2(E_h - E_h^{**}\ln E_h) + \\ a_3(I_h - I_h^{**}\ln I_h) + a_4(S_m - S_m^{**}\ln S_m) + \\ a_5(E_m - E_m^{**}\ln E_m) + a_6(I_m - I_m^{**}\ln I_m). \end{aligned} \tag{3.128}$$

Differentiating Equation (3.128) along system Equation (3.14) gives Equation (3.129)

$$\begin{aligned} L_3'(t) = a_1 S_h'(1 - \frac{S_h^{**}}{S_h}) + a_2 E_h'(1 - \frac{E_h^{**}}{E_h}) + \\ a_3 I_h'(1 - \frac{I_h^{**}}{I_h}) + a_4 S_m'(1 - \frac{S_m^{**}}{S_m}) + \\ a_5 E_m'(1 - \frac{E_m^{**}}{S_m}) + a_6 I_m'(1 - \frac{I_m^{**}}{I_m}). \end{aligned} \tag{3.129}$$

Substituting system Equation (3.14) into Equation (3.129) gives us

$$\begin{aligned} L_3'(t) = a_1\left(\Lambda_h + (1-\omega-\psi)A - \nu\beta_h S_h I_m - \mu_h\right)\left[1 - \frac{S_h^{**}}{S_h}\right] \\ + a_2\left(\omega A + \nu\beta_h I_m - (\mu_h + \alpha_h)E_h\right)\left[1 - \frac{E_h^{**}}{E_h}\right] \\ + a_3\left(\psi A + \alpha_h E_h - (\mu_h + \delta + \rho)I_h\right)\left[1 - \frac{I_h^{**}}{I_h}\right] \\ + a_4\left(\Lambda_m + \nu\beta_m S_m I_h - \mu_m S_m\right)\left[1 - \frac{S_m^{**}}{S_m}\right] \\ + a_5\left(\nu\beta_h S_m I_h - (\mu_m + \alpha_m)E_m\right)\left[1 - \frac{E_m^{**}}{S_m}\right] \\ + a_6\left(\alpha_m E_m - \mu_m I_m\right)\left[1 - \frac{I_m^{**}}{I_m}\right]. \end{aligned} \tag{3.130}$$

Equation (3.130) can be expressed as

$$
\begin{aligned}
L_3'(t) = {} & a_1\left(\nu\beta_h S_h^{**} I_m^{**} + \mu_h S_h^{**} - \nu\beta_h S_h I_m - \mu_h S_h\right)\left[1 - \frac{S_h^{**}}{S_h}\right] \\
& + a_2\left(\omega A + \nu\beta_h S_h I_m - \frac{(\omega A + \nu\beta_h S_m^{**} I_m^{**})}{E_h^{**}} E_h\right)\left[1 - \frac{E_h^{**}}{E_h}\right] \\
& + a_3\left(\psi A + \alpha_h E_h - \frac{(\psi A + \alpha_h E_h^{**})}{I_h^{**}} I_h\right)\left[1 - \frac{I_h^{**}}{I_h}\right] \\
& + a_4\left(\nu\beta_m S_m^{**} I_h^{**} + \mu_m S_m^{**} - \nu\beta_m S_m I_h - \mu_m S_m\right)\left[1 - \frac{S_m^{**}}{S_m}\right] \\
& + a_5\left(\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m^{**} I_h^{**}}{E_m^{**}} E_m\right)\left[1 - \frac{E_m^{**}}{S_m}\right] \\
& + a_6\left(\alpha_m E_m - \frac{\alpha_m E_m^{**}}{I_m^{**}} I_m\right)\left[1 - \frac{I_m^{**}}{I_m}\right].
\end{aligned} \tag{3.131}
$$

$$
\begin{aligned}
L_3'(t) = {} & a_1\Bigg[\mu_h S_h^{**} - \mu_h \frac{S_h^{**2}}{S_h} - \mu_h S_h + \mu_h S_h^{**} + \nu\beta_h S_h^{**} I_m^{**} \\
& - \frac{\nu\beta_h S_h^{**2} I_m^{**}}{S_h} - \nu\beta_h S_h I_m + \nu\beta_h S_h^{**} I_m\Bigg] + a_2\Bigg[\omega \\
& - \frac{\omega A E_h^{**}}{E_h} + \nu\beta_h S_h I_m - \frac{\nu\beta_h I_m E_h^{**}}{E_h} - \omega A \\
& - \frac{\nu\beta_h S_h^{**} I_m^{**} E_h}{E_h^{**}} + \nu\beta_h S_h^{**} I_m^{**}\Bigg] + a_3\Bigg[\psi A \\
& - \frac{\psi A I_h^{**}}{I_h} + \alpha_h E_h - \frac{\alpha_h E_h I_h^{**}}{I_h} - \frac{\psi A I_h}{I_h^{**}} + \psi A \\
& - \frac{\alpha_h E_h I_h}{I_h^{**}} + \alpha_h E_h^{**}\Bigg] + a_4\Bigg[\nu\beta_m S_m^{**} I_h^{**} + \mu_m S_m^{**} \\
& - \frac{\nu\beta_m S_m^{**2} I_h^{**}}{S_m} - \frac{\mu_m S^{**2}}{S_m} - \nu\beta_m S_m I_h + \nu\beta_m S_m^{**} I_h \\
& - \mu_m S_m + \mu_m S_m^{**}\Bigg] + a_5\Bigg[\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m I_h E_m^{**}}{E_m} \\
& - \frac{\nu\beta_m S_m^{**} I_h^{**} E_m}{E_m^{**}} + \nu\beta_m S_m^{**} I_h^{**}\Bigg] + a_6\Bigg[\alpha_m E_m - \\
& - \frac{\alpha_m E m I_m^{**}}{I_m} - \frac{\alpha_m E_m^{**} I_m}{I_m^{**}} + \alpha_m E_m^{**}\Bigg].
\end{aligned} \tag{3.132}
$$

$$
\begin{aligned}
L_3'(t) = a_1 \Bigg[& 2\mu_h S_h^{**} - \mu_h S_h - \mu_h \frac{S_h^{**2}}{S_h} + \nu\beta_h S_h^{**} I_m^{**} \\
& - \frac{\nu\beta_h S_h^{**2} I_m^{**}}{S_h} - \nu\beta_h S_h I_m + \nu\beta_h S_h^{**} I_m \Bigg] + a_2 \Bigg[2\omega A \\
& - \frac{\omega A E_h^{**}}{E_h} + \nu\beta_h S_h I_m - \frac{\nu\beta_h I_m E_h^{**}}{E_h} \\
& - \frac{\nu\beta_h S_h^{**} I_m^{**} E_h}{E_h^{**}} + \nu\beta_h S_h^{**} I_m^{**} \Bigg] + a_3 \Bigg[2\psi A \\
& - \frac{\psi A I_h^{**}}{I_h} + \alpha_h E_h - \frac{\alpha_h E_h I_h^{**}}{I_h} - \frac{\psi A I_h}{I_h^{**}} \\
& - \frac{\alpha_h E_h I_h}{I_h^{**}} + \alpha_h E_h^{**} \Bigg] + a_4 \Bigg[2\mu_m S_m^{**} + \nu\beta_m S_m^{**} I_h^{**} \\
& - \frac{\nu\beta_m S_m^{**2} I_h^{**}}{S_m} - \frac{\mu_m S^{**2}}{S_m} - \nu\beta_m S_m I_h + \nu\beta_m S_m^{**} I_h \\
& - \mu_m S_m \Bigg] + a_5 \Bigg[\nu\beta_m S_m I_h - \frac{\nu\beta_m S_m I_h E_m^{**}}{E_m} \\
& - \frac{\nu\beta_m S_m^{**} I_h^{**} E_m}{E_m^{**}} + \nu\beta_m S_m^{**} I_h^{**} \Bigg] + a_6 \Bigg[\alpha_m E_m - \\
& - \frac{\alpha_m E m I_m^{**}}{I_m} - \frac{\alpha_m E_m^{**} I_m}{I_m^{**}} + \alpha_m E_m^{**} \Bigg].
\end{aligned}
\tag{3.133}
$$

$$L_3'(t) = a_1\left[\mu_h S_h^{**}\left(2 - \frac{S_h}{S_h^{**}} - \frac{S_h^{**}}{S_h}\right) + \nu\beta_h S_h^{**} I_h^{**}\right.$$
$$\left.\left(1 - \frac{S_h^{**}}{S_h} + \frac{I_m}{I_m^{**}} - \frac{s_h I_m}{S_h^{**} I_m^{**}}\right)\right] + a_2\left[\omega A\left(2 - \frac{E_h^{**}}{E_h} - \frac{E_h}{E_h^{**}}\right)\right.$$
$$\left. + \nu\beta_h S_h^{**} I_m^{**}\left(1 - \frac{S_h I_m E_h^{**}}{S_h^{**} I_m^{**} E_h} + \frac{S_h I_m}{S_h^{**} I_m^{**}} - \frac{E_h}{E_h^{**}}\right)\right]$$
$$+ a_3\left[\psi A\left(2 - \frac{I_h^{**}}{I_h} - \frac{I_h}{I_h^{**}}\right) + \alpha_h E_h^{**}\left(1 - \frac{E_h}{E_h^{**}} + \frac{E_h I_h^{**}}{E_h^{**} I_m} - \frac{I_h}{I_h^{**}}\right)\right]$$
$$+ a_4\left[\mu_m S_m^{**}\left(2 - \frac{S_m}{S_m^{**}} - \frac{S_m^{**}}{S_m}\right) + \nu\beta_m S_m^{**} I_m^{**}\left(1 - \frac{S_m^{**}}{S_m}\right.\right.$$
$$\left.\left. + \frac{I_h}{I_h^{**}} - \frac{s_m I_h}{S_m^{**} I_h^{**}}\right)\right] + a_5\left[\nu\beta_m S_m^{**} I_h^{**}\left(1 - \frac{S_m I_h E_m^{**}}{S_m^{**} I_h^{**} E_m} + \frac{S_m I_h}{S_m^{**} I_h^{**}} - \right.\right.$$
$$\left.\left.\frac{E_m}{E_m^{**}}\right)\right] + a_6\left[\alpha_m E_m^{**}\left(1 - \frac{E_m}{E_m^{**}} + \frac{E_m I_m^{**}}{E_m^{**} I_m} - \frac{I_m}{I_m^{**}}\right)\right]. \tag{3.134}$$

We choose the constants a_1 , a_2 , a_3 , a_4 , a_5 and a_6 respectively as

$$1\ ,\ 1,\ \frac{\beta_h S_h^{**} I_m^{**}}{\beta_m S_m^{**} I_h^{**}}\ ,\ \frac{\beta_h S_h^{**} I_m^{**}}{\beta_m S_m^{**} I_h^{**}}\ ,\ \frac{\beta_h S_h^{**} I_m^{**}}{\beta_m S_m^{**} I_h^{**}} \quad \text{and} \quad \frac{\nu\beta_h S_h^{**} I_m^{**}}{\alpha_m E_h^{**}}\ .$$

Sustituting a_1 , a_2 , a_3 , a_4 , a_5 and a_6 into Equation (3.134) gives

$$
\begin{aligned}
L_3'(t) = &\left[\mu_h S_h^{**}\left(2-\frac{S_h}{S_h^{**}}-\frac{S_h^{**}}{S_h}\right)+\nu\beta_h S_h^{**}I_h^{**}\right.\\
&\left.\left(1-\frac{S_h^{**}}{S_h}+\frac{I_m}{I_m^{**}}-\frac{s_h I_m}{S_h^{**}I_m^{**}}\right)\right]+\left[\omega A\left(2-\frac{E_h^{**}}{E_h}-\frac{E_h}{E_h^{**}}\right)\right.\\
&\left.+\nu\beta_h S_h^{**}I_m^{**}\left(1-\frac{S_h I_m E_h^{**}}{S_h^{**}I_m^{**}E_h}+\frac{S_h I_m}{S_h^{**}I_m^{**}}-\frac{E_h}{E_h^{**}}\right)\right]\\
&+\frac{\beta_h S_h^{**}I_m^{**}}{\beta_m S_m^{**}I_h^{**}}\left[\psi A\left(2-\frac{I_h^{**}}{I_h}-\frac{I_h}{I_h^{**}}\right)+\alpha_h E_h^{**}\left(1-\frac{E_h}{E_h^{**}}\right.\right.\\
&\left.\left.+\frac{E_h I_h^{**}}{E_h^{**}I_m}-\frac{I_h}{I_h^{**}}\right)\right]+\frac{\beta_h S_h^{**}I_m^{**}}{\beta_m S_m^{**}I_h^{**}}\left[\mu_m S_m^{**}\left(2-\frac{S_m}{S_m^{**}}-\frac{S_m^{**}}{S_m}\right)\right.\\
&\left.+\nu\beta_m S_m^{**}I_m^{**}\left(1-\frac{S_m^{**}}{S_m}+\frac{I_h}{I_h^{**}}-\frac{s_m I_h}{S_m^{**}I_h^{**}}\right)\right]+\frac{\beta_h S_h^{**}I_m^{**}}{\beta_m S_m^{**}I_h^{**}}\\
&\left[\nu\beta_m S_m^{**}I_h^{**}\left(1-\frac{S_m I_h E_m^{**}}{S_m^{**}I_h^{**}E_m}+\frac{S_m I_h}{S_m^{**}I_h^{**}}-\frac{E_m}{E_m^{**}}\right)\right]\\
&+\frac{\nu\beta_h S_h^{**}I_m^{**}}{\alpha_m E_h^{**}}\left[\alpha_m E_m^{**}\left(1-\frac{E_m}{E_m^{**}}+\frac{E_m I_m^{**}}{E_m^{**}I_m}-\frac{I_m}{I_m^{**}}\right)\right].
\end{aligned}
\tag{3.135}
$$

Simplifying Equation (3.135) further gives us

$$
\begin{aligned}
L_3'(t) = &\mu_h S_h^{**}\left[2-\frac{S_h}{S_h^{**}}-\frac{S_h^{**}}{S_h}\right]+\nu\beta_h S_h^{**}I_m^{**}\left[6-\frac{S_h^{**}}{S_h}\right.\\
&-\frac{S_m^{**}}{S_m}-\frac{E_h I_h^{**}}{E_h^{**}I_h}-\frac{S_h E_h^{**}I_m}{S_h^{**}E_h I_m^{**}}-\frac{S_m E_m^{**}I_h}{S_m^{**}E_m I_h^{**}}\\
&\left.-\frac{E_m I_m^{**}}{E_m^{**}I_m}\right]+\omega A\left[2-\frac{E_h^{**}}{E_h}-\frac{E_h}{E_h^{**}}\right]+\frac{\psi A\beta_h S_h^{**}I_m^{**}}{\beta_m S_m^{**}I_h^{**}}\\
&\left[2-\frac{I_h^{**}}{I_h}-\frac{1_h}{I_h^{**}}\right]+\frac{\mu_m\beta_h S_h^{**}I_m^{**}}{\beta_m I_h^{**}}\left[2-\frac{S_m^{**}}{S_m}-\frac{S_m}{S_m^{**}}\right].
\end{aligned}
\tag{3.136}
$$

From Equation (3.136), it is clear that

$$
\begin{aligned}
2-\tfrac{S_h}{S_h^{**}}-\tfrac{S_h^{**}}{S_h}&\le 0,\\
6-\tfrac{S_h^{**}}{S_h}-\tfrac{S_m^{**}}{S_m}-\tfrac{E_h I_h^{**}}{E_h^{**}I_h}-\tfrac{S_h E_h^{**}I_m}{S_h^{**}E_h I_m^{**}}-\tfrac{S_m E_m^{**}I_h}{S_m^{**}E_m I_h^{**}}-\tfrac{E_m I_m^{**}}{E_m^{**}I_m}&\le 0,\\
2-\tfrac{E_h^{**}}{E_h}-\tfrac{E_h}{E_h^{**}}&\le 0,\\
2-\tfrac{I_h^{**}}{I_h}-\tfrac{1_h}{I_h^{**}}&\le 0,
\end{aligned}
$$

$$2 - \frac{S_m^{**}}{S_m} - \frac{S_m}{S_m^{**}} \leq 0,$$

Hence,

$$L_3'(t) \leq 0, \text{ for } (0 < \omega < 1, 0 < \psi < 1).$$

All the parameters in Equation (3.136) are nonnegative, so $L_2'(t) = 0$ if $S_h = S_h^{**}, E_h = E_h^{**}, I_h = I_h^{**}, S_m = S_m^{**}, E_m = E_m^{**}$ and $I_m = I_m^{**}$.

Therefore, the endemic equilibrium $Q^{**} = (S_h^{**}, E_h^{**}, I_h^{**}, S_m^{**}, E_m^{**}, I_m^{**})$ are globally asymptotically stable whenever $0 < \omega < 1$ and $0 < \psi < 1$. What Theorem (3.10) means epidemiologically is that malaria will persist whenever $0 < \omega < 1$ and $0 < \psi < 1$ regardless of the number of infective immigrants at the initial stage of the population.

Chapter Summary

In this chapter we formulated a deterministic mathematical $SEIR - SEI$ model for malaria transmission with the inclusion of exposed and infective immigrants in the human population. The state variables and the parameters used in the model were displayed explicitly. Since we are dealing with human and animal populations, we have established that the model is well-posed mathematically and epidemiologically correct by showing that all the state variables are positive and bounded. The model system Equation (3.14) has three equilibria namely; disease-free (Q^0), unique endmic (EE_1), (Q^*) and unique endemic (EE_1), (Q^{**}) equilibrium points. We also computed the basic reproduction number $\Re_0$ and use it to investigate the local stabilities of the equilibrium points. We end the chapter using Lyapunov method to determine the global stabilities of the equilibrium points.

CHAPTER FOUR

RESULTS AND DISCCUSION

Introduction

In this chapter, we performed numerical simulations of our model when there are no immigrants ($A = 0$) and when there are immigrants ($A > 0$ such that $0 < \omega < 1$, $0 < \psi < 1$) to know the full extent of the effect of exposed and infected immigrants on malaria transmission and discuss the results obtained. Some of the parameter values were obtained from literature and some were obtained base on assumption. Numerical simulations were carried out using Matlab. The prime aim is to confirm numerically the analytical results we obtained in chapter three. This in our view would enable us to figure out the future trends of malaria transmssion dynamics .

We also performed sensitivity analysis on the basic reproduction number, $\Re_0$ for the following parameters: (β_h), (β_m), (Λ_h), (Λ_m), (μ_h), (μ_m), (ν), (α_h), (α_m), (δ) and (ρ) to know the parameters that influence the transmission dynamics of malaria.

Numerical Analysis

In this section, we carry out numerical simulations of our model when there are no immigrants ($A = 0$) and when there are immigrants ($A > 0$ such that $0 < \omega < 1, 0 < \psi < 1$) to understand effect of exposed and infected immigrants on malaria transmission dynamics. We vary the parameters displayed in Table 3 which resulted in the $\Re_0 < 1$ and $\Re_0 > 1$. The initial conditions of the state variables used in this section are given by $S_h = 2500$, $E_h = 700$, $I_h = 300$, $S_m = 5000$, $E_m = 2500$ and $I_m = 4000$ and the rest of the parameters with its corresponding values are stated in Table 3 below.

Table 3: Parameters and their Values

Parameter	Value/day	Source
β_h	0.002	Assumed
β_m	0.005	(Tilahun, 2017)
Λ_h	25	(Wedajo et al., 2018)
Λ_m	125	(Wedajo et al., 2018)
μ_h	0.01146	(Mojeeb et al., 2017)
μ_m	0.1	(Mojeeb, Ebenezer, Hassan & Yang, 2019)
ν	0.12	(Olaniyi & Obabiyi, 2013)
ω	$[0, 0.4]$	Assumed
ψ	$[0, 0.3]$	Assumed
α_h	0.1	(Mojeeb et al., 2017)
α_m	0.083	(Shah & Gupta, 2013)
δ	0.068	(Mojeeb et al., 2017)
ρ	$\frac{1}{7}$	(Tumwiine et al., 2005)
A	100	Assumed

Source: Wedajo et al. (2018)

Numerical Results when there are no Immigrants $(A = 0)$

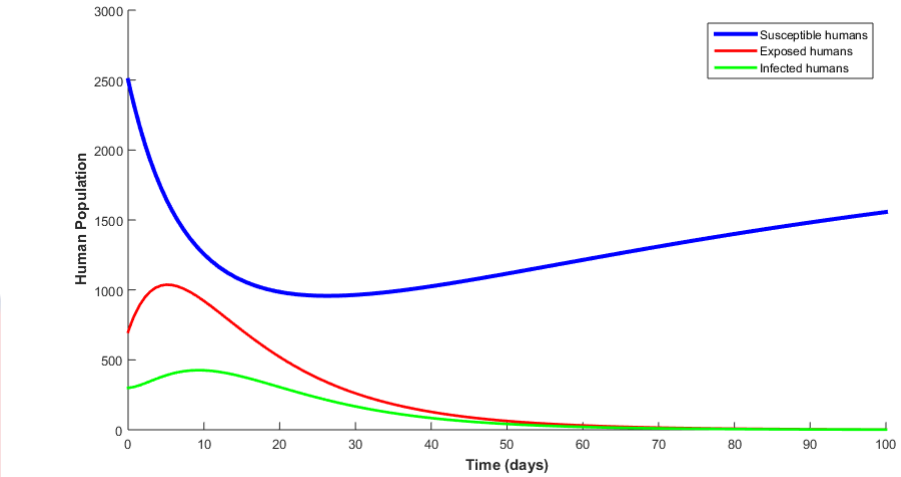

Figure 2: Plot of Human Populations Against Time for $\Re_0 < 1$

Figure 2 display time response of the state variables S_h, E_h and I_h against time. When we vary the values of these two parameters (β_h and β_m) with other paramters used as presented in Table 3, we notice that the population settles at disease free with $\Re_0 = 0.2681 < 1$. At the disease free, all the infected human population (exposed humans and infecteds) move to zero. Also it can be seen that the susceptible population decreases a bit and then increases steadily which is attributed to the recruitment of new individuals into the susceptible compartment by birth. Local and global stabilty of the malaria transmission model at the disease free equilibrium point has also been verified numerically by Figure 2. The epidemiological interpretation of Figure 2 is that in 60 days time, malaria will extinct in the population. The corresponding mosquito population is displayed in Figure 2.

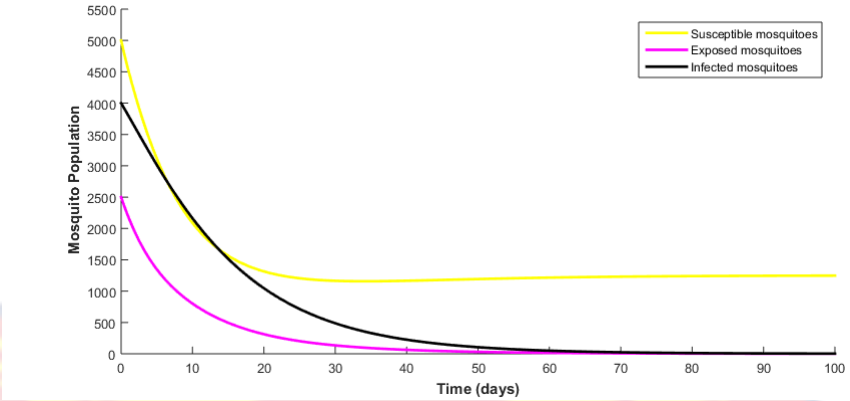

Figure 3: Plot of Mosquito Populations Against Time for $\Re_0 < 1$

Similarly, in the mosquito population, we have observed from Figure 3 that the exposed and the infected mosquitoes compartments tend to zero, while the susceptible mosquitoes compartment decreases to non-zero number at $\Re_0 = 0.2681 < 1$.

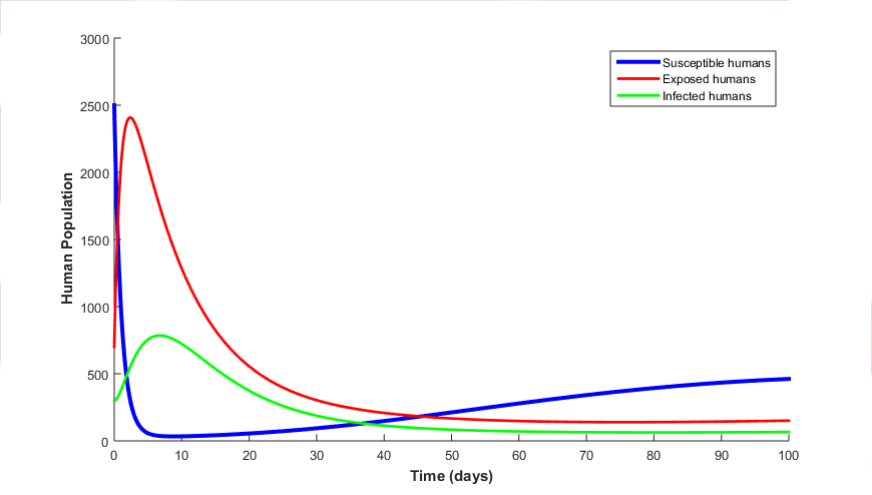

Figure 4: Plot of Human Populations Against Time for $\Re_0 > 1$. The Parameters are as Stated in Table 3.

Also from Figure 4, we note that when we used the all the paramters as stated in Table 3, all the human compartments cotemporal in the population which cornfirmed the endemicity of malaria with the basic reproduction number of $\Re_0 = 2.6809 > 1$. This confirm the presence of malaria in the population.

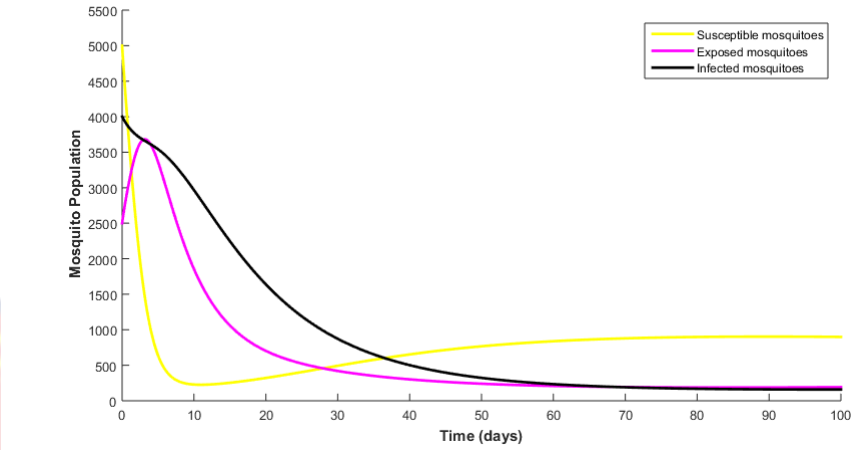

Figure 5: Plot of Mosquito Populations Against Time for $\Re_0 > 1$

From Figure 5, we have seen that all the distinct compartments of mosquito population coexist. The $\Re_0 = 2.6809 > 1$ is calculated using the parameter values as presented in Table 3.

Effect of Exposed and Infected Immigrants on Disease Free Population

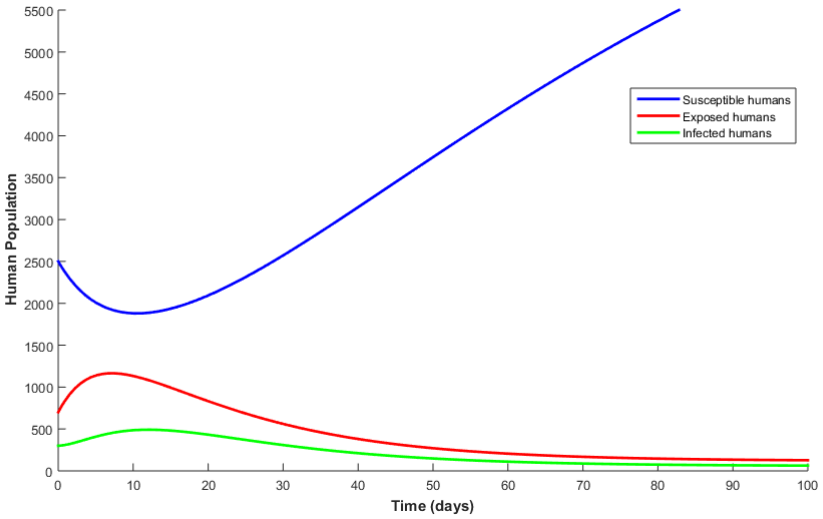

Figure 6: Plot of Human Populations Against Time Showing the Effect of Exposed and Infected Immigrants on Malaria Free Population for $\beta_h = 0.0002$, $\beta_m = 0.0005$, $\omega = 0.1$, $\psi = 0.01$ and $A = 100$ with $\Re_0 < 1$

Figure 6 is the time response of human population against time showing

the effect of exposed and infected immigrants on malaria free population. We noticed that the value of $\Re_0 = 0.2681 < 1$ but the population fails to settle at the disease free. We have all the three distinct compartments of humans coexisting in the population albeit the basic reproduction number is less than unity and this is attributed to the influx of exposed and infected entering the population. This confirm the analytical result in Equation (3.112).

Effect of Exposed Immigrants on Exposed and Infected Human Populations

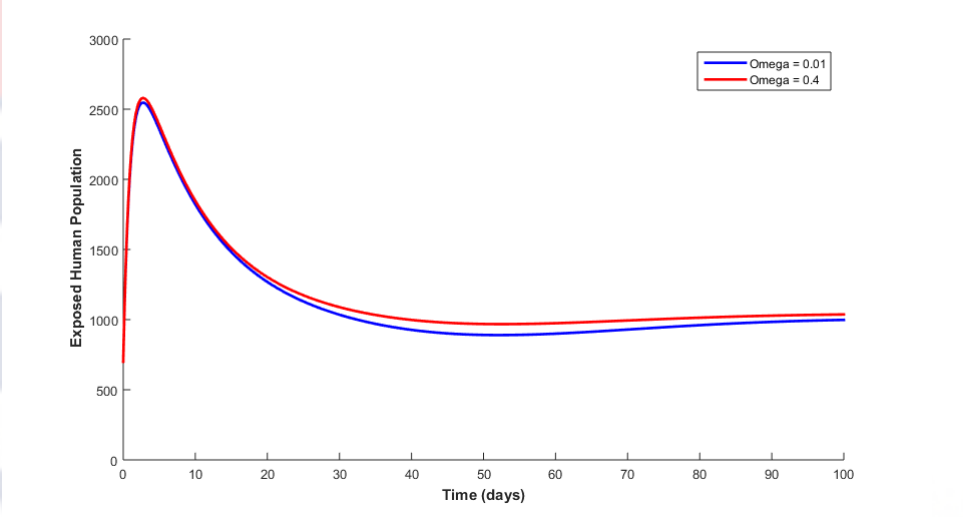

Figure 7: Plot of Exposed Human Populations Against Time Showing the Effect of Varying the Proportion of Exposed Immigrants($\omega = 0.1$, $\omega = 0.4$) with Total Number of Immigrants Say, $A = 100$. The Rest of the Parameters are Maintained as Stated in Table 3

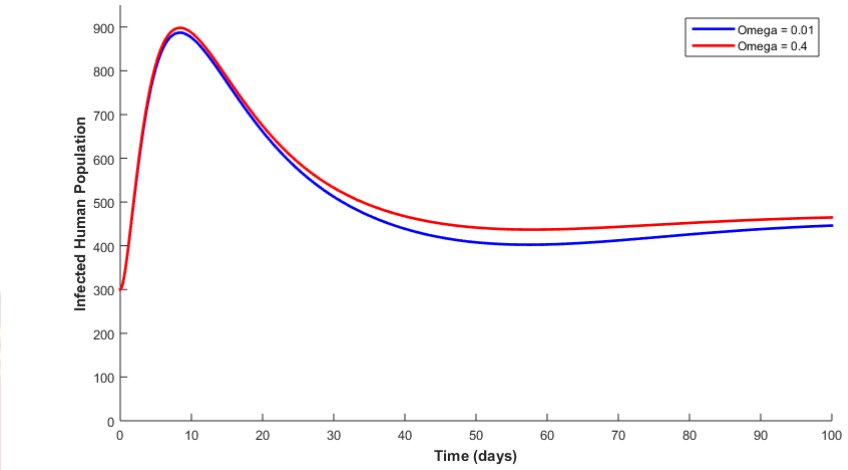

Figure 8: Plot of Infected Human Populations Against Time Showing the Effect of Varying the Proportion of Exposed Immigrants ($\omega = 0.1$, $\omega = 0.4$) with Total Number of Immigrants say, $A = 100$. The Rest of the Parameters are Maintained as Stated in Table 3

In Figure 7 and 8, we noticed that as we increase the proportion of exposed immigrants from $\omega = 0.1$ to $\omega = 0.4$ with a total number of immigrants say, $A = 100$, there is a corresponding increase in both exposed and infected human populations . This means that as the exposed immigrants come into the population, they join the exposed human population and subsequently progress to the infected human population. The simulation results in Figure 7 and 8 demonstrate the crucial role exposed immigrants play in the transmission and spreading of malaria in the population.

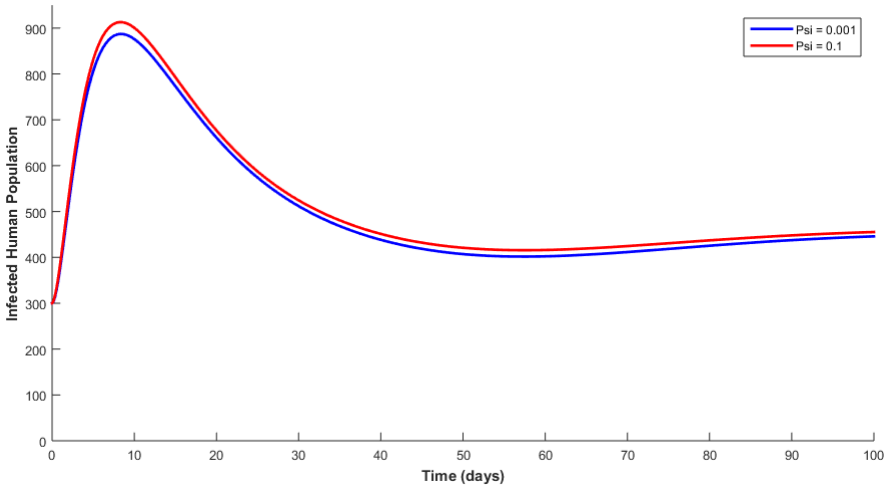

Figure 9: Plot of Infected Human Populations Against Time Showing the Effect of Varying the Proportion of Infected Immigrants($\psi = 0.01$, $\psi = 0.2$) with Total Number of Immigrants Say, $A = 100$. The Rest of the Parameters are Maintained as Stated in Table 3

It was observed in Figure 9 that as the proportion of infected immigrants, ψ increases from $\psi = 0.01$ and $\psi = 0.2$ with total number of immigrants say, $A = 100$, there is a corresponding increase in infected human population . This shows that that infected immigrants has impact on the spread of malaria since those immigrants would be bitten by mosquitoes.

Sensitivity Analysis on the Basic Reproduction Number

Sensitivity indices allow us to measure the relative change in the state variables when a parameter changes. Also, since the basic reproduction number $\Re_0$ is a function of the parameters β_h, β_m, Λ_h, Λ_m, ν, μ_h, μ_m, α_h, α_m, δ, and ρ , we can examine the relative sensitivity for all the parameters that $\Re_0$ depends on. We will not consider μ_h in our analysis. The normalized forward sensitivity index for $\Re_0$, which depends on a parameter, say m , denoted by $\prec_m^{\Re_0}$ is defined by

$$\prec_m^{\Re_0} = \frac{\partial \Re_0}{\partial m} \cdot \frac{m}{\Re_0}, \tag{4.1}$$

where m is a paraterer of interest. In order to determine how to lessen morbidity due to malaria, it is imperative we know the relative significance of distinct parameters that change transmission dynamics of malaria. Subsequently, using Equation (4.1) to evaluate the impact of each parameters on $\Re_0$, we obtained the following derivatives:

$$\frac{\partial \Re_0}{\partial \beta_h} \cdot \frac{\beta_h}{\Re_0} = \frac{\nu^2 \Lambda_h \beta_m \Lambda_m \alpha_m \alpha_h}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\beta_h}{\Re_o} \geq 0,$$

$$\frac{\partial \Re_0}{\partial \beta_m} \cdot \frac{\beta_m}{\Re_0} = \frac{\nu^2 \Lambda_h \beta_h \Lambda_m \alpha_m \alpha_h}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\beta_m}{\Re_0} \geq 0\,,$$

$$\frac{\partial \Re_0}{\partial \Lambda_m} \cdot \frac{\beta_m}{\Re_0} = \frac{\nu^2 \Lambda_h \beta_h \beta_m \alpha_m \alpha_h}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\Lambda_m}{\Re_0} \geq 0.$$

$$\frac{\partial \Re_0}{\partial \nu} \cdot \frac{\nu}{\Re_0} = \frac{2\nu \beta_h \Lambda_h \beta_m \Lambda_m \alpha_m \alpha_h}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\nu}{\Re_0} \geq 0,$$

$$\frac{\partial \Re_0}{\partial \alpha_h} \cdot \frac{\alpha_h}{\Re_0} = \frac{\nu^2 \beta_h \Lambda_h \beta_m \Lambda_m \alpha_m}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)(\mu_h+\alpha_h)^2(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\alpha_h}{\Re_0} \geq 0,$$

$$\frac{\partial \Re_0}{\partial \alpha_m} \cdot \frac{\alpha_m}{\Re_0} = \frac{\nu^2 \beta_h \Lambda_h \beta_m \Lambda_m \alpha_h}{2\mu_h \mu_m^2 (\mu_m+\alpha_m)^2(\mu_h+\alpha_h)(\mu_h+\delta+\rho)\Re_0} \cdot \frac{\alpha_m}{\Re_0} \geq 0.$$

The sensitivity indices of $\Re_0$ to the rest of the parameters can be computed

for in a similar fashion. The results are summarized in Table 4 below.

Table 4: Sensitivity Indices of $\Re_0$ to Parameters

Parameter	Relationship
β_h	$+$
β_m	$+$
Λ_m	$+$
μ_m	$-$
Λ_h	$+$
ν	$+$
α_h	$+$
α_m	$+$
δ	$-$
ρ	$-$

Source: Mojeeb et al. (2019)

Discussion

In this thesis, we developed a mathematical model of malaria with the inclusion of exposed and infected human immigrants that tracks the transmission dynamics of malaria. The model is examined for the existence of disease-free and endemic equilibrium points when there are no immigrants thus, $A = 0$. The basic reproduction number $\Re_0$ was computed. Also, we computed the unique endemic equilibrium when there are immigrants. We also performed numerical simulations using values obtained from published papers and others were assumed. Rigorous analysis of our model system show that due to the entering of exposed and infected immigrants, an equilibrium point with positive proportion of exposed and infected immigrants always exist in the population. In this way, our model can not have a disease-free state and has only unique endemic equilibrium point for which the disease exist for long time in the population. We notice that the basic reproduction number $\Re_0$ becomes irrelevant when there

are exposed and infected immigrants and hence can not be used in determining when malaria will fade out or become endemic in the population. However, when the proportion of exposed and infected immigrants approahes zero, the basic rereproduction number $\Re_0$ assumes its utility in malaria eradication and control.

Numerical results from the model simulations depict that when there are no immigrants and the basic reproduction number $\Re_0 = 0.2681$ which is less that unity, only the susceptible human and mosquito compartments exist while the rest of the compartments go to zero which confirmed disease-free state. Also, when $\Re_0 = 2.6809$ which is greater than unity, our model simulations show that all the distinct compartments coexist in the population which indicated the endemicity of malaria. We see from our numerical simulations that the entry of the expsosed and the infected immigrants increase both exposed and infected human populations, therefore we need to check the influx of immigrants to avoid malaria transmission. The inflow of exposed and infected immigrants do not alter the value of $\Re_0$ and so $\Re_0$ has no application in the transmission dynamics of malaria in the population.

We also performed sensitivity analysis on the parameters that the basic reproduction number $\Re_0$ depends on except human natural death rate μ_h. Our sensitivity analysis indicated that the most sensitive paramter is the mosquito biting rate ν while the less sensitive parameter is the human progression rate from exposed to infected α_h

Chapter Summary

In this chapter, numerical simulations of our model were performed with the use of values from literature and some assumed parameters for the transmission dynamics of malaria with infective immigrants. The results obtained were discussed in details. Matlab as a tool was used in the numerical simulations. It

was shown that when there are some proportion of exposed and infected immigrants, the basic reproduction number $\Re_0$ lost its usefulness in the application of malaria eradiction.

In addtion, sensitivity analysis of our model was perfomed which enable us to know the relationship the parameters have with the basic reproduction number $\Re_0$. It was found out that the parameters: β_h, β_m, Λ_h, Λ_m, ν, α_m and α_h have a positive relationship with $\Re_0$ whiles parameters : μ_m, δ and ρ have a negative relationship with $\Re_0$.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

Application of mathematical modelling as tool was employed in this thesis to analyze the transmission dynamics of malaria. Exposed and infected immigrants were incorporated into the human population to examine their impact on the malaria disease transmission. We have discussed both the analytical and numerical results of our model. We summarize the results obatined in CHAPTER THREE and FOUR below.

Summary

In this thesis we developed an $SEIR - SEI$ model of malaria transmission with influx of exposed and infected immigrants. We esblished that our model has a disease-free and endemic equilibrium points when there are no immigrants.. We employed the use of the next generation matrix to derive the basic reproduction number $\Re_0$, a threshold value that determines whether malaria will die out or become endemic in the population. We also showed that both the disease-free and the endemic equilibrium points are locally and globally asymptotically stable whenever $\Re_0$ is less than unity and greater than unity respectively. Also, when there are immigrants of which proportion of them are exposed and infected , unique endemic equilibrium point exist with the proportion of exposed and infected immigrants. However, it has been established that in the population where the are proportion of exposed and infected immigrants, attaining a disease-free status is impossible even if $\Re_0$ is less than unity. Hence, we can not rely on the value of $\Re_0$ to determine the transmission dynamics of malaria. Our numerical simulations confirmed the analytical results. That is, we can only have a disease free when the proportions of the exposed and infected

immigrants approaches zero, otherewise there will be no disease-free irrespective of the values of the basic reproduction number, $\Re_0$. The results of our study , conseqquently stipulate a framework that should be taken into account by the healthcare providers, stakeholders and policymakers when preparing policies to control and eradicate malaria transmission.

Conclusions

An $SEIR - SEI$ mathematical model of malaria with infective immigrants was developed. The model equations were obtained with the assistance of a flowchart diagram in Figur 2. The state variables and the parameters used in the model were displayed in Table 1 and 2 respectively. We prove that our model is mathematically well-posed and epidemiologically meaningful because all the model solutions were positive and bounded. In the analysis of our model, we looked at two cases; in the first case, we assumed that there were no immigrants $(A = 0)$, we calculated the basic reproduction number, $\Re_0$ and used it to determine local and global stabilities of the disease-free and endemic equilibrium points. In the second case, where there was a constant inflow of immigrants of which proportions are exposed and infected with mararia parasite ($0 < \omega < 1$ and $0 < \psi < 1$) entering the population, we found the unique endemic equilibrium point and its local and global stabilities were determined. We found out in the course of our analysis of the model that the basic reproduction number, $\Re_0$ becomes irrelevant due to the influx of exposed and infected immigrants.

Moreover, we solved our model numerically using ode45 in Matlab and the result from the numerical simulations show that exposed and infected immigrants make disease-free status unattainable even if the $\Re_0$ is less than unity. As a matter of fact, it should be noted that small migratory influx of exposed and infected immigrants plays an eminent role in the transmission dynamics of malaria.

In addtion to that, sensitivity analysis was performed on the parameters that the basic reproduction number, $\Re_0$ depends on and we observed that the most sensitive are ν, β_h, β_m, and Λ_m , these parameters need focus when employing measures to combat and eradicate malaria.

Recommendations

Sensitivity anslysis showed that the mosquito's biting rate, ν is the most sensitive parameter so activities (insecticides-treated nets and residual spray) that inhibit human-mosquito interaction should be intensified.

Human immigrants should be screen at our borders to be sure of malaria free before they are allowed to the country.

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