UNIVERSITY OF CAPE COAST

GENERALIZED AUTO-CORRELATION FUNCTION OF HIGHER ORDER ARMA PROCESSES: APPLICATION TO PANDEMIC DATA

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BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Gyamu-Atta Pius

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Bismark Kwao Nkansah

ABSTRACT

The Autocorrelation Function (ACF) of a time series process reveals the inherent characteristics of the series that may not be visible from the original series. The ACF of the ARMA(p, q) process has been presented in a few studies in understandably rigorous and laborious manner with no explicit form of the function. In this study, the approach of autocovariance generating functions (acvgf) is used to obtain an explicit expression for a series that follows a linear process under condition of distinct real roots of the AR(p) lag operator polynomial. The technique is used to derive ACF of processes as far as ARMA(3,0). The procedure has shown a clear connection among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. It is also observed that the Yule-Walker relation emerges after lag $(q + 2)$ for processes higher than ARMA(2,1). This means that there is the need for the computation of individual $\gamma(k)$ for $k \leq (q + 2)$. The derived approach is applied to daily new Covid-19 cases for three countries with stationary series, and are found to have different ARMA processes. The results are compared with those based on "ARIMAfit" function in R. In each case, the results of the two methods are found to be the same with damp exponential decay, an indication that the pandemic would cease eventually in these countries. The results provide useful relations that may be utilized as diagnostic tests for determining whether a given data follows a specified process.

KEY WORDS

Autocorrelation

Autocovariance generating function

Emperical

Linear Process

Pandemic

Theoretical

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DEDICATION

To my family and my supervisor, Prof. B.K. Nkansah

TABLE OF CONTENTS

CHAPTER THREE: METHODOLOGY

CHAPTER FOUR: DATA ANALYSES AND DISCUSSIONS

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

ACF	Autocorrelation Function
acv	Autocovariance
acvf	Autocovariance Function
agcvf	Autocovariance generating function
Y-W	Yule-Walker

CHAPTER ONE

INTRODUCTION

In studying a phenomenon, Wu and Wei (1989) reveal that we mostly come across datasets with observations taken according to the order of time. Undoubtedly, the evolution and use of time series data has received a lot of attention recently (Tingyan, 2010). A time series is a realization of a stochastic process. In practice, time series arise in areas such as Physical Science, Economics, Marketing, Demography, Process control, and Binary processes among others (Anderson, 1976).

In times series analysis, there are primarily two types of methodologies: frequency domain methods and time-domain methods. The frequency domain approach is based on an extension of the methods of Fourier analysis which originate in the idea that using a weighted sum of sine and cosine functions with harmonically increasing frequencies, any analytic function can be approximated to any level of precision over a finite interval. On the other hand, the time domain methods have their origin from the classical theory of correlation. Such methods primarily focus on the autocovariance function (acvf) and cross-covariance function (ccvf) of the series, and they result in the creation of structural or parametric models of the autoregressive moving average type for single series, and of the transfer-function type for two or more causally related series. This method describes the features of a time series process using time functions such as the autocorrelation function (ACF) and the partial autocorrelation function (PACF), whose dynamics are depicted through various time-lag relationships (Wu and Wei, 1989).

The ACF is the correlation between a time series and a lagged version of itself, while the PACF of a time series process at a particular lag k is the autocorrelation between X_t and X_{t-k} that have not been accounted for by lags 1 through lag $k-1$. Literature outlines the importance of the ACF in studying the dependent structure in a given time series process in addition to identifying inherent characteristics that may not be visible in the original time series observations. In view of this, some researches have looked at obtaining the theoretical ACFs of certain stationary time series process. It is however the case that the higher the order of the time series process, the arduous it is in obtaining the ACF.

The motivation for this study is to contribute to the literature on the derivation of the theoretical ACF of higher order stationary time series process through the autocovariance generating function and apply the derivations to a pandemic data.

Background to the Study

Time series statistical analyses are important in understanding the variability of the series data, identifying the regular and random fluctuations of the series over time, describing the features of these oscillations, and comprehending the physical processes underlying each of these oscillations. Time series processes can broadly be grouped into linear and non-linear. A general Linear Process (LP) is one that assumes that a data series is generated by a linear combination of random errors (Box et al., 2008). Thus, it is the result of a linear filter whose input is a white noise, Z_t. Expressed mathematically,

$$X_t = \sum_{j=0}^{\infty} \Psi_j Z_{t-j} \tag{1.1}$$

where Ψ_j is a series of constants. For X_t to depict a valid stationary process, it is essential that the coefficient Ψ_j be absolutely summable; that is, $\sum_{j=0}^{\infty} |\Psi_j| < \infty$. The white noise process consists of a series of random variables without any correlation. These random variables have zero mean and constant variance, σ^2. Since the random variables Z_t are considered to be uncorrelated, it implies that their autocovariance function is a step function given as

$$\gamma(k) = \mathrm{E}\Big(Z_t \cdot Z_{t+k}\Big) = \begin{cases} \sigma^2, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{1.2}$$

Hence, Z_t has an autocorrelation function given as

$$\rho(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{1.3}$$

Linear time series theory uses three main different model types: Autoregressive of order p (AR(p)), Moving Average of order q (MA(q)), and combined AR and MA called ARMA of orders p,q (ARMA(p, q)). The AR(p) model depicts a linear regression relationship between a series' present value and one or more previous values. The MA(q) model is a regression analysis of the series' current value versus its random shocks. It is assumed that the random errors at each point come from an identical distribution, generally a normal distribution with a zero mean and a steady finite variance. Generally, ARMA models are used when the observations in a given series are stationary. In cases where the data is not stationary, ARMA models can be extended to Autoregressive Integrated Moving Average (ARIMA) of orders p, d, q. In a case where the series is dominated by seasonal effects, ARMA models are yet extended to obtain a Seasonal Autoregressive Integrated Moving Average (SARIMA) models of orders (p,d,q)$\times$ (P,D,Q)$_s$.

Far from linear processes, non-linear time series are generated by nonlinear dynamic equations. Thus, they have features that cannot be modeled by linear processes. Although linear processes are appropriate for describing many real-life phenomena, they do not capture some of the features of time series which have periods of high and low volatility. This indicates that in empirical scenarios with more complicated time series, linear models are unable to cover all of the information. As a result, methods for representing variations in variance over time, commonly known as heteroskedasticity, are introduced. These models are known as Autoregressive Conditional Heteroskedasticity (ARCH) and the collection of this model class has a variety of representations, such as the Generalized AutoRegressive Conditional Heteroskedasticity (GARCH), Exponential Generalized AutoRegressive Conditional Heteroskedasticity (EGARCH), Fractionally Integrated Generalized AutoRegressive Conditional Heteroskedasticity (FIGARCH), among others. These ARCH model classes have been widely utilized in forecasting and predicting various time series data such as inflation, stock prices, exchange rates, and interest rates.

The concept of Correlation and Autocorrelation

Understanding different classes of models in time series analysis depends greatly on correlation. The concept of correlation is generalized to autocorrelation, which is the basic tool for studying a stationary time series. Time series data are much more likely to show some dependence over time than cross sectional data, which makes sense to believe that observations are independent from one another. According to Schlittgen et al. (2008), covariance and correlation are concepts that can be used to evaluate the lack of independence between two adjacent data values, x_s and x_t. The autocorrelation function receives its name by being an extension of the statistical correlation measure between two random variables. In time series analysis, the ACF can either be obtained from the sample data or from the parameter values based on the appropriate model that characterizes the series. ACFs obtained from parameter values are known as the theoretical ACF, while ACFs obtained from the sample data are known as the sample or emperical ACF. In practice, the sample ACF relates directly to the classical correlation.

Assume x and y are two random variables, each with n observations. Then the Pearson correlation between them will be given by

$$\rho = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} \tag{1.4}$$

In the autocorrelation sense, the correlation is computed between one time series and the same series lagged by one or more time units. The first-order autocorrelation coefficient is the simple correlation coefficient between the first $n-1$ observations of x_t where $t = 1, 2, \cdots, n-1$ and the next $n-1$ observations of x_t where $t = 2, 3, \cdots, n$. The correlation between x_t and x_{t-1} is given by

$$\rho_1 = \frac{\sum_{t=1}^{n-1}(x_t - \bar{x}_{(1)})(x_{t-1} - \bar{x}_{(2)})}{\sqrt{\sum_{t=1}^{n-1}(x_t - \bar{x}_{(1)})^2}\sqrt{\sum_{t=1}^{n-1}(x_t - \bar{x}_{(2)})^2}} \tag{1.5}$$

In Equation (1.5), $\bar{x}_{(1)}$ is the mean of the first $n-1$ observations and $\bar{x}_{(2)}$ is the mean of the last $n-1$ observations. For situations where n is reasonably large ($n \geq 30$ according to the Central Limit Theory), the difference between the sub-period means $\bar{x}_{(1)}$ and $\bar{x}_{(2)}$ can be ignored, and Equation (1.5) can be approximated as

$$\rho_1 = \frac{\sum_{t=1}^{n-1}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \tag{1.6}$$

where $\bar{x}$ is the overall mean. Equation (1.6) can be generalized to give the correlation between observations separated by k time steps as

$$\rho_k = \frac{\sum_{t=1}^{n-k}(x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \tag{1.7}$$

ρ_k is the sample ACF at lag k which provides an indication of the degree to which the shift of the time series at one time relates to or can be inferred from its shift at another time.

In general, one can obtain the autocorrelation by first going through the associated autocovariance. The autocovariance between X_t and X_{t-k} is represented as

$$\gamma(t, t+k) = \text{cov}\Big(X_t, X_{t+k}\Big)$$

If the covariance structure is stable, the covariance depends on k but not on t. That is,

$$\gamma(t, t+k) = \gamma(k)$$

If $k = 0$, then

$$\gamma(0) = \text{cov}\Big(X_t, X_t\Big) = \text{var}(X_t) < \infty$$

The acvf and the ACF are related in a way that the correlation function is a normalized covariance function. For a continuous time series process, the autocovariance between two random variables $X_1 = X_t$ and $X_2 = X_{t+k}$ is given as

$$\begin{aligned} R_{XX}(t, t+k) =& \text{E}\Big[X_1 \cdot X_2\Big] \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2) dx_1 dx_2 \end{aligned} \tag{1.8}$$

Similarly autocovariance between two random variables of a discrete time series process is given as

$$R_{XX}(t, t+k) = \text{E}\Big[X_t \cdot X_{t+k}\Big] \qquad k = \cdots, -2, -1, 0, 1, 2, \cdots \tag{1.9}$$

The pertinent characteristics of the ACF are:

1. It is an even function.

$$R_{XX}(k) = R_{XX}(-k)$$

2. It has its maximum value when $k = 0$. Nevertheless, this value can emerge again, for example, at the values of the analogous points in a pe-

riodic function, but it will never be exceeded. Mathematically,

$$R_{XX}(0) \geq |R_{XX}(k)|$$

In addition to the advantages of the ACF mentioned earlier, Haag (2005) add that the ACF provides information on how quickly a time series process changes with respect to the time function, and an appealing knowledge on whether a time series process has a periodic component and what the expected wave might be. It also plays a vital role for statistical inference in time series analysis by providing a vital knowledge of the Moving Average (MA) order.

The next section examines the autocovariance generating function (acvgf) which forms the basis for obtaining the ACFs of higher order ARMA processes in this thesis.

Autocovariance Generating Function

According to Chattamvelli and Shanmugam (2023), a generating function (gf) is a short and simple formula in one or more dummy variables that summarizes the coefficients of a finite or infinite sequence and generates a quantity of interest using calculus or algebra. Generating functions have many useful applications. This include their use in establishing asymptotic formular for the terms of a sequence, solving recurrence relations, and proving combinatorial identities. For a stationary time series process X_t, the sequence of autocovariances γ_k, for $k = 0, 1, \cdots$ can be calculated. According to Hamilton (2020), if the sequence is absolutely summable, then the acvgf is one technique to summarize the autocovariances. The acvgf is represented as

$$c(s) = \sum_{k=-\infty}^{\infty} \gamma(k) s^k \tag{1.10}$$

The function is formed by taking the k^{th} autocovariane and multiplying it by some coefficient s raised to the k^{th} power, and then summing over all the possible values of k. This implies that the variance of the process, $\gamma(0)$, is the coefficient of $s^0 = 1$, whereas the acv of the process at lag k is the coefficient of s^k.

The Covid-19 Pandemic

This section seeks to give a brief on the Covid-19 pandemic. This will be helpful in obtaining an appreciable knowledge about the nature of the disease, its effects and some implementations taken by countries across some parts of the globe. This will aid in the discussion section of this thesis, since application of the derived theoretical ACFs would be made to the pandemic in some selected countries.

Various contagious viral infections and pandemics such as influenza, Zika, Middle East Respiratory Syndrome(MERS), Spanish flu, and Ebola all emerged in the past, which badly affected human lives and economy of the major areas and regions of the world (Khan et al., 2021). At the latter part of 2019, a new coronavirus, SARS-CoV-2, called Corona-virus Disease 2019 (Covid-19) emerged in Wuhan city, China (Zeroual et al., 2020). The primary mode of transmission of the virus is through droplets of saliva or discharge from the nose when an infected person coughs or sneezes. Within a few months, the disease had rapidly spread over all the world. By this, the outbreak was officially declared a Public Health Emergency of International concern by the World Health Organization (WHO) on January 30, 2020, and a global pandemic on March 11, 2020 (Hiscott et al., 2020). Onyema et al. (2020) revealed that the outbreak of the disease affected all aspects of human activities globally ranging from education, entertainment, transportation, worship, social gathering, business, politics and economy. In response, many countries implemented measures such as frequent hand washing and sanitizing, mandatory wearing of nose masks, social distancing, self-isolation, and partial lock-downs to prevent further spread, which proved crucial in maintaining health services to patients most in need of care either for Covid-19 or for other various conditions (Papastefanopoulos et al., 2020). Considering the adversities the disease brought, several researches were carried out to explore the nature of the pandemic, and to make appropriate forecast of the daily new infections and deaths. Petropoulos et al. (2020) note that forecasting the outcome of outbreaks as early and as accurately as possible is crucial for decision making and policy implementations.

Statement of the Problem

The Autocorrelation Function (ACF) plays a major role in time series analysis by identifying inherent characteristics that may not be visible in the original time series observations. In view of this, the literature abounds with the computation of ACF for certain stationary time series processes. Precisely, ACFs are derived (Box et al., 1970; Ma and Genton, 2000; McLeod et al., 1975; Muth, 1978) for lower orders of stationary ARMA models, and a few have focused on higher orders. The presentations of higher order ACFs of the ARMA(p, q) process have been very complex and could be much more depending on the approach used. The problem besides the complexity of the few higher order ACFs is that it is difficult to generalize the procedure by the approach used. It will be apparent from this thesis that the computation of ACF of the process under consideration is an arduous one. This study therefore makes an attempt to explore another alternative that could reduce or possibly eliminate the identified problems (complexity and lack of generalizability).

In the context of the Covid-19 pandemic, models that have made use of the data for certain countries around the globe have focused on predictions which are likely to face challenges with the irregular nature of the wavelike pattern of the pandemic.

Objectives of the Study

The ultimate objective of the study is to contribute to the literature on the derivation of the theoretical autocorrelation functions of higher order stationary time series processes. The study seeks to obtain a generalized autocorrelation function using the autocovariance generating function. The approach is underlined by rigorous mathematical bases to obtain the ACFs of ARMA(1,q), ARMA(2,q), and ARMA(3,q) processes, after which a generalization to ARMA(p, q) will be made.

The study is guided by the following specific objectives:

1. To present an alternative approach for deriving the ACFs for higher orders of ARMA(p,q) processes that ensures generalizability
2. To derive exact analytical expressions for the ACF of specific ARMA processes
3. Use the ACF of the derived ARMA models to deduce the ACFs of specific lower ARMA models
4. Use the derived ACFs to approximate the characteristics of a selected pandemic data

Significance of the Study

ACFs are usually generated from an observed time series data. Functional expressions for the theoretical ACFs are given in the literature for lower orders of AR, MA, and ARMA processes with a few on higher order ACFs, which are not without problems. This is as a result of the complex nature of the ACFs for higher order ARMA(p, q) processes. This study will therefore establish general expressions for higher orders of the ARMA. It will be possible therefore to obtain the ACF for the lower order ARMA processes when the expression for the general ARMA(p, q) is known. The result of this study will deepen understanding of identifying inherent characteristics of an observed time series that follows a linear process. The application of this study will determine the real characteristics of the parameters of the Covid-19 pandemic around some parts of the globe.

Scope and Delimitation of the Study

The study is focused on deriving a generalized ACF of an ARMA(p, q) process using the acvgf. The study will examine the ACFs of the daily new covid-19 cases in some selected countries around the globe. Based on our derived expressions, the ACFs of the daily new Covid-19 cases for the selected countries will be tested. Various comparisons will be made, after which inferences will be drawn. Although time series, the study is not centered on forecasting based on the appropriate models that will be obtained for the selected countries' daily new Covid-19 cases.

Organization of the Study

This study is organized into five chapters: Chapter One covers the introductory part of the study, the problem statement, objectives, and significance, as well as the scope and delimitations of the study. Chapter Two presents the literature review which focuses on works by previous researchers in obtaining the theoritcal ACF of stationary time series processes. Chapter Three reviews some methods employed to accomplish the purpose of the study. Chapters Four and Five, respectively, look at data analysis and discussion on one breadth and summary, conclusion and and suggestions for further studies on the other. The references are also presented in the end matter.

Chapter Summary

In this chapter, the nature and importance of the autocorrelation function of a time series process has been established. Difference between the emperical ACF and the theoretical ACF has also been noted. The chapter has also presented a brief on the autocovariance generating function, a function which will be the basis of the derivation of the theoretical ACF in this study. The arduous process in obtaining the theoretical ACFs of higher order ARMA(p, q) process has been recognized, and the relevance of deriving the theoretical ACF through the autocovariance generating function has been highlighted. Notably, it has established how the results of the study will deepen understanding of identifying inherent characteristics of a time series that follows a linear process.

As an application to the Covid-19 pandemic, the chapter recounts how previous investigations have used mathematical, statistical, and deep machine learning procedures to model and forecast Covid-19 cases across the world. Due to the changing waves of the pandemic however, the chapter points out how good our study will be in examining the real charateristics of the parameters of the covid-19 pandemic around some parts of the globe.

CHAPTER TWO

LITERATURE REVIEW

Introduction

The ultimatum of this research is to obtain a generalized autocorrelation fuctions of higher order stationary time series processes using the autocovariance generating function. This chapter examines studies on the autocovariance and autocorrelation functions of stationary time series processes. In the first section, reviews of the ACFs of lower order ARMA(p, q) processes are examined. The second and final section of this chapter reviews the ACFs of higher order ARMA(p, q) processes.

ACFs of Lower Order ARMA(p, q) Processes

Box et al. (1994) present the ACF of a stationary Moving Average process of order 1. In their study, the MA(1) process is denoted by

$$Z_t = a_t - \theta a_{t-1} = (1 - \theta B)a_t \tag{2.1}$$

where B in Equation (2.1) is the backshift operator, Z_t is the time series process at time t, and a_t and a_{t-1} is the Moving Average Process of order 1. It is noted that $\psi(B) = 1 - \theta B$. Additionally, the study considers the acvgf and denotes it as

$$\gamma(B) = \sigma_a^2 \psi(B)\psi(B^{-1}) \tag{2.2}$$

where $\psi(\cdot)$ is a rational expression explained in Chapter 3 (see Equation 3.19). Then by substituting Equation (2.1) into (2.2) and making comparisons, the variance and the first acv of the MA(1) process is obtained respectively as

$$\gamma_0 = \sigma_a^2(1 + \theta^2) \tag{2.3}$$

and

$$\gamma_1 = \sigma_a^2\theta \tag{2.4}$$

which are influenced directly and in a very simple way by the coefficient of the MA component. It is found that the autocovariance at lag 2 and beyond is equal to zero. By normalizing each autocovariance, the autocorrelations are obtained as

$$\rho_k = \begin{cases} 1, & k = 0 \\ \frac{\theta}{1+\theta^2}, & k = 1 \\ 0, & k \geq 2 \end{cases} \tag{2.5}$$

Yaffee and McGee (2000) present the ACFs of a stationary AR(1) process. The process is represented as

$$y_t = \varphi y_{t-1} + e_t \tag{2.6}$$

It is revealed that to obtain the autocovariance at a particular lag say k, where k is an integer, y_t is multiplied by y_{t-k} after which expectation is taken. The results of the autocovariance obtained after the deductions were summarized as

$$\gamma_k = \begin{cases} \frac{\sigma^2}{1-\varphi}, & k = 0 \\ \varphi^k\gamma(0), & k \geq 1 \end{cases} \tag{2.7}$$

After normalization, the ACF of the AR(1) process is summarized as

$$\rho_k = \begin{cases} 1, & k = 0 \\ \varphi^k, & k \geq 1 \end{cases} \tag{2.8}$$

Triacca (2016) presents the autocovariance function of an ARMA$(1,1)$ process. The causal ARMA(1,1) process is represented as

$$x_t - \phi x_{t-1} = \mu_t + \theta\mu_{t-1} \tag{2.9}$$

where $x.$ represents the Autoregressive component and the $\mu.$ is the Moving Average component. Equation (2.9) is represented in the form

$$x_t = \sum_{j=0}^{\infty} \psi_j \mu_{t-j} \tag{2.10}$$

after which the sequence $\{\psi_0, \psi_1, \cdots\}$ are determined by the relation $(1-\phi_1 z - \cdots - \phi_p z^p)(\psi_0 + \psi_1 z + \cdots) = 1 + \theta_1 z + \cdots + \theta_q z^q$. It is found that $\psi_0 = 1$ and $\psi_j = (\phi+\theta)\phi^{j-1}$ for $j \geq 1$. The variance for the ARMA(1,1) process is deduced afterwards as

$$\gamma_0 = \sigma^2\left[1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right] \tag{2.11}$$

The autocovariance at lag 1 is subsequently found to be

$$\gamma_1 = \sigma^2\left[\phi + \theta + \frac{(\phi+\theta)^2\phi}{1-\phi^2}\right] \tag{2.12}$$

It is revealed that

$$\gamma_k = \phi^{k-1}\gamma(1), k \geq 2 \tag{2.13}$$

A normalization of the autocovariance at each lag gives the respective autocorrelation.

ACFs of Higher Order ARMA(p,q) Processes

Exploration of the literature reveals a ground-breaking work of McLeod et al. (1975) that presents a method for deriving the theoretical autocovarince function of an ARMA model. The derivations helped in obtaining an algorithm suitable for machine computation of the theoretical ACF. The study considers a stationary ARMA(p, q) model given as

$$z_t - \phi_1 z_{t-1} - \cdots - \phi_p z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \tag{2.14}$$

and notes that E(a_t) $= 0$, E(a_t^2) $= \sigma_a^2$ and E($a_t a_s$) $= 0, t \neq s$. The study reviews Box et al. (1970) that the autocovariance function at lag k of Equation (2.14) is given as

$$\gamma_k - \phi_1 \gamma_{k-1} - \cdots - \phi_p \gamma_{k-p} = \gamma_{za}(k) - \theta_1 \gamma_{za}(k-1) - \cdots - \theta_q \gamma_{za}(k-q) \tag{2.15}$$

where $\gamma_k =$ E($z_{t-k} z_t$) and $\gamma_{za}(k) =$ E($z_{t-k} a_t$). Multiplying Equation (2.14) by a_{t-k} and taking expectations,

$$\gamma_{za}(-k) - \phi_1 \gamma_{za}(-k+1) - \cdots - \phi_p \gamma_{za}(-k+p) = -\theta_k \sigma_a^2 \tag{2.16}$$

is obtained. It is noted that in Equation (2.16),

$$[\theta_k] = \begin{cases} \theta_k, & k = 1, \cdots, q \\ -1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

and $\gamma_{za}(k) = 0$ if $k > 0$. If $k > r = \max(p, q)$, Equation (2.16) may be used to calculate γ_k directly from previous values by solving a system of linear equations. The algorithm is underlined by three procedures:

i. Set $\phi_0 = \theta_0 = -1$,$c_0 = 1$ and

$$c_k = -\theta_k + \sum_{i=1}^{\min(p,k)} \phi_i c_{k-i}$$

for $k = 1, \cdots, q$

ii. Set

$$b_k = \sum_{i-k}^{q} \theta_i c_{i-k}$$

for $k = 0, \cdots, q$ and $b_k = 0$ if $k > q$

iii. If $p = 0$, $\gamma_k = b_k \sigma_a^2$, $k = 0, \cdots, q$. If $p > 0$, solve the equations $Ax = y$ where

$$A_{ij} = \begin{cases} [\phi_{i-1}], & j = 1; i = 1, \cdots, r+1, \\ [\phi_{i-j}] + [\phi_{i+j-2}], & j = 2, \cdots, r+1; i = 1, \cdots, r+1 \end{cases}$$

$$[\phi_k] = \begin{cases} \phi_k, & k = 0, 1, \cdots, p \\ 0, & \text{otherwise} \end{cases}$$

$$y_i = -b_{i-1}\sigma_a^2$$

and then set $\gamma_k = x_{k+1}, k = 0, \cdots, r$.

Muth (1978) presents a study on the autocovariance function determined via the z-transform. Special references are made in the paper to the Box-Jenkins forecasting approach, whose underlying process is produced by running a white noise through a linear filter. The equation that characterizes the filter yields the impulse response of the filter as well as its z-transform. The autocovariance function's bilateral z-transforms are then generated from the transfer function and inverted after a partial fraction expansion. From a review of Box et al. (1970), the study states that if X_n is an ARMA(p, q) process, then the input-output relationship is given as

$$X_n - \phi_1 X_{n-1} - \cdots - \phi_p X_{n-p} = a_n - \theta_1 a_{n-1} - \cdots - \theta_q a_{n-q} \tag{2.17}$$

Equation (2.17) is denoted in terms of the impulse response ψ_n of the filter as

$$X_n = \psi_0 a_n + \psi_1 a_{n-1} + \psi_2 a_{n-2} + \cdots \tag{2.18}$$

$$= \psi_n * a_n \tag{2.19}$$

where n represents a time point and $*$ represents convolution. It is noted also that the impulse response is 0 for $n < 0$. Furthermore, the study represents the bilateral $z-$ transform as

$$\bar{X}(z) = \sum_{n=-\infty}^{\infty} X_n z^{-n} \tag{2.20}$$

and $\bar{a}(z)$ and $\bar{\psi}(z)$ are defined analogously. Equation (2.18) is then transformed in accordance with Equation (2.20), and with the convolution rule, becomes

$$\bar{X}(z) = \bar{\psi}(z)(\bar{a})(z) \tag{2.21}$$

The autocovariance functions of a_n and X_n are then respectively denoted as μ_k and γ_k. Thus for lag k,

$$\mu_k = \text{cov}(a_n, a_{n+k})$$

$$\gamma_k = \text{cov}(X_n, X_{n+k})$$

γ_k was represented as an output of a linear filter with impulse response g_k whose input is μ_k. Thus,

$$\gamma_k = g_k * \mu_k \tag{2.22}$$

and in the transform domain,

$$\bar{\gamma}(z) = \bar{g}(z)\bar{\mu}(z) \tag{2.23}$$

Significantly, the g_k was obtained from ψ_k as

$$g_k = \psi_k * \psi_{-k} \tag{2.24}$$

and the transform of this relation was obtained as

$$\bar{g}(z) = \bar{\psi}(z)\bar{\psi}(\frac{1}{z}) \tag{2.25}$$

where $\bar{\psi}(\frac{1}{z})$ is the transform of ψ_{-k}.
By letting $\bar{\psi}(z) = \frac{N(z)}{D(z)}$ and $\bar{\psi}(\frac{1}{z}) = \frac{P(z)}{Q(z)}$,

$$\bar{g}(z) = \frac{N(z)P(z)}{D(z)Q(z)} \tag{2.26}$$

Equation (2.26) is then divided by z and a partial fraction expansion is performed on $\bar{g}(z)/z$. The expansion is then multiplied by z and inverted to obtain the autocovariance function. The results from the approach were used to obtain the autocovariances of certain ARMA(p, q) process. One major drawback of this method was that for partial fraction of cases where the degree of the numerator was higher than the degree of the denominator, the autocovariances were quite arduous to obtain. Specifically, the autocovariances were obtained for ARMA(p, q) processes where $1 \leq p \leq 2$ and $1 \leq q \leq 3$, but no generalizations were made for higher order ARMA processes.

Karanasos (1998) reveals a new approach for obtaining the theoretical acvf of a univariate ARMA model. In the paper, a closed-form solution of the autocovariance is obtained in terms of the roots of the AR polynomial, and the parameters of the moving average part. The acvf of an ARMA$(1, q)$ is derived, after which mathematical induction is used to obtain the acvf of the general ARMA(p, q) process.

The paper represents the ARMA$(1, q)$ process as

$$z_t = \phi_2 z_{t-1} - \sum_{i=0}^{q} \theta_i \epsilon_{t-i} \tag{2.27}$$

where $\theta_0 = -1$ and $\epsilon \sim$ i.i.d$(0, \sigma^2)$. The study showed that

$$\text{cov}(z_t) = \phi_2^j \text{var}(z_t) + \sum_{i=1}^{\min(j,q)} a_i \phi_2^{j-i} \tag{2.28}$$

where

$$a_i = -\sum_{m=0}^{q-i} \theta_{i+m} \phi_2^m + \sum_{n=0}^{q-1-i} \sum_{k=1}^{q-i-n} \theta_k \theta_{n+i+k} \phi_2^n \tag{2.29}$$

It is noted that whenever $i = q$, the preceding double summation vanishes because the lower limit exceeds the upper limit of the summation operator. Specifically, the variance of $z_t(\gamma(0))$ was shown to be obtained from

$$(1 - \phi_2^2)\text{var}(z_t) = \sum_{m=0}^{q} \theta_m^2 + 2\phi_2 \Big(-\sum_{n=1}^{q} \phi_2^{n-1} \theta_n + \sum_{l=1}^{q-1} \sum_{k=1}^{q-l} \theta_k \theta_{k+l} \phi_2^{l-1} \Big) \tag{2.30}$$

Autocovariances of the process at lags other than zero were obtained as

$$\text{cov}_j(z_t) = \begin{cases} \frac{\phi_2^j}{1-\phi_2^2} \lambda_{2j}, & j \leq q-1 \\ \frac{\phi_2^j}{1-\phi_2^2} \lambda_{2q}, & j \geq q \end{cases} \tag{2.31}$$

where

$$\lambda_{2j} = \sum_{i=0}^{q} \theta_i^2 + \sum_{l=1}^{j} \sum_{k=0}^{q-l} \theta_k \theta_{k+l} (\phi_2^l + \phi_2^{-l}) + \sum_{l=j+1}^{q} \sum_{k=0}^{q-l} \theta_k \theta_{k+l} (\phi_2^l + \phi_2^{l-2j}) \tag{2.32}$$

By obtaining the acvf of the ARMA$(1, q)$ process, it is then assumed that if the theorem holds for an ARMA$(p-1, q)$ process, then it will be enough to prove that it holds for an ARMA(p, q) process. The ARMA(p, q) process denoted as y_t is written as an AR(1) process with an ARMA$(p-1, q)$ error term ($y_t =$

$\phi_1 y_{t-1} + z_t$), where z_t is an ARMA($p-1, q$) process given by

$$\prod_{i=2}^{p}(1-\phi_i L)z_t = -\sum_{i=0}^{q}\theta_i \epsilon_{t-i} \tag{2.33}$$

It is noted that since z_t is an ARMA($p-1, q$) process, its autocovariance is represented as

$$\text{cov}_j(z_t) = \begin{cases} \sum_{i=2}^{p} \hat{e_{ij}}\lambda_{ij} = \sum_{i=2}^{p} \hat{e_{ij}}(\lambda_{iq} + \nu_{ij}), & 0 \leq j \leq q-1 \\ \sum_{i=2}^{p} \hat{e_{ij}}\lambda_{iq}, & j \geq q \end{cases} \tag{2.34}$$

After some rigorous mathematics, the autocovariance function of an ARMA(p, q) process is obtained as

$$\text{cov}_j(y_t) = \begin{cases} \sum_{i=1}^{p} e_{ij}\lambda_{ij}, & j \leq q-1 \\ \sum_{i=1}^{p} e_{ij}\lambda_{iq}, & j \geq q \end{cases} \tag{2.35}$$

where

$$e_{ij} = \frac{\phi_i^j \phi_i^{p-1}}{\prod_{l=1}^{p}(1-\phi_l\phi_i)\prod_{k=1,k\neq i}^{p}(\phi_i - \phi_k)} \tag{2.36}$$

and

$$\lambda_{ij} = \sum_{k=0}^{q}\theta_k^2 + \sum_{l=1}^{j}\sum_{k=0}^{q-l}\theta_k\theta_{k+l}(\phi_i^l + \phi_i^{-l}) + \sum_{l=j+1}^{q}\sum_{k=0}^{q-l}\theta_k\theta_{k+l}(\phi_i^l + \phi_i^{l-2j}) \tag{2.37}$$

It is again noted that for $j = q$, the third term of the right-hand side of the Equation (2.37) disappears because the lower limit exceeds the upper limit of the first summation operator. Although the approach is excellent, it is unable to clearly establish the relationships among the autocovariance at the various lags.

Chapter Summary

The literature shows that the derivations of theoretical ACF of higher order ARMA processes have not been left unattempted. However, there has not been so many extensive development of the concept. While this may be due to the ease of access to the autocorrelations in some mathematical software based on McLeod's algorithm, another reason why the theoretical ACF of higher order ARMA processes have not caught the attention of many could be due to the arduous nature of the derivations. It has been seen that the theoretical ACFs of higher order ARMA process are underlined by rigorous mathematical bases, which in most cases, have resulted in the inability to generalize.

CHAPTER THREE

METHODOLOGY

Introduction

The methodology for this study is focused on the mathematical bases for obtaining the theoretical autocorrelation of a stationary time series process. Specifically, the chapter will examine how the Yule-Walker approach, the comparison of coefficients approach, and the autocovariance generating function aid in obtaining the ACF of an ARMA process. In the final section of this chapter, an attention is directed to the data characterization of the Covid-19 cases used as an application to the study, and the software that will aid in the data analyses process.

Autocorrelation Functions through the Yule-Walker Approach

The Yule-Walker equations, developed by George Udny Yule and Gilbert Walker are a set of equations for obtaining the autocovariances and autocorrelations for stationary AR(p) processes. The general AR(p) process is given by

$$\begin{aligned} X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \cdots + \phi_{p-1} X_{t-(p-1)} + \phi_p X_{t-p} + \varepsilon_t \\ &= \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t \end{aligned} \tag{3.1}$$

where t and j are the time and term indices respectively.

The autocovariance at various lags of the process is computed as follows:

At lag 0, both sides of Equation(3.1) are multiplied by X_t to obtain

$$X_t \cdot X_t = \Big(\sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t \Big) \cdot X_t \tag{3.2}$$

Taking expectation on both sides gives

$$\begin{aligned} \mathrm{E}\Big(X_t \cdot X_t\Big) =& \mathrm{E}\Big(\sum_{j=1}^{p} \phi_j X_{t-j} \cdot X_t\Big) + \mathrm{E}\Big(\varepsilon_t \cdot X_t\Big) \\ \mathrm{E}\Big(X_t \cdot X_t\Big) =& \sum_{j=1}^{p} \phi_j \mathrm{E}\Big(X_{t-j} \cdot X_t\Big) + \sigma_\varepsilon^2 \end{aligned} \tag{3.3}$$

Thus, the variance of an AR(p) process is given as

$$\gamma(0) = \sum_{j=1}^{p} \phi_j \gamma(j) + \sigma_\varepsilon^2 \tag{3.4}$$

Similarly, the autocovariance at lag 1 is obtained by multiplying both sides of Equation(3.1) by X_{t-1}. This gives

$$X_t \cdot X_{t-1} = \Big(\sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t\Big) \cdot X_{t-1} \tag{3.5}$$

Taking expectation on both sides

$$\mathrm{E}\Big(X_t \cdot X_{t-1}\Big) = \mathrm{E}\Big(\sum_{j=1}^{p} \phi_j X_{t-j} \cdot X_{t-1}\Big) + \mathrm{E}\Big(\varepsilon_t \cdot X_{t-1}\Big) \tag{3.6}$$

Since the random error (ε_t) of the current time is uncorrelated with the previous values of the process, $\mathrm{E}\Big(\varepsilon_t \cdot X_{t-1}\Big) = 0$. Equation (3.6) simplifies to

$$\begin{aligned} \mathrm{E}\Big(X_t \cdot X_{t-1}\Big) &= \sum_{j=1}^{p} \phi_j \mathrm{E}\Big(X_{t-j} \cdot X_{t-1}\Big) \\ \gamma(1) &= \sum_{j=1}^{p} \phi_j \gamma(j-1) \end{aligned}$$

Subsequently, at lag $p-1$, we multiply both sides of Equation (3.1) by $X_{t-(p-1)}$ to obtain

$$X_t \cdot X_{t-(p-1)} = \Big(\sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t\Big) \cdot X_{t-(p-1)} \tag{3.7}$$

Taking expectation on both sides

$$\mathrm{E}\Big(X_t \cdot X_{t-(p-1)}\Big) = \mathrm{E}\Big(\sum_{j=1}^{p} \phi_j X_{t-j} \cdot X_{t-(p-1)}\Big) + \mathrm{E}\Big(\varepsilon_t \cdot X_{t-(p-1)}\Big) \tag{3.8}$$

$$\begin{aligned} \mathrm{E}\Big(X_t \cdot X_{t-(p-1)}\Big) &= \sum_{j=1}^{p} \phi_j \mathrm{E}\Big(X_{t-j} \cdot X_{t-(p-1)}\Big) \\ \gamma(p-1) &= \sum_{j=1}^{p} \phi_j \gamma(j-p+1) \end{aligned}$$

Similarly, at lag p, we multiply both sides of Equation (3.1) by X_{t-p} to obtain

$$X_t \cdot X_{t-p} = \Big(\sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t\Big) \cdot X_{t-p} \tag{3.9}$$

Taking expectation on both sides

$$\mathrm{E}\Big(X_t \cdot X_{t-p}\Big) = \mathrm{E}\Big(\sum_{j=1}^{p} \phi_j X_{t-j} \cdot X_{t-p}\Big) + \mathrm{E}\Big(\varepsilon_t \cdot X_{t-p}\Big) \tag{3.10}$$

$$\begin{aligned} \mathrm{E}\Big(X_t \cdot X_{t-p}\Big) &= \sum_{j=1}^{p} \phi_j \mathrm{E}\Big(X_{t-j} \cdot X_{t-p}\Big) \\ \gamma(p) &= \sum_{j=1}^{p} \phi_j \gamma(j-p) \end{aligned}$$

Putting $\gamma(1)$ to $\gamma(p)$ together, we obtain the system of equations

$$\begin{aligned}
\gamma(1) =& \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_{p-1}\gamma(p-2) + \phi_p\gamma(p-1) \\
\gamma(2) =& \phi_1\gamma(-1) + \phi_2\gamma(0) + \cdots + \phi_{p-1}\gamma(p-3) + \phi_p\gamma(p-2) \\
\gamma(3) =& \phi_1\gamma(-2) + \phi_2\gamma(-1) + \cdots + \phi_{p-1}\gamma(p-4) + \phi_p\gamma(p-3) \\
\vdots =& \quad \vdots \quad + \quad \vdots \quad + \quad \vdots \quad + \ddots + \quad \vdots \quad + \quad \vdots \\
\gamma(p-1) =& \phi_1\gamma(-p+2) + \phi_2\gamma(-p+3) + \cdots + \phi_{p-1}\gamma(0) + \phi_p\gamma(1) \\
\gamma(p) =& \phi_1\gamma(-p+1) + \phi_2\gamma(-p+2) + \cdots + \phi_{p-1}\gamma(-1) + \phi_p\gamma(0)
\end{aligned}$$

Since the autocovariance function is symmetric, $\gamma(-h) = \gamma(h)$. The system of equations can then be simplified into a matrix as

$$\begin{aligned}
\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \\ \vdots \\ \gamma(p-1) \\ \gamma(p) \end{bmatrix} =& \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(p-2) \\ \gamma(2) & \gamma(1) & \cdots & \gamma(p-3) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(p-2) & \gamma(p-3) & \cdots & \gamma(1) \\ \gamma(p-1) & \gamma(p-2) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix} \qquad (3.11) \\
\gamma =& \Gamma\Phi
\end{aligned}$$

In Equation (3.11), $\mathbf{\Gamma}$ is full-rank and symmetric.

The autocovariances is replaced with the autocorrelatons when normalized by the variance. In that way, Equation (3.11) summarizes to

$$\begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(p-1) \\ \rho(p) \end{bmatrix} = \begin{bmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & \cdots & \rho(p-3) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \cdots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix} \tag{3.12}$$

To obtain functional expression for the autocorrelation at various lags, one will have to solve a system of linear equations in Equation (3.12).

Autocorrelation Functions through the Relationships between the AR and MA weights

Consider a causal time series process given as

$$\Phi(L)X_t = \Theta(L)Z_t \tag{3.13}$$

where $\Phi(L)$ and $\Theta(L)$ are linear filters, and L is the lag operator represented as B in some texts. Equation (3.13) can be simplified as

$$X_t = \frac{\Theta(L)}{\Phi(L)} Z_t \tag{3.14}$$

which is the Moving Average representation of the process. Equation (3.14) can be written as

$$X_t = \sum_{j=0}^{\infty} \Psi_j L^j Z_t$$

where

$$\begin{aligned}\Psi_j L^j =& \frac{\Theta_j L^j}{\Phi_j L^j}\\ \Theta_j L^j =& \Psi_j L^j \Phi_j L^j\end{aligned}$$

$$\begin{aligned}\Theta(L) =& \Psi(L)\Phi(L)\\ \left(\theta_0 + \theta_1 L + \cdots + \theta_r L^r + \cdots\right) =& \Bigg\{\left(\psi_0 + \psi_1 L + \cdots + \psi_r L^r + \cdots\right)\times\\ & \left(\phi_0 - \phi_1 L + \cdots + \phi_r L^r + \cdots\right)\Bigg\}\end{aligned}$$

Noting that $\theta_0 = \phi_0 = 1$, we expand and match coefficients on both sides

At $j = 0$, $\psi_0 = 1$

At $j = 1$,

$$\begin{aligned}\psi_1 L - \psi_0\phi_1 L =& \theta_1 L\\ \psi_1 - \psi_0\phi_1 =& \theta_1\end{aligned}$$

At $j = 2$,

$$\begin{aligned}\psi_2 L^2 - \psi_1\phi_1 L^2 - \psi_0\phi_2 L^2 =& \theta_2 L^2\\ \psi_2 - \psi_1\phi_1 - \psi_0\phi_2 =& \theta_2\end{aligned}$$

At $j = 3$,

$$\begin{aligned}\psi_3 L^3 - \psi_2\phi_1 L^3 - \psi_1\phi_2 L^3 - \psi_0\phi_3 L^3 =& \theta_3 L^3\\ \psi_3 - \psi_2\phi_1 - \psi_1\phi_2 - \psi_0\phi_3 =& \theta_3\end{aligned}$$

and so on.

Generalizing for $j = 1, 2, \cdots$,

$$\begin{aligned}\theta_j =& \psi_j - \psi_{j-1}\phi_1 - \psi_{j-2}\phi_2 - \psi_{j-3}\phi_3 - \cdots - \psi_0\phi_j \\ =& \psi_j - \sum_{k=1}^{\infty} \psi_{j-k}\phi_k\end{aligned}$$

$$\psi_j = \theta_j + \sum_{k=1}^{\infty} \psi_{j-k}\phi_k \tag{3.15}$$

Equation (3.15) gives a recursive method to calculate the elements of ψ_j, starting with ψ_0, ψ_1, ψ_2 and so on.

The autocovariance at lag k ($\gamma(k)$) is therefore obtained as $\psi_j\psi_{j+k}$. A normalization of $\psi_j\psi_{j+k}$ gives the autocorrelation at lag k.

Autocorrelation Functions through the Autocovariance Generating Function Approach

For a stationary time series process X_t, the sequence of autocovariances γ_k, for $k = 0, 1, \cdots$ can be calculated through a scalar valued function called the autocovariance generating function defined as

$$c(s) = \sum_{k=-\infty}^{\infty} \gamma(k)s^k \tag{3.16}$$

The function is constructed by taking the k^{th} autocovariane and multiplying it by some number s raised to the k^{th} power, and then summing over all the possible values of k. This implies that the variance of the process, $\gamma(0)$, is the coefficient of $s^0 = 1$, while $\gamma(k)$, the autocovariance of the process at lag k is the coefficient of both s^k and s^{-k}.

From Equation (3.14), it is notied that if X_t is a linear process, then

$$X_t = \sum_{r=0}^{\infty} \Psi_r Z_{t-r} \tag{3.17}$$

where Ψ_r are constants and $\sum \Psi_r^2 < \infty$

The autocovariance at lag k of Equation (3.17) is obtained as

$$\begin{aligned} \text{cov}\Big(X_t, X_{t+k}\Big) =& \text{E}\Big(X_t \cdot X_{t+k}\Big) \\ =& \text{E}\Big[\sum \Psi_r Z_{t-r} \cdot \sum \Psi_j Z_{t+k-j}\Big] \\ =& \sigma^2 \sum_{r=0}^{\infty} \Psi_r \Psi_{r+k} \end{aligned}$$

Inferring from Equation (3.17), we consider a case where

$$c(s) = \sum_{r=0}^{\infty} \Psi_r s^r \tag{3.18}$$

By multiplying Equation (3.18) by another power series $c(s^{-1})$, we shall obtain

$$\begin{aligned} c(s) \cdot c(s^{-1}) =& \sum_{r=0}^{\infty} \Psi_r s^r \sum_{j=0}^{\infty} \Psi_j s^{-j} \\ =& \sigma^2 \sum_{r=0}^{\infty} \Psi_r \Psi_{r+k} \end{aligned}$$

which is the same as the covariance of a time series at lag k.

Thus, $\gamma(k)$ is the coefficient of s^k in the expansion of the power series given by

$$\begin{aligned} c(s)c(s^{-1}) =& \sigma^2 \frac{\Theta(s)\Theta(s^{-1})}{\Phi(s)\Phi(s^{-1})} \\ =& \sigma^2 \frac{\Big(\theta_0 + \cdots + \theta_p s^p\Big)\Big(\theta_0 + \cdots + \theta_p s^{-p}\Big)}{\Big(\phi_0 - \cdots - \phi_p s^p\Big)\Big(\phi_0 - \cdots - \phi_p s^{-p}\Big)} \end{aligned} \tag{3.19}$$

In Equation (3.19), $\theta_0 = \phi_0 = 1$.

Illustrative Dataset

As an application to the derivations that will be obtained in this study in the last part of Chapter Four, data on daily new Covid-19 cases for some selected countries across the globe is obtained from the official website of the Johns Hopkins University Center for Systems Science Engineering (JHU CSSE). The choice of the countries were based on the availability of complete (non-missing) data and the condition that the series exhibit stationarity. The starting point of each series was different, since the Corona virus were identified on different days for most countries. However, to make a more reliable inference, it was ensured that the data points for all the selected countries exceeded 365. More specifically, data points were from January 2020 to March 2022.

Data obtained will be processed and analyzed using Microsoft Excel, Minitab 19, and R statistical software.

Chapter Summary

This chapter has presented the most dominant mathematical bases for obtaining the theoretical autocorrelation of a stationary time series process. Specifically, the chapter has examined how the Yule-Walker approach, the comparison of coefficients approach, and the autocovariance generating function aid in obtaining the ACF of an ARMA process. To help in the implementation of the formulas that will be derived, the chapter further looks at the characteristics of the data selected, and the software that will be used to aid in this regard.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

The focus of this chapter is to present extensions and generalizations of the ACFs of ARMA(p, q) processes. The ACFs of lower order processes which includes the ARMA$(1, 0)$, ARMA$(0, 1)$, as well as ARMA$(1, 1)$ are well-documented in the literature and are therefore presented in Chapter Two. This chapter therefore begins with a study of the ACFs of ARMA(1,2) and ARMA(1,3) are derived, after which a generalization is made for ARMA(1,q) process. Generalizations of higher order ACFs are subsequently considered. Applications of the results are made to relevant stationary time series data in the end. Similarly, the ACFs of ARMA$(2, 0)$, ARMA$(2, 1)$, ARMA$(2, 2)$ and ARMA$(2, 3)$ are derived, after which a generalization is made for ARMA(2,q). The variance and the first autocorrelation function of an ARMA$(3, 0)$ process are also derived. In the last section of this chapter, the generalizations are verified, and applications are made to a pandemic data.

ACF of an ARMA(1,2) Process

In this section, the ACF of an ARMA(1,2) process is derived. Firstly, the autocovariance generating function (acgf) is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation functions.

An ARMA (1,2) process is given by

$$X_t = \phi_1 X_{t-1} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} \tag{4.1}$$

By introducing a lag operator, Equation(4.1) can be simplified as

$$(1-\phi_1 L)X_t = (1+\theta_1 L+\theta_2 L^2)Z_t$$

Further simplification yields

$$X_t = \frac{(1+\theta_1 L+\theta_2 L^2)}{(1-\phi_1 L)}Z_t \tag{4.2}$$

From acgf, the autocovariance of an ARMA (1,2) process can be written as

$$c(s)c(s^{-1}) = \sigma^2\frac{(1+\theta_1 s+\theta_2 s^2)(1+\theta_1 s^{-1}+\theta_2 s^{-2})}{(1-\phi_1 s)(1-\phi_1 s^{-1})} \tag{4.3}$$

Simplifying further, we obtain

$$\begin{aligned} c(s)c(s^{-1}) = \sigma^2\Big[&(1+\theta_1 s^{-1}+\theta_2 s^{-2})+(\theta_1 s+\theta_1^2+\theta_1\theta_2 s^{-1})+ \\ &(\theta_2 s^2+\theta_1\theta_2 s+\theta_2^2)\Big]\times\sum_{r=0}^{\infty}(\phi_1 s)^r\cdot\sum_{r=0}^{\infty}(\phi_1 s^{-1})^r \end{aligned}$$

Further groupings yield

$$\begin{aligned} c(s)c(s^{-1}) = \sigma^2\Big[&(1+\theta_1^2+\theta_2^2)+(\theta_1+\theta_1\theta_2)s+\theta_2 s^2+ \\ &(\theta_1+\theta_1\theta_2)s^{-1}+\theta_2 s^{-2}\Big]\times\sum_{r=0}^{\infty}(\phi_1 s)^r\cdot\sum_{r=0}^{\infty}(\phi_1 s^{-1})^r \end{aligned} \tag{4.4}$$

Now,

$$\sum_{r=0}^{\infty}(\phi_1 s)^r\cdot\sum_{r=0}^{\infty}(\phi_1 s^{-1})^r = \sum_{r=0}^{\infty}\phi_1^{2r}\Big[\sum_{r=0}^{\infty}(\phi_1 s)^r+\sum_{r=1}^{\infty}(\phi_1 s^{-1})^r\Big] \tag{4.5}$$

At lag 0, we consider terms in s^0 in Equation (4.4) and obtains the variance of the series as

$$\begin{aligned}
\gamma(0) =& \sigma^2 \Bigg\{ (1+\theta_1^2+\theta_2^2)\Big[\sum_{r=0}^{\infty}\phi_1^{2r}\Big] + (\theta_1+\theta_1\theta_2)s\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-1} + \\
& \theta_2 s^2\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-2} + (\theta_1+\theta_1\theta_2)s^{-1}\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s + \\
& \theta_2 s^{-2}\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2 \Bigg\} \\
=& \sigma^2 \Bigg\{ (1+\theta_1^2+\theta_2^2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1^2\theta_2\sum_{r=0}^{\infty}\phi_1^{2r} + \\
& \phi_1(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1^2\theta_2\sum_{r=0}^{\infty}\phi_1^{2r} \Bigg\} \\
=& \sigma^2 \Bigg\{ \Big(1+\theta_1^2+\theta_2^2\Big) + \phi_1\Big(\theta_1+\theta_1\theta_2\Big) + \phi_1^2\theta_2 + \phi_1\Big(\theta_1+\theta_1\theta_2\Big) + \\
& \phi_1^2\theta_2 \Bigg\} \sum_{r=0}^{\infty}\phi_1^{2r}
\end{aligned}$$

After some further simplifications,

$$\gamma(0) = \sigma^2 \Bigg\{ 1 + \theta_1\Big(\theta_1 + 2\phi_1\Big) + \theta_2\Big(\theta_2 + 2\phi_1\theta_1 + 2\phi_1^2\Big) \Bigg\} \frac{1}{1-\phi_1^2} \tag{4.6}$$

We consider terms in s and obtain the autocovariance (acv) at lag 1 as

$$\begin{aligned}
\gamma(1) =& \sigma^2 \Bigg\{ (1+\theta_1^2+\theta_2^2)\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s + (\theta_1+\theta_1\theta_2)s\Big[\sum_{r=0}^{\infty}\phi_1^{2r}\Big] + \\
& \theta_2 s^2\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-1} + (\theta_1+\theta_1\theta_2)s^{-1}\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2 + \\
& \theta_2 s^{-2}\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^3 \Bigg\} \\
=& \sigma^2 \Bigg\{ \phi_1(1+\theta_1^2+\theta_2^2)\sum_{r=0}^{\infty}\phi_1^{2r} + (\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1\theta_2\sum_{r=0}^{\infty}\phi_1^{2r} + \\
& \phi_1^2(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1^3\theta_2\sum_{r=0}^{\infty}\phi_1^{2r} \Bigg\}
\end{aligned}$$

Further simplification gives

$$\gamma(1) = \sigma^2 \left\{ \phi_1 \Big(1 + \theta_1^2 + \theta_2^2 \Big) + (\theta_1 + \theta_1\theta_2) \Big[1 + \phi_1^2 \Big] + \theta_2 \Big[\phi_1^3 + \phi_1 \Big] \right\} \frac{1}{1 - \phi_1^2}$$

At lag 2, we consider terms in $\mathbf{s^2}$ and obtain

$$\begin{aligned}
\gamma(2) =& \sigma^2 \left\{ (1 + \theta_1^2 + \theta_2^2) \Big[\phi_1^2 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^2 + (\theta_1 + \theta_1\theta_2) s \Big[\phi_1 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s + \right. \\
& \left. \theta_2 s^2 \Big[\sum_{r=0}^{\infty} \phi_1^{2r} \Big] + (\theta_1 + \theta_1\theta_2) s^{-1} \Big[\phi_1^3 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^3 + \theta_2 s^{-2} \Big[\phi_1^4 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^4 \right\} \\
=& \sigma^2 \left\{ \phi_1^2 (1 + \theta_1^2 + \theta_2^2) \sum_{r=0}^{\infty} \phi_1^{2r} + \phi_1 (\theta_1 + \theta_1\theta_2) \sum_{r=0}^{\infty} \phi_1^{2r} + \theta_2 \sum_{r=0}^{\infty} \phi_1^{2r} + \right. \\
& \left. \phi_1^3 (\theta_1 + \theta_1\theta_2) \sum_{r=0}^{\infty} \phi_1^{2r} + \phi_1^4 \theta_2 \sum_{r=0}^{\infty} \phi_1^{2r} \right\}
\end{aligned}$$

Therefore,

$$\gamma(2) = \sigma^2 \left\{ \phi_1^2 \Big(1 + \theta_1^2 + \theta_2^2 \Big) + (\theta_1 + \theta_1\theta_2) \Big[\phi_1^3 + \phi_1 \Big] + \theta_2 \Big[1 + \phi_1^4 \Big] \right\} \frac{1}{1 - \phi_1^2}$$

At lag 3, we consider terms in $\mathbf{s^3}$ and obtain

$$\begin{aligned}
\gamma(3) =& \sigma^2 \left\{ (1 + \theta_1^2 + \theta_2^2) \Big[\phi_1^3 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^3 + (\theta_1 + \theta_1\theta_2) s \Big[\phi_1^2 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^2 + \right. \\
& \theta_2 s^2 \Big[\phi_1 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s + (\theta_1 + \theta_1\theta_2) s^{-1} \Big[\phi_1^4 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^4 + \\
& \left. \theta_2 s^{-2} \Big[\phi_1^5 \sum_{r=0}^{\infty} \phi_1^{2r} \Big] s^5 \right\} \\
=& \sigma^2 \left\{ \phi_1^3 (1 + \theta_1^2 + \theta_2^2) \sum_{r=0}^{\infty} \phi_1^{2r} + \phi_1^2 (\theta_1 + \theta_1\theta_2) \sum_{r=0}^{\infty} \phi_1^{2r} + \phi_1 \theta_2 \sum_{r=0}^{\infty} \phi_1^{2r} + \right. \\
& \left. \phi_1^4 (\theta_1 + \theta_1\theta_2) \sum_{r=0}^{\infty} \phi_1^{2r} + \phi_1^5 \theta_2 \sum_{r=0}^{\infty} \phi_1^{2r} \right\} \\
=& \sigma^2 \left\{ \phi_1^3 \Big(1 + \theta_1^2 + \theta_2^2 \Big) + (\theta_1 + \theta_1\theta_2) \Big[\phi_1^4 + \phi_1^2 \Big] + \theta_2 \Big[\phi_1^5 + \phi_1 \Big] \right\} \frac{1}{1 - \phi_1^2}
\end{aligned}$$

Therefore

$$\begin{aligned}\gamma(3) =&\phi_1\sigma^2\left\{\phi_1^2\left(1+\theta_1^2+\theta_2^2\right)+(\theta_1+\theta_1\theta_2)\left[\phi_1^3+\phi_1\right]+\theta_2\left[1+\phi_1^4\right]\right\}\frac{1}{1-\phi_1^2}\\ =&\phi_1\gamma(2)\end{aligned}$$

At lag 4, we consider terms in s^4 and obtain

$$\begin{aligned}\gamma(4) =&\sigma^2\left\{(1+\theta_1^2+\theta_2^2)\left[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^4+(\theta_1+\theta_1\theta_2)s\left[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^3+\right.\\ &\theta_2 s^2\left[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^2+(\theta_1+\theta_1\theta_2)s^{-1}\left[\phi_1^5\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^5+\\ &\left.\theta_2 s^{-2}\left[\phi_1^6\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^6\right\}\\ =&\sigma^2\left\{\phi_1^4(1+\theta_1^2+\theta_2^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^3(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^2\theta_2\sum_{r=0}^{\infty}\phi_1^{2r}+\right.\\ &\left.\phi_1^5(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^6\theta_2\sum_{r=0}^{\infty}\phi_1^{2r}\right\}\\ =&\sigma^2\left\{\phi_1^4\left(1+\theta_1^2+\theta_2^2\right)+(\theta_1+\theta_1\theta_2)\left[\phi_1^5+\phi_1^3\right]+\theta_2\left[\phi_1^6+\phi_1^2\right]\right\}\frac{1}{1-\phi_1^2}\end{aligned}$$

Therefore

$$\begin{aligned}\gamma(4) =&\phi_1^2\sigma^2\left\{\phi_1^2\left(1+\theta_1^2+\theta_2^2\right)+(\theta_1+\theta_1\theta_2)\left[\phi_1^3+\phi_1\right]+\theta_2\left[1+\phi_1^4\right]\right\}\frac{1}{1-\phi_1^2}\\ =&\phi_1^2\gamma(2)\\ =&\phi_1\gamma(3)\end{aligned}$$

The autocovariances of ARMA(1,2) process obtained so far shows a clear pattern after lag 2.

Therefore, at lag k, we consider terms in s^k

$$\begin{aligned}
\gamma(k) =&\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^k s^k+\phi_1^{k+2}s^k+\phi_1^{k+4}s^k+\cdots\Big]+\\
&(\theta_1+\theta_1\theta_2)s\Big[\phi_1^{k-1}s^{k-1}+\phi_1^{k+1}s^{k-1}+\phi_1^{k+3}s^{k-1}+\cdots\Big]+\\
&\theta_2 s^2\Big[\phi_1^{k-2}s^{k-2}+\phi_1^{k}s^{k-2}+\phi_1^{k+2}s^{k-2}+\cdots\Big]+\\
&(\theta_1+\theta_1\theta_2)s^{-1}\Big[\phi_1^{k+1}s^{k+1}+\phi_1^{k+3}s^{k+1}+\phi_1^{k+5}s^{k+1}+\cdots\Big]+\\
&\theta_2 s^{-2}\Big[\phi_1^{k+2}s^{k+2}+\phi_1^{k+4}s^{k+2}+\phi_1^{k+6}s^{k+2}+\cdots\Big]\Bigg\}\\
=&\sigma^2\Bigg\{\phi_1^k(1+\theta_1^2+\theta_2^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k-1}(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k-2}\theta_2\sum_{r=0}^{\infty}\phi_1^{2r}\\
&+\phi_1^{k+1}(\theta_1+\theta_1\theta_2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k+2}\theta_2\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}\\
=&\sigma^2\Bigg\{\phi_1^k\Big(1+\theta_1^2+\theta_2^2\Big)+\phi_1^{k-1}\Big(\theta_1+\theta_1\theta_2\Big)+\phi_1^{k-2}\theta_2+\\
&\phi_1^{k+1}\Big(\theta_1+\theta_1\theta_2\Big)+\phi_1^{k+2}\theta_2\Bigg\}\sum_{r=0}^{\infty}\phi_1^{2r}\\
=&\sigma^2\Bigg\{\phi_1^k\Big(1+\theta_1^2+\theta_2^2\Big)+\phi_1^{k-1}\Big(\theta_1+\theta_1\theta_2\Big)+\phi_1^{k-2}\theta_2+\\
&\phi_1^{k+1}\Big(\theta_1+\theta_1\theta_2\Big)+\phi_1^{k+2}\theta_2\Bigg\}\frac{1}{1-\phi_1^2}
\end{aligned}$$

$$\begin{aligned}
=&\sigma^2\Bigg\{\phi_1^k\Big(1+\theta_1^2+\theta_2^2\Big)+(\theta_1+\theta_1\theta_2)\Big[\phi_1^{k+1}+\phi_1^{k-1}\Big]+\\
&\theta_2\Big[\phi_1^{k+2}+\phi_1^{k-2}\Big]\Bigg\}\frac{1}{1-\phi_1^2}\\
=&\phi_1^{k-2}\sigma^2\Bigg\{\phi_1^2\Big(1+\theta_1^2+\theta_2^2\Big)+(\theta_1+\theta_1\theta_2)\Big[\phi_1^3+\phi_1\Big]+\theta_2\Big[1+\phi_1^4\Big]\Bigg\}\frac{1}{1-\phi_1^2}\\
=&\phi_1^{k-2}\gamma(2)\\
=&\phi_1\gamma(k-1)
\end{aligned}$$

Therefore, the ACF of an ARMA(1,2) process can be summarized as

$$\rho(k) = \begin{cases} 1 & ,k=0 \\ \frac{\phi_1(1+\theta_1^2+\theta_2^2)+(\theta_1+\theta_1\theta_2)[1+\phi_1^2]+\theta_2[\phi_1^3+\phi_1]}{1+\theta_1(\theta_1+2\phi_1)+\theta_2(\theta_2+2\phi_1\theta_1+2\phi_1^2)} & ,k=1 \\ \frac{(1+\theta_1^2+\theta_2^2)+(\theta_1+\theta_1\theta_2)[\phi_1^3+\phi_1]+\theta_2[1+\phi_1^4]}{1+\theta_1(\theta_1+2\phi_1)+\theta_2(\theta_2+2\phi_1\theta_1+2\phi_1^2)} & ,k=2 \\ \phi^{k-2}\rho(2) & ,k\geq 3 \end{cases} \tag{4.7}$$

From the relation among the $\gamma(k)$s, it can be verified that

$$\rho^2(3) = \rho(2) \times \rho(4) \tag{4.8}$$

ACF of an ARMA(1,3) Process

This section examines the ACF of an ARMA(1,3) process. The acgf is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation function.

An ARMA (1,3) process is given by

$$X_t = \phi_1 X_{t-1} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3} \tag{4.9}$$

By introducing a lag operator, Equation(4.9) can be simplified as

$$(1-\phi_1 L)X_t = (1+\theta_1 L + \theta_2 L^2 + \theta_3 L^3)Z_t$$

Further simplification yields

$$X_t = \frac{(1+\theta_1 L + \theta_2 L^2 + \theta_3 L^3)}{(1-\phi_1 L)} Z_t \tag{4.10}$$

From autocovariance generating functions, the autocovariance generating function can thus be written as

$$c(s)c(s^{-1}) = \sigma^2\left[\frac{1+\theta_1 s+\theta_2 s^2+\theta_3 s^3}{(1-\phi_1 s)} \times \frac{1+\theta_1 s^{-1}+\theta_2 s^{-2}+\theta_3 s^{-3}}{(1-\phi_1 s^{-1})}\right]$$

This simplifies to

$$\frac{1}{(1-\phi_1 s)(1-\phi_1 s^{-1})}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s+$$
$$(\theta_2+\theta_1\theta_3)s^2+\theta_3 s^3+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}+$$
$$(\theta_2+\theta_1\theta_1\theta_3)s^{-2}+\theta_3 s^{-3}\Big\}$$

Equivalently,

$$c(s)c(s^{-1}) = \sigma^2\Big\{\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)s+\Big(\theta_2+\theta_1\theta_3\Big)s^2+$$
$$\theta_3 s^3+\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)s^{-1}+\Big(\theta_2+\theta_1\theta_1\theta_3\Big)s^{-2}+\theta_3 s^{-3}\Big\}\times$$
$$\sum_{r=0}^{\infty}(\phi_1 s)^r\cdot\sum_{r=0}^{\infty}(\phi_1 s^{-1})^r \tag{4.11}$$

At lag 0, we consider terms in s^0 in Equation (4.11) and using Equation (4.5), the variance of the process is given by

$$\gamma(0) =\sigma^2\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\sum_{r=0}^{\infty}\phi_1^{2r}\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-1}$$
$$+(\theta_2+\theta_1\theta_3)s^2\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-2}+\theta_3 s^3\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-3}+$$
$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s+(\theta_2+\theta_1\theta_3)s^{-2}\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2+$$
$$\theta_3 s^{-3}\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^3\Big\}$$

$$=\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^2(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^3\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^2(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^3\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}$$

After some further simplifications, the variance of an ARMA(1,3) is obtained as

$$\begin{aligned}\gamma(0)=&\sigma^2\Bigg\{1+\theta_1\Big(\theta_1+2\phi_1\Big)+\theta_2\Big(\theta_2+2\phi_1\theta_1+2\phi_1^2\Big)+\\&\theta_3\Big(\theta_3+2\phi_1\theta_2+2\phi_1^2\theta_1+2\phi_1^3\Big)\Bigg\}\frac{1}{1-\phi_1^2}\end{aligned}\tag{4.12}$$

At lag 1, we consider terms in s and obtain

$$\gamma(1)=\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big[\sum_{r=0}^{\infty}\phi_1^{2r}\Big]+$$
$$(\theta_2+\theta_1\theta_3)s^2\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-1}+\theta_3s^3\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-2}+$$
$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2+(\theta_2+\theta_1\theta_3)s^{-2}\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^3+$$
$$\theta_3s^{-3}\Big[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^4\Bigg\}$$
$$=\sigma^2\Bigg\{\phi_1(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^2\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^3(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^4\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}$$
$$=\sigma^2\Bigg\{\phi_1\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1\Big(\theta_2+\theta_1\theta_3\Big)+$$
$$\phi_1^2\theta_3+\phi_1^2\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1^3\Big(\theta_2+\theta_1\theta_3\Big)+\phi_1^4\theta_3\Bigg\}\sum_{r=0}^{\infty}\phi_1^{2r}$$

Therefore, the autocovariance at lag 1 summarizes to

$$\gamma(1) = \sigma^2\Bigg\{\phi_1\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big) + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[1+\phi_1^2\Big]+ (\theta_2+\theta_1\theta_3)\Big[\phi_1^3+\phi_1\Big] + \theta_3\Big[\phi_1^4+\phi_1^2\Big]\Bigg\}\frac{1}{1-\phi_1^2}$$

Similarly,

$$\begin{aligned}\gamma(2) =&\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2 + (\theta_1+\theta_1\theta_2+ \\ &\theta_2\theta_3)s\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s + (\theta_2+\theta_1\theta_3)s^2\Big[\sum_{r=0}^{\infty}\phi_1^{2r}\Big] + \theta_3 s^3\Big[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^{-1}+ \\ &(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^3 + (\theta_2+\theta_1\theta_3)s^{-2}\Big[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^4+ \\ &\theta_3 s^{-3}\Big[\phi_1^5\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^5\Bigg\} \\ =&\sigma^2\Bigg\{\phi_1^2(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+ \\ &(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1\theta_3\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1^3(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+ \\ &\phi_1^4(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r} + \phi_1^5\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}\end{aligned}$$

Therefore,

$$\gamma(2) = \sigma^2\Bigg\{\phi_1^2\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big) + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^3+\phi_1\Big]+ (\theta_2+\theta_1\theta_3)\Big[1+\phi_1^4\Big] + \theta_3\Big[\phi_1^5+\phi_1\Big]\Bigg\}\frac{1}{1-\phi_1^2}$$

At lag 3, we consider terms in $\mathbf{s^3}$ and obtain

$$\begin{aligned}\gamma(3) =&\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\left[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^3+(\theta_1+\theta_1\theta_2+\\&\theta_2\theta_3)s\left[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^2+(\theta_2+\theta_1\theta_3)s^2\left[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\right]s+\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\\&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\left[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^4+(\theta_2+\theta_1\theta_3)s^{-2}\left[\phi_1^5\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^5+\\&\theta_3 s^{-3}\left[\phi_1^6\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^6\Bigg\}\\=&\sigma^2\Bigg\{\phi_1^3(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\\&\phi_1(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^4(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\\&\phi_1^5(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^6\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(3) =&\sigma^2\Bigg\{\phi_1^3\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\left[\phi_1^4+\phi_1^2\right]+\\&(\theta_2+\theta_1\theta_3)\left[\phi_1^5+\phi_1\right]+\theta_3\left[1+\phi_1^6\right]\Bigg\}\frac{1}{1-\phi_1^2}\end{aligned}$$

At lag 4, we consider terms in $\mathbf{s^4}$ and obtain

$$\begin{aligned}\gamma(4) =&\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\left[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^4+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\\&\left[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^3+(\theta_2+\theta_1\theta_3)s^2\left[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^2+\theta_3\left[\phi_1\sum_{r=0}^{\infty}\phi_1^{2r}\right]s+\\&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\left[\phi_1^5\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^5+(\theta_2+\theta_1\theta_3)s^{-2}\left[\phi_1^6\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^6+\\&\theta_3 s^{-3}\left[\phi_1^7\sum_{r=0}^{\infty}\phi_1^{2r}\right]s^7\Bigg\}\end{aligned}$$

$$=\sigma^2\Bigg\{\phi_1^4(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^3(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^2(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^5(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^6(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^7\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}$$

Therefore, $\gamma(4)$ can be obtained as

$$\gamma(4)=\sigma^2\phi_1\Bigg\{\phi_1^3\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^4+\phi_1^2\Big]+$$
$$(\theta_2+\theta_1\theta_3)\Big[\phi_1^5+\phi_1\Big]+\theta_3\Big[1+\phi_1^6\Big]\Bigg\}\frac{1}{1-\phi_1^2}$$
$$=\phi_1\gamma(3)$$

Similarly,

$$\gamma(5)=\sigma^2\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1^5\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^5+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big[\phi_1^4\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^4$$
$$+(\theta_2+\theta_1\theta_3)s^2\Big[\phi_1^3\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^3+\theta_3\Big[\phi_1^2\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^2+$$
$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big[\phi_1^6\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^6+(\theta_2+\theta_1\theta_3)s^{-2}\Big[\phi_1^7\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^7+$$
$$\theta_3s^{-3}\Big[\phi_1^8\sum_{r=0}^{\infty}\phi_1^{2r}\Big]s^8\Bigg\}$$
$$=\sigma^2\Bigg\{\phi_1^5(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^4(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^3(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^2\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^6(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+$$
$$\phi_1^7(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^8\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Bigg\}$$

$$
\begin{aligned}
&=\sigma^2\phi_1^2\Big\{\phi_1^3\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^4+\phi_1^2\Big]+\\
&\quad(\theta_2+\theta_1\theta_3)\Big[\phi_1^5+\phi_1\Big]+\theta_3\Big[1+\phi_1^6\Big]\Big\}\frac{1}{1-\phi_1^2}\\
&=\phi_1^2\gamma(3)\\
&=\phi_1\gamma(4)
\end{aligned}
$$

At lag k, we consider terms in s^k and obtain

$$
\begin{aligned}
\gamma(k)=&\sigma^2\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1^k s^k+\phi_1^{k+2}s^k+\phi_1^{k+4}s^k+\cdots\Big]+\\
&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big[\phi_1^{k-1}s^{k-1}+\phi_1^{k+1}s^{k-1}+\phi_1^{k+3}s^{k-1}+\cdots\Big]+\\
&(\theta_2+\theta_1\theta_3)s^2\Big[\phi_1^{k-2}s^{k-2}+\phi_1^{k}s^{k-2}+\phi_1^{k+2}s^{k-2}+\cdots\Big]+\\
&\theta_3 s^3\Big[\phi_1^{k-3}s^{k-3}+\phi_1^{k-1}s^{k-3}+\phi_1^{k+1}s^{k-3}+\cdots\Big]+\\
&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big[\phi_1^{k+1}s^{k+1}+\phi_1^{k+3}s^{k+1}+\phi_1^{k+5}s^{k+1}+\cdots\Big]+\\
&(\theta_2+\theta_1\theta_3)s^{-2}\Big[\phi_1^{k+2}s^{k+2}+\phi_1^{k+4}s^{k+2}+\phi_1^{k+6}s^{k+2}+\cdots\Big]+\\
&\theta_3 s^{-3}\Big[\phi_1^{k+3}s^{k+3}+\phi_1^{k+5}s^{k+3}+\phi_1^{k+7}s^{k+3}+\cdots\Big]\Big\}\\
=&\sigma^2\Big\{\phi_1^k(1+\theta_1^2+\theta_2^2+\theta_3^2)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k-1}(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\\
&\phi_1^{k-2}(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k-3}\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k+1}(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}\\
&+\phi_1^{k+2}(\theta_2+\theta_1\theta_3)\sum_{r=0}^{\infty}\phi_1^{2r}+\phi_1^{k+3}\theta_3\sum_{r=0}^{\infty}\phi_1^{2r}\Big\}\\
=&\sigma^2\Big\{\phi_1^k\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+\phi_1^{k-1}\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1^{k-2}\Big(\theta_2+\theta_1\theta_3\Big)+\\
&\phi_1^{k-3}\theta_3+\phi_1^{k+1}\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1^{k+2}\Big(\theta_2+\theta_1\theta_3\Big)+\phi_1^{k+3}\theta_3\Big\}\frac{1}{1-\phi_1^2}
\end{aligned}
$$

$$
\begin{aligned}
=&\phi_1^{k-3}\sigma^2\Bigg\{\phi_1^3\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+\phi_1^2\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1\Big(\theta_2+\theta_1\theta_3\Big) \\
&+\theta_3+\phi_1^4\Big(\theta_1+\theta_1\theta_2+\theta_2\theta_3\Big)+\phi_1^5\Big(\theta_2+\theta_1\theta_3\Big)+\phi_1^6\theta_3\Bigg\}\frac{1}{1-\phi_1^2} \\
=&\sigma^2\phi_1^{k-3}\Bigg\{\phi_1^3\Big(1+\theta_1^2+\theta_2^2+\theta_3^2\Big)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^4+\phi_1^2\Big]+ \\
&(\theta_2+\theta_1\theta_3)\Big[\phi_1^5+\phi_1\Big]+\theta_3\Big[1+\phi_1^6\Big]\Bigg\}\frac{1}{1-\phi_1^2} \\
=&\phi_1^{k-3}\gamma(3) \\
=&\phi_1\gamma(k-1)
\end{aligned}
$$

Therefore, the ACF of an ARMA(1,3) process can be summarized as

$$
\rho(k)=\begin{cases}
1 & ,k=0 \\
\frac{\phi_1(1+\theta_1^2+\theta_2^2+\theta_3^2)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)[1+\phi_1^2]+(\theta_2+\theta_1\theta_3)[\phi_1^3+\phi_1]+\theta_3[\phi_1^4+\phi_1^2]}{1+\theta_1(\theta_1+2\phi_1)+\theta_2(\theta_2+2\phi_1\theta_1+2\phi_1^2)+\theta_3(\theta_3+2\phi_1\theta_2+2\phi_1^2\theta_1+2\phi_1^3)} & ,k=1 \\
\frac{\phi_1^2(1+\theta_1^2+\theta_2^2+\theta_3^2)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)[\phi_1^3+\phi_1]+(\theta_2+\theta_1\theta_3)[1+\phi_1^4]+\theta_3[\phi_1^5+\phi_1]}{1+\theta_1(\theta_1+2\phi_1)+\theta_2(\theta_2+2\phi_1\theta_1+2\phi_1^2)+\theta_3(\theta_3+2\phi_1\theta_2+2\phi_1^2\theta_1+2\phi_1^3)} & ,k=2 \\
\frac{\phi_1^3(1+\theta_1^2+\theta_2^2+\theta_3^2)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)[\phi_1^4+\phi_1^2]+(\theta_2+\theta_1\theta_3)[\phi_1^5+\phi_1]+\theta_3[1+\phi_1^6]}{1+\theta_1(\theta_1+2\phi_1)+\theta_2(\theta_2+2\phi_1\theta_1+2\phi_1^2)+\theta_3(\theta_3+2\phi_1\theta_2+2\phi_1^2\theta_1+2\phi_1^3)} & ,k=3 \\
\phi^{k-3}\rho(3) & ,k\geq 4
\end{cases}
\tag{4.13}
$$

From the relation among the $\gamma(k)$s of the ARMA(1,3) process, it is clear that

$$
\rho^2(4)=\rho(3)\times\rho(5) \tag{4.14}
$$

The processes so far has shown a clear pattern among autocovariance at consecutive lags of the respective process. It is observed that for any $\gamma(k)$ of a given ARMA($1,q$) process,

$$
\gamma(k)=\phi^{k-q}\gamma(q) \quad \text{for } k\geq q+1 \tag{4.15}
$$

The pattern also suggests that separate ACFs should be obtained for individual lags prior to the order q. These observations will be helpful to obtain a generalization of an expression for the ARMA$(1, q)$ process.

ACF of an ARMA(1,q) Process

In this section, the approach used to obtain the ACFs of the ARMA(1,2) and ARMA(1,3) time series processes is extended to derive the generalized autocorrelation function of an ARMA$(1, q)$ process. The ARMA(1,q) process is given by

$$X_t - \phi X_{t-1} = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3} + \cdots + \theta_{q-1} Z_{t-(q-1)} + \theta_q Z_{t-q} + Z_t$$

which may be re-written as

$$X_t - \phi X_{t-1} = \sum_{j=0}^{q} \theta_j Z_{t-j} \tag{4.16}$$

The ARMA(1,q) process can be written in a lag form as

$$X_t(1 - \phi L) = \sum_{j=0}^{q} \theta_j L^j Z_t \tag{4.17}$$

It is worth noting from Equation (4.17) that $\theta_0 = 1$

Equation (4.17) can be simplified as

$$X_t = \frac{\sum_{j=0}^{q} \theta_j L^j}{1 - \phi L} Z_t \tag{4.18}$$

The acvgf of an ARMA(1,q) process is obtained as

$$c(s)c(s^{-1}) = \sigma^2 \frac{\sum_{j=0}^{q} \theta_j s^j \sum_{j=0}^{q} \theta_j s^{-j}}{(1 - \phi_1 s)(1 - \phi_1 s^{-1})} \tag{4.19}$$

Equation (4.19) can be simplified as

$$c(s)c(s^{-1}) = \sigma^2 \sum_{r=0}^{\infty} (\phi_1 s)^r \sum_{r=0}^{\infty} (\phi_1 s^{-1})^r \sum_{j=0}^{q} \theta_j s^j \sum_{j=0}^{q} \theta_j s^{-j} \tag{4.20}$$

From Equation (4.20),

$$\begin{aligned}
\sum_{j=0}^{q} \theta_j s^j \sum_{j=0}^{q} \theta_j s^{-j} &= \sum_{j=0}^{q} \theta_j^2 + \left[\sum_{j=0}^{q-1} \theta_j \theta_{j+1}\right] s + \left[\sum_{j=0}^{q-2} \theta_j \theta_{j+2}\right] s^2 + \\
&\left[\sum_{j=0}^{q-3} \theta_j \theta_{j+3}\right] s^3 + \cdots \left[\sum_{j=0}^{q-5} \theta_j \theta_{j+5}\right] s^5 + \cdots + \left[\sum_{j=0}^{2} \theta_j \theta_{j+(q-2)}\right] s^{q-2} + \\
&\left[\sum_{j=0}^{1} \theta_j \theta_{j+(q-1)}\right] s^{q-1} + \left[\sum_{j=0}^{0} \theta_j \theta_{j+q}\right] s^q + \left[\sum_{j=0}^{q-1} \theta_j \theta_{j+1}\right] s^{-1} + \\
&\left[\sum_{j=0}^{q-2} \theta_j \theta_{j+2}\right] s^{-2} + \left[\sum_{j=0}^{q-3} \theta_j \theta_{j+3}\right] s^{-3} + \cdots + \left[\sum_{j=0}^{2} \theta_j \theta_{j+(q-2)}\right] s^{-q+2} + \\
&\left[\sum_{j=0}^{1} \theta_j \theta_{j+(q-1)}\right] s^{-q+1} + \left[\sum_{j=0}^{0} \theta_j \theta_q\right] s^{-q}
\end{aligned} \tag{4.21}$$

and

$$\sum_{r=0}^{\infty} (\phi s)^r \cdot \sum_{r=0}^{\infty} (\phi s^{-1})^r = \sum_{r=0}^{\infty} \phi^{2r} \left[\sum_{r=0}^{\infty} (\phi_1 s)^r + \sum_{r=1}^{\infty} (\phi s^{-1})^r\right] \tag{4.22}$$

At lag 0, we consider terms in s^0 in Equation (4.20)

$$
\begin{aligned}
\gamma(0) =&\sigma^2\Bigg\{\left[\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\right]+\left[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\\
&\left[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\left[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\cdots+\\
&\left[\phi^{q-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\left[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\
&\left[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\cdots+\left[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\\
&\left[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\left[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\cdots+\\
&\left[\phi^{q-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\left[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\
&\left[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]\Bigg\}\\
=&\sigma^2\Bigg\{\left[\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\right]+2\left[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\\
&2\left[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+2\left[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\cdots+\\
&2\left[\phi^{q-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+2\left[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\
&2\left[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]\Bigg\}\\
=&\sigma^2\Bigg\{\left[\sum_{j=0}^{q}\theta_j^2\right]+2\phi\left[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+2\phi^2\left[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\
&2\phi^3\left[\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\cdots+2\phi^{q-2}\left[\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\\
&2\phi^{q-1}\left[\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+2\phi^{q}\left[\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]\Bigg\}\sum_{r=0}^{\infty}\phi^{2r}
\end{aligned}
$$

Further simplification gives

$$\begin{aligned}
\gamma(0) =&\sigma^2\Bigg\{\Big[\sum_{j=0}^{q}\theta_j^2\Big]+2\phi\Big[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Big]+2\phi^2\Big[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Big]+\\
&2\phi^3\Big[\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Big]+\cdots+2\phi^{q-2}\Big[\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\Big]+\\
&2\phi^{q-1}\Big[\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Big]+2\phi^{q}\Big[\sum_{j=0}^{0}\theta_j\theta_{j+q}\Big]\Bigg\}\frac{1}{1-\phi^2}\\
=&\sigma^2\Bigg\{\sum_{j=0}^{q}\theta_j^2+\sum_{n=1}^{q}2\phi^n\sum_{j=0}^{q-n}\theta_j\theta_{j+n}\Bigg\}\frac{1}{1-\phi^2}
\end{aligned}$$

Therefore,

$$\gamma(0)=\sigma^2\Bigg\{\sum_{j=0}^{q}\theta_j^2+2\sum_{n=1}^{q}\sum_{j=0}^{q-n}\phi^n\theta_j\theta_{j+n}\Bigg\}\frac{1}{1-\phi^2} \tag{4.23}$$

At lag 1 , considering terms in s gives

$$\begin{aligned}
\gamma(1) =&\sigma^2\Bigg\{\Bigg[\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\Bigg]+\\
&\Bigg[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\\
&\Bigg[\phi^4\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\cdots+\Bigg[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\Bigg]+\\
&\Bigg[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\cdots+\\
&\Bigg[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\\
&\Bigg[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\cdots+\Bigg[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}
\end{aligned}$$

This further simplifies to

$$
\begin{aligned}
\gamma(1) =&\sigma^2\Bigg\{\Bigg[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi\sum_{j=0}^{q}\theta_j^2\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\\
&\Bigg[\phi^4\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\cdots+\Bigg[\phi^q\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+1}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\\
&\Bigg[\phi\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\cdots\\
&+\Bigg[\phi^{q-2}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q-1}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}\sum_{r=0}^{\infty}\phi^{2r}\\
=&\sigma^2\Bigg\{\Bigg[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi\sum_{j=0}^{q}\theta_j^2\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\\
&\Bigg[\phi^4\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\cdots+\Bigg[\phi^q\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+1}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\\
&\Bigg[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\cdots\\
&+\Bigg[\phi^{q-2}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q-1}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}\frac{1}{1-\phi^2}\\
=&\sigma^2\Bigg\{\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+\sum_{n=0}^{q}\phi^{n+1}\sum_{j=0}^{q-n}\theta_j\theta_{j+n}+\sum_{n=0}^{q-2}\phi^{n+1}\sum_{j=0}^{q-(n+2)}\theta_j\theta_{j+(n+2)}\Bigg\}\frac{1}{1-\phi^2}
\end{aligned}
$$

Therefore,

$$
\gamma(1)=\sigma^2\Bigg\{\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+\sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{n+1}\theta_j\theta_{j+n}+\sum_{n=0}^{q-2}\sum_{j=0}^{q-(n+2)}\phi^{n+1}\theta_j\theta_{j+(n+2)}\Bigg\}\frac{1}{1-\phi^2}
$$

At lag 2 , we consider terms in s^2

$$
\begin{aligned}
\gamma(2) =&\sigma^2\Bigg\{\Bigg[\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\\
&\Bigg[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\Bigg]+\Bigg[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\\
&\Bigg[\phi^4\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\cdots+\Bigg[\phi^q\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\Bigg]+\\
&\Bigg[\phi^{q+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\\
&\Bigg[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\Bigg[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\\
&\Bigg[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-5}\theta_j\theta_{j+5}\Bigg]+\cdots+\Bigg[\phi^{q-3}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\\
&\Bigg[\phi^{q-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}\\
=&\sigma^2\Bigg\{\Bigg[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-1}\theta_j^2\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\\
&\Bigg[\phi^4\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\cdots+\Bigg[\phi^{q+1}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+2}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\\
&\Bigg[\phi\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-5}\theta_j\theta_{j+5}\Bigg]+\cdots\\
&+\Bigg[\phi^{q-3}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q-2}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}\sum_{r=0}^{\infty}\phi^{2r}\\
=&\sigma^2\Bigg\{\Bigg[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\Bigg[\phi\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-1}\theta_j^2\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Bigg]+\\
&\Bigg[\phi^4\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Bigg]+\cdots+\Bigg[\phi^{q+1}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q+2}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]+\\
&\Bigg[\phi\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Bigg]+\Bigg[\phi^2\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Bigg]+\Bigg[\phi^3\sum_{j=0}^{q-5}\theta_j\theta_{j+5}\Bigg]+\cdots\\
&+\Bigg[\phi^{q-3}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Bigg]+\Bigg[\phi^{q-2}\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg]\Bigg\}\frac{1}{1-\phi^2}
\end{aligned}
$$

Further simplification yields

$$\gamma(2) = \sigma^2 \left\{ \sum_{j=0}^{q-2} \theta_j \theta_{j+2} + \phi \sum_{j=0}^{q-1} \theta_j \theta_{j+1} + \sum_{n=0}^{q} \phi^{n+2} \sum_{j=0}^{q-n} \theta_j \theta_{j+n} + \sum_{n=0}^{q-3} \phi^{n+1} \sum_{j=0}^{q-(n+3)} \theta_j \theta_{j+(n+3)} \right\} \frac{1}{1-\phi^2}$$

$$= \sigma^2 \left\{ \sum_{n=0}^{1} \phi^{1-n} \sum_{j=0}^{q-(n+1)} \theta_j \theta_{j+(n+1)} + \sum_{n=0}^{q} \phi^{n+2} \sum_{j=0}^{q-n} \theta_j \theta_{j+n} + \sum_{n=0}^{q-3} \phi^{n+1} \sum_{j=0}^{q-(n+3)} \theta_j \theta_{j+(n+3)} \right\} \frac{1}{1-\phi^2}$$

Therefore,

$$\gamma(2) = \sigma^2 \left\{ \sum_{n=0}^{1} \sum_{j=0}^{q-(n+1)} \phi^{1-n} \theta_j \theta_{j+(n+1)} + \sum_{n=0}^{q} \sum_{j=0}^{q-n} \phi^{n+2} \theta_j \theta_{j+n} + \sum_{n=0}^{q-3} \sum_{j=0}^{q-(n+3)} \phi^{n+1} \theta_j \theta_{j+(n+3)} \right\} \frac{1}{1-\phi^2}$$

Subsequently, at lag h, for all $1 \leq h \leq (q-1)$,

$$\gamma(h) = \sigma^2 \left\{ \sum_{n=0}^{h-1} \phi^{h-1-n} \sum_{j=0}^{q-(n+1)} \theta_j \theta_{j+(n+1)} + \sum_{n=0}^{q} \phi^{h+n} \sum_{j=0}^{q-n} \theta_j \theta_{j+n} + \sum_{n=0}^{q-h-1} \phi^{n+1} \sum_{j=0}^{q-(h+n+1)} \theta_j \theta_{j+(h+n+1)} \right\} \frac{1}{1-\phi^2}$$

$$= \sigma^2 \left\{ \sum_{n=0}^{h-1} \sum_{j=0}^{q-(n+1)} \phi^{h-1-n} \theta_j \theta_{j+(n+1)} + \sum_{n=0}^{q} \sum_{j=0}^{q-n} \phi^{h+n} \theta_j \theta_{j+n} + \sum_{n=0}^{q-h-1} \sum_{j=0}^{q-(h+n+1)} \phi^{n+1} \theta_j \theta_{j+(h+n+1)} \right\} \frac{1}{1-\phi^2}$$

At lag q , we consider terms in s^q and obtain

$$
\begin{aligned}
\gamma(q) =&\sigma^2\Bigg\{\left[\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\left[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\
&\left[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots+\left[\phi^{q-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\
&\left[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+2}\right]+\left[\phi^{q+2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\
&\left[\phi^{q+3}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\left[\phi^{q+4}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\right]+\cdots\Bigg\}\\
=&\sigma^2\Bigg\{\left[\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\left[\phi\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\left[\phi^2\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots\\
&+\left[\phi^{q-2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\left[\phi^{q-1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q+2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\cdots\Bigg\}\sum_{r=0}^{\infty}\phi^{2r}\\
=&\sigma^2\Bigg\{\left[\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\left[\phi\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\left[\phi^2\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots\\
&+\left[\phi^{q-2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\left[\phi^{q-1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q+2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\cdots\Bigg\}\frac{1}{1-\phi^2}\\
=&\left\{\sum_{n=0}^{q}\phi^{q-n}\sum_{j=0}^{q-n}\theta_j\theta_{j+n}+\sum_{n=0}^{q-1}\phi^{q+n+1}\sum_{j=0}^{q-(n+1)}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}
\end{aligned}
$$

Therefore,

$$
\gamma(q)=\left\{\sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{q-n}\theta_j\theta_{j+n}+\sum_{n=0}^{q-1}\sum_{j=0}^{q-(n+1)}\phi^{q+n+1}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}
$$

At lag $q+1$, we consider terms in s^{q+1} and obtain

$$\begin{aligned}
\gamma(q+1) =& \left[\phi\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right] + \left[\phi^2\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\
&\left[\phi^3\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right] + \cdots + \left[\phi^{q-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\
&\left[\phi^{q}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right] + \left[\phi^{q+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+2}\right] + \left[\phi^{q+3}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\
&\left[\phi^{q+4}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right] + \left[\phi^{q+5}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\right]+\cdots\\
\gamma(q+1) =& \Bigg\{\left[\phi\sum_{j=0}^{0}\theta_j\theta_{j+q}\right] + \left[\phi^2\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right] + \left[\phi^3\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots\\
&+\left[\phi^{q-1}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right] + \left[\phi^{q}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right] + \left[\phi^{q+1}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+2}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right] + \left[\phi^{q+3}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\cdots\Bigg\}\sum_{r=0}^{\infty}\phi^{2r}\\
=&\Bigg\{\left[\phi\sum_{j=0}^{0}\theta_j\theta_{j+q}\right] + \left[\phi^2\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right] + \left[\phi^3\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots\\
&+\left[\phi^{q-1}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right] + \left[\phi^{q}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right] + \left[\phi^{q+1}\sum_{j=0}^{q}\theta_j^2\right]+\\
&\left[\phi^{q+2}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right] + \left[\phi^{q+3}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\cdots\Bigg\}\frac{1}{1-\phi^2}\\
=&\left\{\sum_{n=0}^{q}\phi^{q-n+1}\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=0}^{q-1}\phi^{q+n+2}\sum_{j=0}^{q-(n+1)}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}\\
=&\phi\left\{\sum_{n=0}^{q}\phi^{q-n}\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=0}^{q-1}\phi^{q+n+1}\sum_{j=0}^{q-(n+1)}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(q+1) =&\phi\left\{\sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{q-n}\theta_j\theta_{j+n}+\sum_{n=0}^{q-1}\sum_{j=0}^{q-(n+1)}\phi^{q+n+1}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}\\ =&\phi\gamma(q)\end{aligned}$$

Subsequently, at lag $q+h$, for $h\geq 1$ we consider terms in s^{q+h}

$$\begin{aligned}\gamma(q+h) =& \left[\phi^h\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\left[\phi^{h+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\\ &\left[\phi^{h+2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\cdots+\left[\phi^{q+h-2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]\\ &+\left[\phi^{q+h-1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q+h}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q}\theta_j^2\right]+\\ &\left[\phi^{q+h+1}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-1}\theta_j\theta_{j+2}\right]+\left[\phi^{q+h+2}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\\ &\left[\phi^{q+h+3}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\right]+\left[\phi^{q+h+4}\sum_{r=0}^{\infty}\phi^{2r}\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\right]+\cdots\\ =&\left\{\phi^h\left[\sum_{j=0}^{0}\theta_j\theta_{j+q}\right]+\left[\phi^{h+1}\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\right]+\left[\phi^{h+2}\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\right]+\right.\\ &\cdots+\left[\phi^{q+h-2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\left[\phi^{q+h-1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q+h}\sum_{j=0}^{q}\theta_j^2\right]+\\ &\left.\left[\phi^{q+h+1}\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\right]+\left[\phi^{q+h+2}\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\right]+\cdots\right\}\sum_{r=0}^{\infty}\phi^{2r}\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(q+h) =&\phi^h\left\{\sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{q-n}\theta_j\theta_{j+n}+\sum_{n=0}^{q-1}\sum_{j=0}^{q-(n+1)}\phi^{q+n+1}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2}\\ =&\phi^h\gamma(q) \qquad \text{for } h\geq 1\\ =&\phi^{k-q}\gamma(q) \quad \text{for } k\geq q+1\end{aligned}$$

The acvf of an ARMA(1,q) process can be summarized as

$$\gamma(0) =\sigma^2\left\{\sum_{j=0}^{q}\theta_j^2 + 2\sum_{n=1}^{q}\sum_{j=0}^{q-n}\phi^n\theta_j\theta_{j+n}\right\}\frac{1}{1-\phi^2}$$

For all $h = 1, 2, \cdots, q-1$

$$\begin{aligned}\gamma(h) =\sigma^2\Bigg\{&\sum_{n=0}^{h-1}\sum_{j=0}^{q-(n+1)}\phi^{h-1-n}\theta_j\theta_{j+(n+1)} + \sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{h+n}\theta_j\theta_{j+n}+\\ &\sum_{n=0}^{q-h-1}\sum_{j=0}^{q-(h+n+1)}\phi^{n+1}\theta_j\theta_{j+(h+n+1)}\Bigg\}\frac{1}{1-\phi^2}\end{aligned} \tag{4.24}$$

$$\gamma(q) =\left\{\sum_{n=0}^{q}\sum_{j=0}^{q-n}\phi^{q-n}\theta_j\theta_{j+n} + \sum_{n=0}^{q-1}\sum_{j=0}^{q-(n+1)}\phi^{q+n+1}\theta_j\theta_{j+(n+1)}\right\}\frac{1}{1-\phi^2} \tag{4.25}$$

For all $h \geq 1$

$$\gamma(q+h) = \phi^h\gamma(q) \tag{4.26}$$

It then follows that a relation for three consecutive ACF of ARMA$(1, q)$ is given by

$$\rho^2(q+1) = \rho(q) \times \rho(q+2) \tag{4.27}$$

ACF of an ARMA(2,0) Process

In this section, the ACF of an ARMA(2,0) process is derived. The autocovariance generating function (acgf) is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation functions.

An ARMA (2,0) process is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t \tag{4.28}$$

By introducing a lag operator, Equation (4.28) can be simplified as

$$(1-\phi_1 L-\phi_2 L^2)X_t = Z_t \tag{4.29}$$

Further simplification yields

$$X_t = \frac{1}{1-\phi_1 L-\phi_2 L^2} Z_t \tag{4.30}$$

Assuming the quadratic $1-\phi_1 L-\phi_2 L^2$ has two different real roots, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, then

$$1-\phi_1 L-\phi_2 L^2$$

can be written as

$$(1-\alpha L)(1-\beta L)$$

It can be verified that $(\alpha+\beta)$=ϕ_1 and $\alpha\beta=-\phi_2$

If

$$X_t = \frac{1}{(1-\alpha L)(1-\beta L)} Z_t$$

then

$$c(s) = \frac{1}{(1-\alpha s)(1-\beta s)}$$

The acvgf can thus, be written as

$$c(s)c(s^{-1}) = \sigma^2\left[\frac{1}{(1-\alpha s)(1-\beta s)} \times \frac{1}{(1-\alpha s^{-1})(1-\beta s^{-1})}\right] \tag{4.31}$$

This simplifies to

$$c(s)c(s^{-1}) = \sigma^2\left[\frac{1}{(1-\alpha s)(1-\alpha s^{-1})(1-\beta s)(1-\beta s^{-1})}\right] \tag{4.32}$$

Equation (4.32) simplifies to

$$\begin{aligned}
c(s)c(s^{-1}) =&\sigma^2 \sum_{r=0}^{\infty}(\alpha s)^r \cdot \sum_{r=0}^{\infty}(\alpha s^{-1})^r \cdot \sum_{r=0}^{\infty}(\beta s)^r \cdot \sum_{r=0}^{\infty}(\beta s^{-1})^r \\
=&\sigma^2 \sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\left[\sum_{r=0}^{\infty}(\alpha s)^r + \sum_{r=1}^{\infty}(\alpha s^{-1})^r\right]\times \\
&\left[\sum_{r=0}^{\infty}(\beta s)^r + \sum_{r=1}^{\infty}(\beta s^{-1})^r\right]
\end{aligned} \tag{4.33}$$

Equation (4.33) simplifies to

$$c(s)c(s^{-1}) = \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\mathbf{T}(\mathbf{s^r}) \tag{4.34}$$

where $\mathrm{T}(\mathrm{s^r})$ are expressions in terms of s^r obtained from Equation (4.33)

At lag0, we consider terms in Equation (4.33) that results in s^0 and obtain

$$\begin{aligned}
\gamma(0) =&\sigma^2\left[\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right]\mathbf{T}(\mathbf{s^0}) \\
=&\sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\left\{\Big[1(1+\alpha\beta)\Big]\right\} \\
=&\sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\left\{\Big[1(1+\alpha\beta)\Big]\right\} \\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\left\{\Big[1(1-\phi_2)\Big]\right\}
\end{aligned}$$

Therefore, the variance function of the ARMA(2,0) which is denoted by $\gamma_{2,0}$ is given as

$$\gamma_{2,0}(0) = \frac{\sigma^2}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big(1-\phi_2\Big) \tag{4.35}$$

At lag1, we consider terms that give s

$$\begin{aligned}
\gamma(1) =&\sigma^2\Big[\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\Big]\mathbf{T(s)}\\
=&\sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{1\Big[(\alpha+\beta)\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(\alpha+\beta)\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[\phi_1\Big]\Bigg\}
\end{aligned}$$

$\gamma(1)$ is obtained in terms of $\gamma_{2,0}(0)$ as

$$\begin{aligned}
\gamma(1) &= \sigma^2\left[\frac{\phi_1(1-\phi_2)}{(1-\phi_2)(1+\phi_2)\Big((1-\phi_2)^2-\phi_1^2\Big)}\right]\\
&= \sigma^2\left(\frac{\phi_1}{1-\phi_2}\right)\left[\frac{1-\phi_2}{(1+\phi_2)\Big((1-\phi_2)^2-\phi_1^2\Big)}\right]\\
&= \left(\frac{\phi_1}{1-\phi_2}\right)\gamma_{2,0}(0)
\end{aligned}$$

At lag 2, we consider terms in s^2 and obtain

$$\begin{aligned}
\gamma(2) =&\sigma^2\Big[\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\Big]\mathbf{T(s^2)}\\
=&\sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{1\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]\Bigg\}
\end{aligned}$$

Regrouping, we shall obtain

$$
\begin{aligned}
\gamma(2) =&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[\phi_1^2\Big]\Bigg\}+\\
&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[\phi_2-\phi_2^2\Big]\Bigg\}\\
=&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{\Big[\phi_1\Big]\Bigg\}+\\
&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{\Big[1-\phi_2\Big]\Bigg\}\\
=&\phi_1\gamma(1)+\phi_2\gamma_{2,0}(0)\\
=&\phi_1\Big(\frac{\phi_1}{1-\phi_2}\Big)\gamma_{2,0}(0)+\phi_2\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}(\phi_1^2+\phi_2-\phi_2^2)\gamma_{2,0}(0)\\
=&\frac{c_{1,20}}{1-\phi_2}\gamma_{2,0}(0)
\end{aligned}
$$

where $c_{1,20}=\phi_1^2-\phi_2^2+\phi_2$

The autocovariance at lag 3 is obtained as

$$
\begin{aligned}
\gamma(3) =&\sigma^2\Big[\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\Big]\mathbf{T}(\mathbf{s^3})\\
=&\sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{\Big[(\alpha^3+\beta^3)+(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-\\
&(\alpha\beta)^2(\alpha+\beta)\Big]\Bigg\}\\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]\Bigg\}
\end{aligned}
$$

Expanding some of the terms and regrouping, we shall obtain

$$\begin{aligned}
\gamma(3) =&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{\Big[\phi_1^2+\phi_2-\phi_2^2\Big]\Bigg\}+\\
&\sigma^2 \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{\Big[\phi_1\Big]\Bigg\}\\
=&\phi_1\gamma(2)+\phi_2\gamma(1)\\
=&\phi_1\Big(\frac{c_{1,20}}{1-\phi_2}\Big)\gamma(0)+\phi_2\Big(\frac{\phi_1}{1-\phi_2}\Big)\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg[\phi_1 c_{1,20}+\phi_1\phi_2\Bigg]\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg[\phi_1\Big(c_{1,20}+\phi_2\Big)\Bigg]\gamma_{2,0}(0)\\
=&\frac{c_{2,20}}{1-\phi_2}\gamma_{2,0}(0)
\end{aligned}$$

Where $c_{2,20} = \phi_1 c_{1,20} + \phi_1\phi_2$

It is obvious that our simplification of the coefficient of s^k so far leads to the Yule-Walker equations.

Subsequently,

$$\begin{aligned}
\gamma(4) &= \phi_1\gamma(3)+\phi_2\gamma(2)\\
&= \phi_1\Big(\frac{c_{2,0}}{1-\phi_2}\Big)\gamma_{2,0}(0)+\phi_2\Big(\frac{c_{1,0}}{1-\phi_2}\Big)\gamma_{2,0}(0)\\
&= \frac{1}{1-\phi_2}\Bigg[\phi_1\Big(\phi_1 c_{1,20}+\phi_1\phi_2\Big)+\phi_2 c_{1,20}\Bigg]\gamma_{2,0}(0)\\
&= \frac{1}{1-\phi_2}\Bigg[\phi_1^2 c_{1,20}+\phi_1^2\phi_2+\phi_2 c_{1,20}\Bigg]\gamma_{2,0}(0)\\
&= \frac{1}{1-\phi_2}\Bigg[\Big(\phi_1^2+\phi_2\Big)c_{1,20}+\phi_1^2\phi_2\Bigg]\gamma_{2,0}(0)\\
&= \frac{c_{3,20}}{1-\phi_2}\gamma_{2,0}(0)
\end{aligned}$$

Where $c_{3,20} = \left(\phi_1^2 + \phi_2\right)c_{1,20} + \phi_1^2\phi_2$

The autocovariance at lag 5 will be obtained as

$$\begin{aligned}
\gamma(5) &= \phi_1\gamma(4) + \phi_2\gamma(3) \\
&= \phi_1\left(\frac{c_{3,20}}{1-\phi_2}\right)\gamma_{2,0}(0) + \phi_2\left(\frac{c_{2,20}}{1-\phi_2}\right)\gamma_{2,0}(0) \\
&= \frac{1}{1-\phi_2}\left\{\left[\phi_1\left(\phi_1^2+\phi_2\right)c_{1,20} + \phi_1^2\phi_2\right] + \phi_2\left[\phi_1 c_{1,20} + \phi_1\phi_2\right]\right\}\gamma_{2,0}(0) \\
&= \frac{1}{1-\phi_2}\left\{\phi_1 c_{1,20}\left(\phi_1^2+\phi_2\right) + \phi_1^3\phi_2 + \phi_1\phi_2 c_{1,20} + \phi_1\phi_2^2\right\}\gamma_{2,0}(0) \\
&= \frac{1}{1-\phi_2}\left\{\phi_1 c_{1,20}\left(\phi_1^2+\phi_2\right) + \phi_1\phi_2\left(\phi_1^2+\phi_2\right) + \phi_1\phi_2 c_{1,20}\right\}\gamma_{2,0}(0) \\
&= \frac{1}{1-\phi_2}\left\{\left(\phi_1^2+\phi_2\right)\left(\phi_1 c_{1,20} + \phi_1\phi_2\right) + \phi_1\phi_2 c_{1,20}\right\}\gamma_{2,0}(0)
\end{aligned}$$

Where $c_{4,20} = \left(\phi_1^2 + \phi_2\right)\left(\phi_1 c_{1,20} + \phi_1\phi_2\right) + \phi_1\phi_2 c_{1,20}$

Similarly,

$$\begin{aligned}
\gamma(6) =&\phi_1\gamma(5) + \phi_2\gamma(4) \\
=&\phi_1\left(\frac{c_{4,20}}{1-\phi_2}\right)\gamma_{2,0}(0) + \phi_2\left(\frac{c_{3,20}}{1-\phi_2}\right)\gamma_{2,0}(0) \\
=&\frac{1}{1-\phi_2}\left\{\phi_1\left[\left(\phi_1^2+\phi_2\right)\left(\phi_1 c_{1,20}+\right.\right.\right. \\
&\left.\left.\left.\phi_1\phi_2\right) + \phi_1\phi_2 c_{1,20}\right] + \phi_2\left[\left(\phi_1^2+\phi_2\right)c_{1,20} + \phi_1^2\phi_2\right]\right\}\gamma_{2,0}(0) \\
=&\frac{1}{1-\phi_2}\left\{\left[\left(\phi_1^2+\phi_2\right)\left(\phi_1^2 c_{1,20} + \phi_1^2\phi_2\right) + \phi_1^2\phi_2 c_{1,20}\right]+\right. \\
&\left.\left[\left(\phi_1^2+\phi_2\right)\phi_2 c_{1,20} + \phi_1^2\phi_2^2\right]\right\}\gamma_{2,0}(0)
\end{aligned}$$

Further simplification gives

$$\begin{aligned}
\gamma(6) =& \frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2 c_{1,20}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2 c_{1,20}+ \\
& \Big(\phi_1^2+\phi_2\Big)\phi_2 c_{1,20}+\phi_1^2\phi_2^2\Bigg\}\gamma(0) \\
=& \frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2 c_{1,20}+\phi_2 c_{1,20}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2 c_{1,20}+\phi_1^2\phi_2^2\Bigg\}\gamma_{2,0}(0)
\end{aligned}$$

Where $c_{5,20} = \Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2 c_{1,20}+\phi_2 c_{1,20}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2 c_{1,20}+\phi_1^2\phi_2^2$

At lag 7, the autocovariance is obtained as

$$\begin{aligned}
\gamma(7) =& \phi_1\gamma(6)+\phi_2\gamma(5) \\
=& \phi_1\Big(\frac{c_{5,20}}{1-\phi_2}\Big)\gamma_{2,0}(0)+\phi_2\Big(\frac{c_{4,20}}{1-\phi_2}\Big)\gamma_{2,0}(0) \\
=& \frac{1}{1-\phi_2}\Bigg\{\phi_1\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2 c_{1,20}+\phi_2 c_{1,20}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2 c_{1,20}+\phi_1^2\phi_2^2\Big] \\
& +\phi_2\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,20}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,20}\Big]\Bigg\}\gamma_{2,0}(0) \\
=& \frac{1}{1-\phi_2}\Bigg\{\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^3 c_{1,20}+\phi_1\phi_2 c_{1,20}+\phi_1^3\phi_2\Big)+\phi_1^3\phi_2 c_{1,20}+ \\
& \phi_1^3\phi_2^2\Big]+\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1\phi_2 c_{1,20}+\phi_1\phi_2^2\Big)+\phi_1\phi_2^2 c_{1,20}\Big]\Bigg\}\gamma_{2,0}(0) \\
=& \frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^3 c_{1,20}+\phi_1\phi_2 c_{1,20}+\phi_1^3\phi_2\Big)+\phi_1^3\phi_2 c_{1,20}+\phi_1^3\phi_2^2 \\
& +\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1\phi_2 c_{1,20}+\phi_1\phi_2^2\Big)+\phi_1\phi_2^2 c_{1,20}\Bigg\}\gamma_{2,0}(0) \\
=& \frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^3 c_{1,20}+2\phi_1\phi_2 c_{1,20}+\phi_1^3\phi_2+\phi_1\phi_2^2\Big)+ \\
& \phi_1^3\phi_2 c_{1,20}+\phi_1\phi_2^2 c_{1,20}+\phi_1^3\phi_2^2\Bigg\}\gamma_{2,0}(0)
\end{aligned}$$

Where $c_{6,20} = \Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^3 c_{1,20}+2\phi_1\phi_2 c_{1,20}+\phi_1^3\phi_2+\phi_1\phi_2^2\Big)+\phi_1^3\phi_2 c_{1,20}+$ $\phi_1\phi_2^2 c_{1,20}+\phi_1^3\phi_2^2$

At lag 8, the autocovariance is obtained as

$$
\begin{aligned}
\gamma(8) =&\phi_1\gamma(7) + \phi_2\gamma(6)\\
=&\phi_1\Big(\frac{c_{6,20}}{1-\phi_2}\Big)\gamma(0) + \phi_2\Big(\frac{c_{5,20}}{1-\phi_2}\Big)\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg\{\phi_1\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^3c_{1,20}+2\phi_1\phi_2c_{1,20}+\phi_1^3\phi_2+\phi_1\phi_2^2\Big)+\\
&\phi_1^3\phi_2c_{1,20}+\phi_1\phi_2^2c_{1,20}+\phi_1^3\phi_2^2\Big]+\phi_2\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,20}+\phi_2c_{1,20}+\\
&\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,20}+\phi_1^2\phi_2^2\Big]\Bigg\}\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg\{\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^4c_{1,20}+2\phi_1^2\phi_2c_{1,20}+\phi_1^4\phi_2+\phi_1^2\phi_2^2\Big)+\\
&\phi_1^4\phi_2c_{1,20}+\phi_1^2\phi_2^2c_{1,20}+\phi_1^4\phi_2^2\Big]+\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2\phi_2c_{1,20}+\\
&\phi_2^2c_{1,20}+\phi_1^2\phi_2^2\Big)+\phi_1^2\phi_2^2c_{1,20}+\phi_1^2\phi_2^3\Big]\Bigg\}\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^4c_{1,20}+2\phi_1^2\phi_2c_{1,20}+\phi_1^4\phi_2+\phi_1^2\phi_2^2\Big)+\\
&\phi_1^4\phi_2c_{1,20}+\phi_1^2\phi_2^2c_{1,20}+\phi_1^4\phi_2^2+\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2\phi_2c_{1,20}+\phi_2^2c_{1,20}+\\
&\phi_1^2\phi_2^2\Big)+\phi_1^2\phi_2^2c_{1,20}+\phi_1^2\phi_2^3\Bigg\}\gamma_{2,0}(0)\\
=&\frac{1}{1-\phi_2}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^4c_{1,20}+3\phi_1^2\phi_2c_{1,20}+\phi_2^2c_{1,20}+\phi_1^4\phi_2+2\phi_1^2\phi_2^2\Big)+\\
&\phi_1^4\phi_2c_{1,20}+2\phi_1^2\phi_2^2c_{1,20}+\phi_1^4\phi_2^2+\phi_1^2\phi_2^3\Bigg\}\gamma_{2,0}(0)
\end{aligned}
$$

Where $c_{7,20} = \Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^4c_{1,20}+3\phi_1^2\phi_2c_{1,20}+\phi_2^2c_{1,20}+\phi_1^4\phi_2+2\phi_1^2\phi_2^2\Big)+ \phi_1^4\phi_2c_{1,20}+2\phi_1^2\phi_2^2c_{1,20}+\phi_1^4\phi_2^2+\phi_1^2\phi_2^3$

An expression for $\gamma(k)$ may similarly be expressed as

$$
\gamma(k) = \frac{1}{1-\phi_2}c_{k-1,20}\gamma_{2,0}(0) \tag{4.36}
$$

Let $k-1=r$. Then $c_{r,20}$ is given by

$$\begin{aligned}
c_{r,20} =&\left(\phi_1^2+\phi_2\right)\left\{c_{1,20}\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s}+\right.\\
&\left.\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\right\}+\\
&c_{1,20}\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}+\\
&\sum_{r-5\geq 2s}\binom{(r-5-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+2}
\end{aligned} \tag{4.37}$$

for all $r\geq 4$

Thus, for an ARMA$(2,0)$ process,

$$\begin{aligned}
\gamma_{2,0}(k) =&\sigma^2\left[\left(\phi_1^2+\phi_2\right)\left\{c_{1,20}\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s}+\right.\right.\\
&\left.\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\right\}+\\
&c_{1,20}\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}+\\
&\left.\sum_{r-5\geq 2s}\binom{(r-5-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+2}\right]\frac{\gamma_{2,0}(0)}{1-\phi_2}
\end{aligned} \tag{4.38}$$

By normalizing Equation (4.38), the ACF of an ARMA(2,0) process is obtained as

$$\rho_{2,0}(1)=\frac{\phi_1}{1-\phi_2} \tag{4.39}$$

$$\rho_{2,0}(2)=\frac{c_{1,20}}{1-\phi_2} \quad \text{where } c_{1,20}=\phi_1^2-\phi_2^2+\phi_2 \tag{4.40}$$

For lags greater or equal to 4, the autocorrelation is given as

$$\begin{aligned}\rho_{2,0}(k) = \Bigg[\left(\phi_1^2+\phi_2\right)\Bigg\{c_{1,20}\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s}+\\ \sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\Bigg\}+\\ c_{1,20}\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}+\\ \sum_{r-5\geq 2s}\binom{(r-5-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+2}\Bigg]\frac{1}{1-\phi_2}\end{aligned} \tag{4.41}$$

The results in Equation (4.41) shows that the ACF of the ARMA(2,0) is a function of a coefficient $c_{1,20} = \phi_1^2 - \phi_2^2 + \phi_2$, the numerator of the autocorrelation at lag 2. It also involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$ for which $r - t \geq 2s$, $3 \leq t \leq 5$. It therefore suggests that the computation of $\rho_{2,0}(k)$ will be sensitive to the lag order.

ACF of an ARMA(2,1) Process

In this section, the ACF of an ARMA(2,1) process is derived. The autocovariance generating function (acgf) is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation functions.

An ARMA (2,1) process is represented as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 Z_{t-1} + Z_t \tag{4.42}$$

By introducing a lag operator, Equation (4.42) can be simplified as

$$(1 - \phi_1 L - \phi_2 L^2)X_t = (1 + \theta_1 L)Z_t$$

Further simplification yields

$$X_t = \frac{(1+\theta_1 L)}{1-\phi_1 L-\phi_2 L^2} Z_t \tag{4.43}$$

Assuming the quadratic $1-\phi_1 L-\phi_2 L^2$ has two different real roots, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, then

$$1-\phi_1 L-\phi_2 L^2$$

can be written as

$$(1-\alpha L)(1-\beta L)$$

It can be verified that $(\alpha+\beta)$=ϕ_1 and $\alpha\beta = -\phi_2$

If

$$X_t = \frac{1+\theta_1 L}{(1-\alpha L)(1-\beta L)} Z_t$$

then

$$c(s) = \frac{1+\theta_1 s}{(1-\alpha s)(1-\beta s)}$$

The acvgf can thus, be written as

$$c(s)c(s^{-1}) = \sigma^2 \left[\frac{1+\theta_1 s}{(1-\alpha s)(1-\beta s)} \times \frac{1+\theta_1 s^{-1}}{(1-\alpha s^{-1})(1-\beta s^{-1})} \right] \tag{4.44}$$

This simplifies to

$$\frac{(1+\theta_1^2)+\theta_1 s+\theta_1 s^{-1}}{(1-\alpha s)(1-\alpha s^{-1})(1-\beta s)(1-\beta s^{-1})} \tag{4.45}$$

Equation (4.45) simplifies to

$$c(s)c(s^{-1}) = \sigma^2 \left[(1+\theta_1^2)+\theta_1 s+\theta_1 s^{-1} \right] \sum_{r=0}^{\infty} (\alpha s)^r \cdot \sum_{r=0}^{\infty} (\alpha s^{-1})^r \times \sum_{r=0}^{\infty} (\beta s)^r \cdot \sum_{r=0}^{\infty} (\beta s^{-1})^r$$

$$\begin{aligned} c(s)c(s^{-1}) =& \sigma^2 \left[(1+\theta_1^2) + \theta_1 s + \theta_1 s^{-1} \right] \sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \times \\ & \left[\sum_{r=0}^{\infty} (\alpha s)^r + \sum_{r=1}^{\infty} (\alpha s^{-1})^r \right] \left[\sum_{r=0}^{\infty} (\beta s)^r + \sum_{r=1}^{\infty} (\beta s^{-1})^r \right] \end{aligned} \tag{4.46}$$

Equation (4.46) simplifies to

$$c(s)c(s^{-1}) = \sigma^2 \left(\sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \right) \mathrm{T}(\mathbf{s^r}) \tag{4.47}$$

where $\mathrm{T}(\mathbf{s^r})$ are expressions in terms of s^r obtained from Equation (4.46)

At lag 0, we consider terms in Equation (4.46) that results in s^0

$$\begin{aligned} \gamma(0) =& \sigma^2 \left(\sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \right) \left\{ \left[(1+\theta_1^2) \right] \mathrm{T}(\mathbf{s^0}) + \left[\theta_1 s \right] \mathrm{T}(\mathbf{s^{-1}}) + \left[\theta_1 s^{-1} \right] \mathrm{T}(\mathbf{s}) \right\} \\ =& \sigma^2 \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \left\{ \left[(1+\theta_1^2)(1+\alpha\beta) \right] + \left[\theta_1(\alpha+\beta) \right] \right. \\ & \left. + \left[\theta_1(\alpha+\beta) \right] \right\} \end{aligned}$$

This simplifies to

$$\begin{aligned} \gamma(0) =& \sigma^2 \frac{1}{(1-\alpha\beta)\left[(1+\alpha\beta)^2 - (\alpha+\beta)^2 \right]} \left\{ \left[(1+\theta_1^2)(1+\alpha\beta) \right] + \right. \\ & \left. 2\left[\theta_1(\alpha+\beta) \right] \right\} \\ =& \sigma^2 \frac{1}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2 \right]} \left\{ \left[(1+\theta_1^2)(1-\phi_2) \right] + \left[2\phi_1\theta_1 \right] \right\} \\ =& \sigma^2 \frac{1}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2 \right]} \left\{ \left(1-\phi_2 \right) + \theta_1 \left(2\phi_1 + \theta_1 - \theta_1\phi_2 \right) \right\} \end{aligned}$$

Let $\chi = 2\phi_1 + \theta_1 - \theta_1\phi_2$

The variance function of an ARMA(2,1) process represented as $\gamma_{2,1}(0)$ is given by

$$\gamma_{2,1}(0) = \frac{\sigma^2}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Big\{1 - \phi_2 + \theta_1\chi\Big\} \tag{4.48}$$

It is clear from Equation (4.48) that if $\theta_1 = 0$, we obtain $\gamma_{2,0}(0)$, the variance function of the ARMA(2,0) process. At lag1, we consider terms in s and obtain

$$\begin{aligned}
\gamma(1) =& \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Big\{\Big[(1+\theta_1^2)\Big]\mathbf{T(s)} + \Big[\theta_1 s\Big]\mathbf{T(s^0)} + \Big[\theta_1 s^{-1}\Big]\mathbf{T(s^2)}\Big\}\\
=&\sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Big\{(1+\theta_1^2)\Big[(\alpha+\beta)\Big]+\\
&\theta_1\Big[(1+\alpha\beta)\Big] + \theta_1\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]\Big\}\\
=&\sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Big\{(1+\theta_1^2)\Big[(\alpha+\beta)\Big]+\\
&\theta_1\Big[(1+\alpha\beta)\Big] + \theta_1\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]\Big\}
\end{aligned}$$

A simplification gives

$$\begin{aligned}
\gamma(1) =& \sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big\{(1+\theta_1^2)\Big[\phi_1\Big] + \theta_1\Big[(1-\phi_2)\Big]+\\
&\theta_1\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]\Big\}\\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big\{(1+\theta_1^2)\Big[\phi_1\Big] + \theta_1\Big[1+\phi_1^2-\phi_2^2\Big]\Big\}\\
=&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big\{\phi_1 + \theta_1\Big(1+\theta_1\phi_1+\phi_1^2-\phi_2^2\Big)\Big\}
\end{aligned}$$

Let $\varpi = 1 + \theta_1\phi_1 + \phi_1^2 - \phi_2^2$

Then,

$$\begin{aligned}
\gamma(1) =& \frac{\sigma^2}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Big\{\phi_1 + \theta_1\varpi\Big\} \\
=& \frac{\phi_1 + \theta_1\varpi}{1 - \phi_2 + \theta_1\chi}\gamma_{2,1}(0)
\end{aligned}$$

The autocovariance at lag 2 is obtained as

$$\begin{aligned}
\gamma(2) =& \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Big\{\Big[(1+\theta_1^2)\Big]\mathbf{T}(\mathbf{s^2}) + \Big[\theta_1 s\Big]\mathbf{T}(\mathbf{s}) + \Big[\theta_1 s^{-1}\Big]\mathbf{T}(\mathbf{s^3})\Big\} \\
=& \sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Big\{(1+\theta_1^2)\Big[(\alpha^2+\beta^2) + \alpha\beta(1-\alpha\beta)\Big] + \\
& \theta_1\Big[(\alpha+\beta)\Big] + \theta_1\Big[(\alpha^3+\beta^3) + (\alpha^2\beta + \alpha\beta^2)(1-\alpha\beta)\Big]\Big\} \\
=& \sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2 - (\alpha+\beta)^2\Big]}\Big\{(1+\theta_1^2)\Big[(\alpha+\beta)^2 - \alpha\beta - (\alpha\beta)^2\Big] + \\
& \theta_1\Big[(\alpha+\beta)\Big] + \theta_1\Big[(\alpha+\beta)^3 - 2\alpha\beta(\alpha+\beta) - (\alpha\beta)^2(\alpha+\beta)\Big]\Big\}
\end{aligned}$$

Further simplification yields

$$\begin{aligned}
\gamma(2) =& \sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Big\{(1+\theta_1^2)\Big[(\phi_1^2 + \phi_2 - \phi_2^2)\Big] + \theta_1\Big[\phi_1\Big] \\
& + \theta_1\Big[(\phi_1^3 + 2\phi_1\phi_2 - \phi_1\phi_2^2)\Big]\Big\} \\
=& \sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Big\{(1+\theta_1^2)\Big[\phi_1^2 + \phi_2 - \phi_2^2\Big] + \\
& \theta_1\Big[\phi_1^3 + \phi_1 + 2\phi_1\phi_2 - \phi_1\phi_2^2\Big]\Big\}
\end{aligned}$$

Regrouping, we shall obtain

$$\begin{aligned}\gamma(2) =&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2)\Big[\phi_1^2\Big]+\theta_1\Big[\phi_1^3+\phi_1-\phi_1\phi_2^2\Big]\Bigg\}+\\ &\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2)\Big[\phi_2-\phi_2^2\Big]+\theta_1\Big[2\phi_1\phi_2\Big]\Bigg\}\\ =&\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{(1+\theta_1^2)\Big[\phi_1\Big]+\theta_1\Big[1+\phi_1^2-\phi_2^2\Big]\Bigg\}+\\ &\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{\Big[(1+\theta_1^2)(1-\phi_2)\Big]+\Big[2\phi_1\theta_1\Big]\Bigg\}\end{aligned}$$

It can be observed that

$$\begin{aligned}\gamma(2) =&\phi_1\gamma(1)+\phi_2\gamma(0)\\ =&\phi_1\Big(\frac{\phi_1+\theta_1\varpi}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0)+\phi_2\gamma_{2,1}(0)\\ =&\frac{1}{1-\phi_2+\theta_1\chi}\Big(\phi_1^2+\phi_2-\phi_2^2+\theta_1\phi_2\chi+\theta_1\phi_1\varpi\Big)\gamma_{2,1}(0)\\ =&\frac{c_{1,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)\end{aligned}$$

where $c_{1,21}=\phi_1^2+\phi_2-\phi_2^2+\theta_1\phi_2\chi+\theta_1\phi_1\varpi$

From ARMA$(2,0)$, we know that $c_{1,20}=\phi_1^2+\phi_2-\phi_2^2$

Thus,

$$\begin{aligned}c_{1,21} =&c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi\\ =&c_{1,20}+\theta_1(\phi_2\chi+\phi_1\varpi)\end{aligned}$$

At lag 3, we consider terms in s^3 and obtain

$$\gamma(3) = \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2)\Big]\mathbf{T(s^3)} + \Big[\theta_1 s\Big]\mathbf{T(s^2)} + \Big[\theta_1 s^{-1}\Big]\mathbf{T(s^4)}\Bigg\}$$

$$= \sigma^2\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2)\Big[(\alpha^3+\beta^3)+$$

$$(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big] + \theta_1\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]+$$

$$\theta_1\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big]\Bigg\}$$

$$= \sigma^2\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha+\beta)^3-$$

$$2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big] + \theta_1\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]+$$

$$\theta_1\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]\Bigg\}$$

$$= \sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+$$

$$\theta_1\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+\theta_1\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]\Bigg\}$$

$$= \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2)\Big[\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+$$

$$\theta_1\Big[\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3\Big]\Bigg\}$$

Expanding some of the terms and regrouping, we shall obtain

$$\gamma(3) = \sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{(1+\theta_1^2)\Big[\phi_1^2+\phi_2-\phi_2^2\Big]+$$

$$\theta_1\Big[\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big]\Bigg\}+$$

$$\sigma^2\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{(1+\theta_1^2)\Big[\phi_1\Big]+\theta_1\Big[1+\phi_1^2-\phi_2^2\Big]\Bigg\}$$

Therefore,

$$
\begin{aligned}
\gamma(3) =& \phi_1\gamma(2) + \phi_2\gamma(1) \\
=& \phi_1\Big(\frac{c_{1,21}}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0) + \phi_2\Big(\frac{\phi_1+\theta_1\varpi}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0) \\
=& \frac{1}{1-\phi_2+\theta_1\chi}\Bigg[\phi_1 c_{1,21} + \phi_1\phi_2 + \theta_1\phi_2\varpi\Bigg]\gamma_{2,1}(0) \\
=& \frac{1}{1-\phi_2+\theta_1\chi}\Bigg[\phi_1\Big(c_{1,21}+\phi_2\Big) + \theta_1\phi_2\varpi\Bigg]\gamma_{2,1}(0) \\
=& \frac{c_{2,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)
\end{aligned}
$$

where $c_{2,21} = \phi_1\Big(c_{1,21}+\phi_2\Big) + \theta_1\phi_2\varpi$

From ARMA$(2, 20)$, we know that $c_{2,20} = \phi_1\Big(c_{1,20}+\phi_2\Big)$

We also know that $c_{1,21} = c_{1,20} + \theta_1\phi_2\chi + \theta_1\phi_1\varpi$

Thus,

$$
\begin{aligned}
c_{2,21} =& \phi_1 c_{1,21} + \phi_1\phi_2 + \theta_1\phi_2\varpi \\
=& \phi_1\Big(c_{1,20} + \theta_1\phi_2\chi + \theta_1\phi_1\varpi\Big) + \phi_1\phi_2 + \theta_1\phi_2\varpi \\
=& \phi_1\Big(c_{1,20}+\phi_2\Big) + \theta_1\phi_1\phi_2\chi + \theta_1\phi_1^2\varpi + \theta_1\phi_2\varpi \\
=& c_{2,20} + \theta_1\phi_1\phi_2\chi + \theta_1\varpi\Big(\phi_1^2+\phi_2\Big)
\end{aligned}
$$

Just like the ARMA$(2, 0)$, it is obvious from this process also that our simplification of the coefficient of s^k so far leads to the Yule-Walker equations.

Subsequently,

$$
\begin{aligned}
\gamma(4) =& \phi_1\gamma(3) + \phi_2\gamma(2) \\
=& \phi_1\Big(\frac{c_{2,21}}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0) + \phi_2\Big(\frac{c_{1,21}}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi}\left[\phi_1\left(\phi_1 c_{1,21}+\phi_1\phi_2+\theta_1\phi_2\varpi\right)+\phi_2 c_{1,21}\right]\gamma_{2,1}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi}\left[\left(\phi_1^2+\phi_2\right)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi\right]\gamma_{2,1}(0)\\
=&\frac{c_{3,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)
\end{aligned}
$$

where $c_{3,21}=\left(\phi_1^2+\phi_2\right)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi$

From ARMA$(2,20)$, we know that $c_{3,20}=\left(\phi_1^2+\phi_2\right)c_{1,20}+\phi_1^2\phi_2$

We also know that $c_{1,21}=c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi$

Thus,

$$
\begin{aligned}
c_{3,21}=&\left(\phi_1^2+\phi_2\right)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi\\
=&\left(\phi_1^2+\phi_2\right)\left(c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi\right)+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi\\
=&c_{3,20}+\theta_1\chi\left(\phi_1^2\phi_2+\phi_2^2\right)+\theta_1\varpi\left(\phi_1^3+2\phi_1\phi_2\right)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\gamma(5)=&\phi_1\gamma(4)+\phi_2\gamma(3)\\
=&\phi_1\left(\frac{c_{3,21}}{1-\phi_2+\theta_1\chi}\right)\gamma_{2,1}(0)+\phi_2\left(\frac{c_{2,21}}{1-\phi_2+\theta_1\chi}\right)\gamma_{2,1}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi}\left\{\phi_1\left[\left(\phi_1^2+\phi_2\right)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi\right]+\right.\\
&\left.\phi_2\left[\phi_1 c_{1,21}+\phi_1\phi_2+\theta_1\phi_2\varpi\right]\right\}\gamma_{2,1}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi}\left\{\phi_1 c_{1,21}\left(\phi_1^2+\phi_2\right)+\phi_1\phi_2\left(\phi_1^2+\phi_2\right)+\phi_1\phi_2 c_{1,21}+\right.\\
&\left.\theta_1\varpi(\phi_1^2\phi_2+\phi_2^2)\right\}\gamma_{2,1}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi}\Bigg\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+\\
&\theta_1\varpi\Big(\phi_1^2\phi_2+\phi_2^2\Big)\Bigg\}\gamma_{2,1}(0)\\
=&\frac{c_{4,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)
\end{aligned}
$$

where $c_{4,21}=\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+\theta_1\varpi\Big(\phi_1^2\phi_2+\phi_2^2\Big)$

From ARMA$(2,0)$, we know that $c_{4,20}=\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,20}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,20}$

We also know that $c_{1,21}=c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi$

Thus,

$$
\begin{aligned}
c_{4,21}=&\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+\theta_1\varpi\Big(\phi_1^2\phi_2+\phi_2^2\Big)\\
=&\Big(\phi_1^2+\phi_2\Big)\Big[\phi_1(c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi)+\phi_1\phi_2\Big]+\\
&\phi_1\phi_2\Big(c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi\Big)+\theta_1\varpi\Big(\phi_1^2\phi_2+\phi_2^2\Big)\\
=&c_{4,20}+\theta_1\chi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)+\theta_1\varpi\Big(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2\Big)
\end{aligned}
$$

The autocovariance at lag 6 is also obtained as

$$
\begin{aligned}
\gamma(6)=&\phi_1\gamma(5)+\phi_2\gamma(4)\\
=&\phi_1\Big(\frac{c_{4,21}}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0)+\phi_2\Big(\frac{c_{3,21}}{1-\phi_2+\theta_1\chi}\Big)\gamma_{2,1}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi}\Bigg\{\phi_1\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+\\
&\theta_1\varpi\Big(\phi_1^2\phi_2+\phi_2^2\Big)\Big]+\phi_2\Big[\Big(\phi_1^2+\phi_2\Big)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2^2\varpi\Big]\Bigg\}\gamma_{2,1}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi}\Bigg\{\Big[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2 c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2 c_{1,21}+\\
&\theta_1\varpi\Big(\phi_1^3\phi_2+\phi_1\phi_2^2\Big)\Big]+\Big[\Big(\phi_1^2+\phi_2\Big)\phi_2 c_{1,21}+\phi_1^2\phi_2^2+\theta_1\phi_1\phi_2^2\varpi\Big]\Bigg\}\gamma_{2,1}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi}\Big\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_2c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,21}+\\
&\phi_1^2\phi_2^2+\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)\Big\}\gamma_{2,1}(0)\\
=&\frac{c_{5,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)
\end{aligned}
$$

where $c_{5,21} = \Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_2c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,21}+\phi_1^2\phi_2^2+\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)$

From ARMA$(2,20)$, we know that

$c_{5,20} = \Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,20}+\phi_2c_{1,20}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,20}+\phi_1^2\phi_2^2$

We also know that $c_{1,21} = c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi$

Thus,

$$
\begin{aligned}
c_{5,21} =&\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_2c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,21}+\phi_1^2\phi_2^2+\\
&\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)
\end{aligned}
$$

A simplification gives

$$
\begin{aligned}
c_{5,21} =&\Big(\phi_1^2+\phi_2\Big)\Big[\phi_1^2(c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi)\Big]+\Big(\phi_1^2+\phi_2\Big)\Big[\phi_2(c_{1,20}+\\
&\theta_1\phi_2\chi+\theta_1\phi_1\varpi)\Big]+\Big(\phi_1^2+\phi_2\Big)\phi_1^2\phi_2+\phi_1^2\phi_2\Big(c_{1,20}+\theta_1\phi_2\chi+\theta_1\phi_1\varpi\Big)\\
&+\phi_1^2\phi_2^2+\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)\\
=&c_{5,20}+\theta_1\chi\Big(\phi_1^4\phi_2+3\phi_1^2\phi_2^2+\phi_2^3\Big)+\theta_1\varpi\Big(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2\Big)
\end{aligned}
$$

Subsequently,

$$
\gamma(7) =\frac{c_{6,21}}{1-\phi_2+\theta_1\chi}\gamma_{2,1}(0)
$$

Where

$$\begin{aligned}
c_{6,21} =& \left(\phi_1^2 + \phi_2\right)\left(\phi_1^3 c_{1,21} + 2\phi_1\phi_2 c_{1,21} + \phi_1^3\phi_2 + \phi_1\phi_2^2\right) + \phi_1^3\phi_2 + \phi_1\phi_2^2 c_{1,21} \\
& + \phi_1^3\phi_2^2 + \theta_1\left(\phi_1^4\phi_2 + 3\phi_1^2\phi_2^2 + \phi_2^3\right) \\
=& c_{6,20} + \theta_1\chi\left(\phi_1^5\phi_2 + 4\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3\right) + \theta_1\varpi\left(\phi_1^6 + 5\phi_1^4\phi_2 + 6\phi_1^2\phi_2^2 + \phi_2^3\right)
\end{aligned}$$

$$\gamma(8) = \frac{c_{7,21}}{1 - \phi_2 + \theta_1\chi}\gamma_{2,1}(0)$$

Where

$$\begin{aligned}
c_{7,21} =& \left(\phi_1^2 + \phi_2\right)\left(\phi_1^4 c_{1,21} + 3\phi_1^2\phi_2 c_{1,21} + \phi_2^2 c_{1,21} + \phi_1^4\phi_2\right. \\
& \left. + 2\phi_1^2\phi_2^2\right) + \phi_1^4\phi_2 c_{1,21} + 2\phi_1^2\phi_2^2 c_{1,21} + \phi_1^4\phi_2^2 + \phi_1^2\phi_2^3 + \\
& \theta_1\left(\phi_1^5\phi_2 + 4\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3\right) \\
=& c_{7,20} + \theta_1\chi\left(\phi_1^6\phi_2 + 5\phi_1^4\phi_2^2 + 6\phi_1^2\phi_2^3 + \phi_2^4\right) + \\
& \theta_1\varpi\left(\phi_1^7 + 6\phi_1^5\phi_2 + 10\phi_1^3\phi_2^2 + 4\phi_1\phi_2^3\right)
\end{aligned}$$

In terms of $c_{r,21}$, an expression for $\gamma(k)$ may be expressed as

$$\gamma(k) = \frac{c_{k-1,21}}{1 - \phi_2 + \theta_1\chi}\gamma_{2,1}(0) \tag{4.49}$$

Let $k - 1 = r$. Then

$$\begin{aligned}
c_{r,21} =& \left(\phi_1^2 + \phi_2\right)\left\{c_{1,21}\sum_{r-3\geq 2s}\binom{r-3-s}{s}\phi_1^{r-3-2s}\phi_2^s + \right. \\
& \left. \sum_{r-4\geq 2s}\binom{r-4-s}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\right\} + \\
& c_{1,21}\sum_{r-4\geq 2s}\binom{r-4-s}{s}\phi_1^{r-3-2s}\phi_2^{s+1} + \\
& \sum_{r-5\geq 2s}\binom{r-5-s}{s}\phi_1^{r-3-2s}\phi_2^{s+2}
\end{aligned} \tag{4.50}$$

for all $r \geq 4$

Thus, for an ARMA(2,1) process,

$$\begin{aligned}\gamma(k) = \sigma^2 \Bigg[\left(\phi_1^2 + \phi_2\right) \Bigg\{ c_{1,21} \sum_{r-3 \geq 2s} \binom{(r-3-s)}{s} \phi_1^{r-3-2s} \phi_2^{s} + \\ \sum_{r-4 \geq 2s} \binom{(r-4-s)}{s} \phi_1^{r-3-2s} \phi_2^{s+1} \Bigg\} + \\ c_{1,21} \sum_{r-4 \geq 2s} \binom{(r-4-s)}{s} \phi_1^{r-3-2s} \phi_2^{s+1} + \\ \sum_{r-5 \geq 2s} \binom{(r-5-s)}{s} \phi_1^{r-3-2s} \phi_2^{s+2} \Bigg] \frac{1}{1-\phi_2+\theta_1\chi}\end{aligned} \tag{4.51}$$

In terms of $c_{r,20}$,

$$\begin{aligned}c_{r,21} = & c_{r,20} + \theta_1 \chi \left[\sum_{r-1 \geq 2s} \binom{(r-1-s)}{s} \phi_1^{r-1-2s} \phi_2^{s+1} \right] + \\ & \theta_1 \varpi \left[\sum_{r \geq 2s} \binom{(r-s)}{s} \phi_1^{r-2s} \phi_2^{s} \right]\end{aligned} \tag{4.52}$$

for all $r \geq 4$

Equivalently for an ARMA(2,1) process,

$$\begin{aligned}\gamma(k) = & \sigma^2 \Bigg\{ c_{r,20} + \theta_1 \chi \left[\sum_{r-1 \geq 2s} \binom{(r-1-s)}{s} \phi_1^{r-1-2s} \phi_2^{s+1} \right] + \\ & \theta_1 \varpi \left[\sum_{r \geq 2s} \binom{(r-s)}{s} \phi_1^{r-2s} \phi_2^{s} \right] \Bigg\} \frac{1}{1-\phi_2+\theta_1\chi} \gamma_{2,1}(0)\end{aligned} \tag{4.53}$$

The results in Equation (4.53) shows that the ACF of the ARMA(2,1) is a function of a coefficient $c_{1,21} = c_{1,20} + \theta_1\phi_2\chi + \theta_1\phi_1\varpi$, , which may further be given in terms of a coefficient $c_{r,20}$, a general coefficient for ARMA(2,0). The results then involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$ for which $r - t \geq 2s$, $0 \leq t \leq 1$.

Another important observation is that two more constants (χ and ϖ) have been introduced than the constants in the general expression for ARMA(2,0) process.

ACF of an ARMA(2,2) Process

In this section, the ACF of an ARMA(2,2) process is derived. The autocovariance generating function (acgf) is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation functions.

An ARMA (2,2) process is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t \tag{4.54}$$

By introducing the lag operator, Equation (4.54) can be simplified as

$$(1 - \phi_1 L - \phi_2 L^2) X_t = (1 + \theta_1 L + \theta_2 L^2) Z_t$$

Further simplification yields

$$X_t = \frac{(1 + \theta_1 L + \theta_2 L^2)}{1 - \phi_1 L - \phi_2 L^2} Z_t \tag{4.55}$$

Assuming the quadratic $1 - \phi_1 L - \phi_2 L^2$ has two different real roots, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, then

$$1 - \phi_1 L - \phi_2 L^2 = 0$$

can be written as

$$Q(L) = (1 - \alpha L)(1 - \beta L) = 0$$

If

$$X_t = \frac{1 + \theta_1 L + \theta_2 L^2}{(1 - \alpha L)(1 - \beta L)} Z_t$$

then

$$c(s) = \frac{1 + \theta_1 s + \theta_2 s^2}{(1 - \alpha s)(1 - \beta s)}$$

The agcf can thus, be written as

$$c(s)c(s^{-1}) = \sigma^2 \left[\frac{1+\theta_1 s + \theta_2 s^2}{(1-\alpha s)(1-\beta s)} \times \frac{1+\theta_1 s^{-1} + \theta_2 s^{-2}}{(1-\alpha s^{-1})(1-\beta s^{-1})} \right] \tag{4.56}$$

This simplifies to

$$c(s)c(s^{-1}) = \frac{\theta_2 s^2 + (\theta_1+\theta_1\theta_2)s + (1+\theta_1^2+\theta_2^2) + (\theta_1+\theta_1\theta_2)s^{-1} + \theta_2 s^{-2}}{(1-\alpha s)(1-\alpha s^{-1})(1-\beta s)(1-\beta s^{-1})} \tag{4.57}$$

Equation (4.57) simplifies to

$$\begin{aligned} c(s)c(s^{-1}) =& \sigma^2 \Big[(1+\theta_1^2+\theta_2^2) + (\theta_1+\theta_1\theta_2)s + \theta_2 s^2 + (\theta_1+\theta_1\theta_2)s^{-1} \\ &+ \theta_2 s^{-2} \Big] \sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \left[\sum_{r=0}^{\infty} (\alpha s)^r + \sum_{r=1}^{\infty} (\alpha s^{-1})^r \right] \times \\ &\left[\sum_{r=0}^{\infty} (\beta s)^r + \sum_{r=1}^{\infty} (\beta s^{-1})^r \right] \end{aligned} \tag{4.58}$$

Equation (4.58) simplifies to

$$c(s)c(s^{-1}) = \sigma^2 \left(\sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \right) \mathrm{T}(\mathbf{s^r}) \tag{4.59}$$

where $\mathrm{T}(\mathbf{s^r})$ are expression in terms of s^r obtained from Equation (4.58).

At lag0, we consider terms in Equation (4.58) that results in s^0 and obtain

$$\begin{aligned} \gamma(0) =& \sigma^2 \left(\sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \right) \Big\{ \Big[(1+\theta_1^2+\theta_2^2) \Big] \mathrm{T}(\mathbf{s^0}) + \Big[(\theta_1+\theta_1\theta_2)s \Big] \mathrm{T}(\mathbf{s^{-1}}) + \\ & \Big[\theta_2 s^2 \Big] \mathrm{T}(\mathbf{s^{-2}}) + \Big[(\theta_1+\theta_1\theta_2)s^{-1} \Big] \mathrm{T}(\mathbf{s}) + \Big[\theta_2 s^{-2} \Big] \mathrm{T}(\mathbf{s^2}) \Big\} \end{aligned}$$

$$=\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(1+\alpha\beta)\Big]+$$

$$\Big[(\theta_1+\theta_1\theta_2)(\alpha+\beta)\Big]+\theta_2\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]+$$

$$\Big[(\theta_1+\theta_1\theta_2)(\alpha+\beta)\Big]+\theta_2\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]\Bigg\}$$

$$=\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(1+\alpha\beta)\Big]+$$

$$2\Big[(\alpha+\beta)(\theta_1+\theta_1\theta_2)\Big]+2\theta_2\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]\Bigg\}$$

$$=\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(1+\alpha\beta)\Big]+$$

$$2\Big[(\alpha+\beta)(\theta_1+\theta_1\theta_2)\Big]+2\theta_2\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(1-\phi_2)\Big]+\Big[2\phi_1(\theta_1+\theta_1\theta_2)\Big]$$

$$+\Big[2\theta_2(\phi_1^2+\phi_2-\phi_2^2)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(1-\phi_2\Big)+\theta_1\Big(2\phi_1+\theta_1-\theta_1\phi_2\Big)+$$

$$\theta_2\Big(2\phi_1\theta_1+2\phi_1^2+2\phi_2-2\phi_2^2+\theta_2-\theta_2\phi_2\Big)\Bigg\}$$

Let

$$\begin{aligned}\chi =&2\phi_1+\theta_1-\theta_1\phi_2\\ =&2\phi_1+\theta_1(1-\phi_2)\end{aligned}$$

and

$$\begin{aligned}\tau =&2\phi_1\theta_1+2\phi_1^2+2\phi_2-2\phi_2^2+\theta_2-\theta_2\phi_2\\ =&2c_{1,20}+2\phi_1\theta_1+\theta_2(1-\phi_2)\end{aligned}$$

Therefore, the variance function of ARMA(2,2) denoted as $\gamma_{2,2}(0)$ is given as

$$\gamma_{2,2}(0) = \frac{\sigma^2}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Bigg\{1 - \phi_2 + \theta_1\chi + \theta_2\tau\Bigg\} \tag{4.60}$$

It is clear from Equation (4.60) that if $\theta_2 = 0$, we obtain $\gamma_{2,1}(0)$, the variance function of the ARMA(2,1) process. Additionally, if $\theta_1 = \theta_2 = 0$, we obtain $\gamma_{2,0}(0)$, the variance function of the ARMA(2,0) process.

At lag 1, we consider terms in s and obtain

$$\begin{aligned}
\gamma(1) =& \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\Big]\mathbf{T}(\mathbf{s}) + \Big[(\theta_1+\theta_1\theta_2)s\Big]\mathbf{T}(\mathbf{s^0}) + \\
& \Big[\theta_2 s^2\Big]\mathbf{T}(\mathbf{s^{-1}}) + \Big[(\theta_1+\theta_1\theta_2)s^{-1}\Big]\mathbf{T}(\mathbf{s^2}) + \Big[\theta_2 s^{-2}\Big]\mathbf{T}(\mathbf{s^3})\Bigg\} \\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(\alpha+\beta)\Big] + \\
& \Big[(\theta_1+\theta_1\theta_2)(1+\alpha\beta)\Big] + \Big[\theta_2(\alpha+\beta)\Big] + \Big[(\theta_1+\theta_1\theta_2)\{(\alpha^2+\beta^2) + \\
& \alpha\beta(1-\alpha\beta)\}\Big] + \theta_2\Big[(\alpha^3+\beta^3) + (\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]\Bigg\} \\
=& \frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)(\alpha+\beta)\Big] + \\
& \Big[(\theta_1+\theta_1\theta_2)(1+\alpha\beta)\Big] + \Big[\theta_2(\alpha+\beta)\Big] + \Big[(\theta_1+\theta_1\theta_2)\{(\alpha+\beta)^2 - \\
& \alpha\beta - (\alpha\beta)^2\}\Big] + \theta_2\Big[(\alpha+\beta)^3 - 2\alpha\beta(\alpha+\beta) - (\alpha\beta)^2(\alpha+\beta)\Big]\Bigg\} \\
=& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\phi_1\Big] + \Big[(\theta_1+\theta_1\theta_2)(1-\phi_2)\Big] + \\
& \Big[\theta_2\phi_1\Big] + \Big[(\theta_1+\theta_1\theta_2)(\phi_1^2+\phi_2-\phi_2^2)\Big] + \Big[\theta_2(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]\Bigg\}
\end{aligned}$$

Further simplification gives

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\phi_1\Big]+(\theta_1+\theta_1\theta_2)\Big[1+\phi_1^2-\phi_2^2\Big]$$

$$+\theta_2\Big[\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\phi_1+\theta_1\Big(1+\theta_1\phi_1+\phi_1^2-\phi_2^2\Big)+\theta_2\Big(\theta_1+\theta_2\phi_1$$

$$+\theta_1\phi_1^2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big)\Bigg\}$$

Let $\varpi = 1+\theta_1\phi_1+\phi_1^2-\phi_2^2$ and $\pi = \theta_1+\theta_2\phi_1+\theta_1\phi_1^2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2$

Therefore,

$$\gamma(1) =\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Bigg\{\phi_1+\theta_1\varpi+\theta_2\pi\Bigg\}$$

$$=\frac{\phi_1+\theta_1\varpi+\theta_2\pi}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)$$

Similarly, the autocovariance at lag 2 is obtained as

$$\gamma(2) =\sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\Big]\mathbf{T}(\mathbf{s^2})+\Big[(\theta_1+\theta_1\theta_2)s\Big]\mathbf{T}(\mathbf{s})+$$

$$\Big[\theta_2 s^2\Big]\mathbf{T}(\mathbf{s^0})+\Big[(\theta_1+\theta_1\theta_2)s^{-1}\Big]\mathbf{T}(\mathbf{s^3})+\Big[\theta_2 s^{-2}\Big]\mathbf{T}(\mathbf{s^4})\Bigg\}$$

$$=\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]$$

$$+(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)\Big]+\theta_2\Big[(1+\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha^3+\beta^3)+$$

$$(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+\theta_2\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)$$

$$(1-\alpha\beta)\Big]\Bigg\}$$

$$=\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha+\beta)^2-\alpha\beta-$$

$$(\alpha\beta)^2\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)\Big]+\theta_2\Big[(1+\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^3-$$

$$2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]+\theta_2\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-$$

$$(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+$$

$$(\theta_1+\theta_1\theta_2)\Big[\phi_1\Big]+\theta_2\Big[(1-\phi_2)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+$$

$$\theta_2\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^2+\phi_2-\phi_2^2\Big]+(\theta_1+\theta_1\theta_2)$$

$$\Big[\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+\theta_2\Big[1+\phi_1^4+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2+\phi_2^2-\phi_2^3\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(\phi_1^2+\phi_2-\phi_2^2\Big)+\theta_1\Big(\theta_1\phi_1^2+\theta_1\phi_2-\theta_1\phi_2^2+\phi_1^3+$$

$$\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big)+\theta_2\Big(1+\phi_1^4+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2+\phi_2^2-\phi_2^3+\theta_2\phi_1^2+$$

$$\theta_2\phi_2-\theta_2\phi_2^2+\theta_1\phi_1^3+\theta_1\phi_1+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2\Big)\Bigg\}$$

Let $\eta=\theta_1\phi_1^2+\theta_1\phi_2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2$ and $\lambda=1+\phi_1^4+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2+\phi_2^2-\phi_2^3+\theta_2\phi_1^2+\theta_2\phi_2-\theta_2\phi_2^2+\theta_1\phi_1^3+\theta_1\phi_1+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2$

Then,

$$\begin{aligned}\gamma(2)=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Bigg\{\phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda\Bigg\}\\=&\frac{\phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)\\=&\frac{c_{1,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)\end{aligned}$$

where $c_{1,22} = \phi_1^2 + \phi_2 - \phi_2^2 + \theta_1\eta + \theta_2\lambda$

From ARMA$(2,1)$, it can be verified that $\eta = \phi_2\chi + \phi_1\varpi$.

Again, $c_{1,21} = \phi_1^2 + \phi_2 - \phi_2^2 + \theta_1(\phi_2\chi + \phi_1\varpi)$ and $c_{1,20} = \phi_1^2 + \phi_2 - \phi_2^2$

Thus,

$$\begin{aligned}
c_{1,22} =& \phi_1^2 + \phi_2 - \phi_2^2 + \theta_1(\phi_2\chi + \phi_1\varpi) + \theta_2\lambda \\
=& c_{1,21} + \theta_2\lambda \\
=& c_{1,0} + \theta_1\Big(\phi_2\chi + \phi_1\varpi\Big) + \theta_2\lambda
\end{aligned}$$

At lag 3, we consider terms in s^3 and obtain

$$\begin{aligned}
\gamma(3) =& \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\Big]\mathbf{T(s^3)} + \Big[(\theta_1+\theta_1\theta_2)s\Big]\mathbf{T(s^2)} + \\
& \Big[\theta_2 s^2\Big]\mathbf{T(s)} + \Big[(\theta_1+\theta_1\theta_2)s^{-1}\Big]\mathbf{T(s^4)} + \Big[\theta_2 s^{-2}\Big]\mathbf{T(s^5)}\Bigg\} \\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha^3+\beta^3)+ \\
& (\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big] + (\theta_1+\theta_1\theta_2)\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big]+ \\
& \theta_2\Big[(\alpha+\beta)\Big] + (\theta_1+\theta_1\theta_2)\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big] \\
& + \theta_2\Big[(\alpha^5+\beta^5)+(\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big]\Bigg\} \\
=& \frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha+\beta)^3 - 2\alpha\beta(\alpha+\beta) \\
& - (\alpha\beta)^2(\alpha+\beta)\Big] + (\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^2 - \alpha\beta - (\alpha\beta)^2\Big] + \theta_2\Big[(\alpha+\beta)\Big] \\
& + (\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^4 - 3\alpha\beta(\alpha+\beta)^2 + (\alpha\beta)^2 - (\alpha\beta)^2\{(\alpha+\beta)^2+ \\
& \alpha\beta\}\Big] + \theta_2\Big[(\alpha+\beta)^5 - 4\alpha\beta(\alpha+\beta)^3 + 3(\alpha\beta)^2(\alpha+\beta) - (\alpha\beta)^2(\alpha+\beta)^3 \\
& + 2(\alpha\beta)^3(\alpha+\beta)\Big]\Bigg\}
\end{aligned}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+$$

$$(\theta_1+\theta_1\theta_2)\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+\theta_2\Big[\phi_1\Big]+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^4+3\phi_1^2\phi_2+$$

$$\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]+\theta_2\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+$$

$$(\theta_1+\theta_1\theta_2)\Big[\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3\Big]+\theta_2\Big[\phi_1^5+\phi_1+$$

$$4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big)+\theta_1\Big(\phi_1^4+\phi_1^2+$$

$$3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2\Big)+\theta_2\Big(\phi_1^5+\phi_1+$$

$$4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_2\phi_1^3+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\theta_1\phi_1^4+$$

$$\theta_1\phi_1^2+3\theta_1\phi_1^2\phi_2-\theta_1\phi_1^2\phi_2^2+\theta_1\phi_2-\theta_1\phi_2^3\Big)\Bigg\}$$

Let $\omega=\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2$ and $\kappa=\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_2\phi_1^3+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\theta_1\phi_1^4+\theta_1\phi_1^2+3\theta_1\phi_1^2\phi_2-\theta_1\phi_1^2\phi_2^2+\theta_1\phi_2-\theta_1\phi_2^3$

Then,

$$\begin{aligned}\gamma(3)=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Bigg\{\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2+\theta_1\omega+\theta_2\kappa\Bigg\}\\
=&\frac{\phi_1^3+\phi_1\phi_2-\phi_1\phi_2^2+\phi_1\phi_2+\theta_1\omega+\theta_2\kappa}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)\\
=&\frac{\phi_1\Big(\phi_1^2+\phi_2-\phi_2^2\Big)+\phi_1\phi_2+\theta_1\omega+\theta_2\kappa}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)\\
=&\frac{c_{2,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)\end{aligned}$$

where $c_{2,22} = \phi_1\left(\phi_1^2 + \phi_2 - \phi_2^2\right) + \phi_1\phi_2 + \theta_1\omega + \theta_2\kappa$

From ARMA$(2,1)$, it can be verified that $c_{2,21} = \phi_1 c_{1,21} + \phi_1\phi_2 + \theta_1\phi_2\varpi = \phi_1\left(\phi_1^2 + \phi_2 - \phi_2^2\right) + \phi_1\phi_2 + \theta_1\omega$

Thus,

$$\begin{aligned}
c_{2,22} =& \phi_1\left(\phi_1^2 + \phi_2 - \phi_2^2\right) + \phi_1\phi_2 + \theta_1\omega + \theta_2\kappa \\
=& \phi_1 c_{1,21} + \phi_1\phi_2 + \theta_1\phi_2\varpi + \theta_2\kappa \\
=& c_{2,21} + \theta_2\kappa \\
=& c_{2,0} + \theta_1\phi_1\phi_2\chi + \theta_1\varpi\left(\phi_1^2 + \phi_2\right) + \theta_2\kappa
\end{aligned}$$

At lag 4, the autocovariance is obtained as

$$\begin{aligned}
\gamma(4) =& \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\Big]\mathbf{T}(\mathbf{s^4}) + \Big[(\theta_1+\theta_1\theta_2)s\Big]\mathbf{T}(\mathbf{s^3}) + \\
& \Big[\theta_2 s^2\Big]\mathbf{T}(\mathbf{s^2}) + \Big[(\theta_1+\theta_1\theta_2)s^{-1}\Big]\mathbf{T}(\mathbf{s^5}) + \Big[\theta_2 s^{-2}\Big]\mathbf{T}(\mathbf{s^6})\Bigg\} \\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha^4+\beta^4) + (\alpha^3\beta + \\
& \alpha^2\beta^2 + \alpha\beta^3)(1-\alpha\beta)\Big] + (\theta_1+\theta_1\theta_2)\Big[(\alpha^3+\beta^3) + (\alpha^2\beta+\alpha\beta^2) \\
& (1-\alpha\beta)\Big] + \theta_2\Big[(\alpha^2+\beta^2) + \alpha\beta(1-\alpha\beta)\Big] + (\theta_1+\theta_1\theta_2)\Big[(\alpha^5+\beta^5) \\
& + (\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big] + \theta_2\Big[(\alpha^6+\beta^6) + (\alpha^5\beta + \\
& \alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5)(1-\alpha\beta)\Big]\Bigg\}
\end{aligned}$$

$$=\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha+\beta)^4-$$

$$3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]+(\theta_1+\theta_1\theta_2)$$

$$\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]+\theta_2\Big[(\alpha+\beta)^2-\alpha\beta-$$

$$(\alpha\beta)^2\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^5-4\alpha\beta(\alpha+\beta)^3+3(\alpha\beta)^2(\alpha+\beta)-$$

$$(\alpha\beta)^2(\alpha+\beta)^3+2(\alpha\beta)^3(\alpha+\beta)\Big]+\theta_2\Big[(\alpha+\beta)^6-5\alpha\beta(\alpha+\beta)^4+$$

$$6(\alpha\beta)^2(\alpha+\beta)^2-(\alpha\beta)^2(\alpha+\beta)^4+3(\alpha\beta)^3(\alpha+\beta)^2-(\alpha\beta)^3-(\alpha\beta)^4\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]$$

$$+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\theta_2\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+(\theta_1+\theta_1\theta_2)$$

$$\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\theta_2\Big[(\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-$$

$$\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2^3-\phi_2^4)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]$$

$$+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\theta_2\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+(\theta_1+\theta_1\theta_2)$$

$$\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\theta_2\Big[(\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-$$

$$\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2^3-\phi_2^4)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3\Big]$$

$$+(\theta_1+\theta_1\theta_2)\Big[\phi_1^5+\phi_1^3+4\phi_1^3\phi_2+2\phi_1\phi_2^2+2\phi_1\phi_2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3\Big]+$$

$$\theta_2\Big[\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2+\phi_2-\phi_2^2+\phi_2^3-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2^4\Big]\Bigg\}$$

Expanding some of the terms and regrouping, we shall obtain

$$
\begin{aligned}
\gamma(4) =& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^4+2\phi_1^2\phi_2-\phi_1^2\phi_2^2\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^5+\phi_1^3+3\phi_1^3\phi_2+\phi_1\phi_2-\phi_1^3\phi_2^2-\phi_1\phi_2^3\Big]+\theta_2\Big[\phi_1^6+\phi_1^2+ \\
& 4\phi_1^4\phi_2+3\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-2\phi_1^2\phi_2^3\Big]\Bigg\}+ \\
& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^2\phi_2+\phi_2^2-\phi_2^3\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^3\phi_2+\phi_1\phi_2+2\phi_1\phi_2^2-\phi_1\phi_2^3\Big]+\theta_2\Big[\phi_2+\phi_1^4\phi_2+ \\
& 3\phi_1^2\phi_2^2-\phi_1^2\phi_2^3-\phi_2^2+\phi_2^3-\phi_2^4\Big]\Bigg\}
\end{aligned}
$$

Simplifying further, the autocovariance at lag 4 is given as

$$
\begin{aligned}
\gamma(4) =& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3\Big]+\theta_2\Big[\phi_1^5+\phi_1+ \\
& 4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3\Big]\Bigg\}+ \\
& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^2+\phi_2-\phi_2^2\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+\theta_2\Big[1+\phi_1^4+3\phi_1^2\phi_2- \\
& \phi_1^2\phi_2^2-\phi_2+\phi_2^2-\phi_2^3\Big]\Bigg\} \\
=& \phi_1\gamma(3)+\phi_2\gamma(2) \\
=& \phi_1\left(\frac{c_{2,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0)+\phi_2\left(\frac{c_{1,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\phi_1\Big(\phi_1 c_{1,21}+\phi_1\phi_2+\theta_1\phi_2\varpi\Big)+\\
&\phi_2\Big(c_{1,21}+\theta_2\lambda\Big)\Bigg]\gamma_{2,2}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big(\phi_1^2 c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+\theta_2\phi_1\kappa\Big)+\\
&\Big(\phi_2 c_{1,21}+\theta_2\phi_2\lambda\Big)\Bigg]\gamma_{2,2}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big(\phi_1^2+\phi_2\Big)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+\\
&\theta_2\phi_1\kappa+\theta_2\phi_2\lambda\Bigg]\gamma_{2,2}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[c_{3,21}+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda\Bigg]\gamma_{2,2}(0)\\
=&\frac{c_{3,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)
\end{aligned}
$$

Similarly, the autocovariance at lag 5 is obtained as

$$
\begin{aligned}
\gamma(5)=&\sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\Bigg\{\Big[(1+\theta_1^2+\theta_2^2)\Big]\mathbf{T}(\mathbf{s^5})+\Big[(\theta_1+\theta_1\theta_2)s\Big]\mathbf{T}(\mathbf{s^4})+\\
&\Big[\theta_2 s^2\Big]\mathbf{T}(\mathbf{s^3})+\Big[(\theta_1+\theta_1\theta_2)s^{-1}\Big]\mathbf{T}(\mathbf{s^6})+\Big[\theta_2 s^{-2}\Big]\mathbf{T}(\mathbf{s^7})\Bigg\}\\
=&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha^5+\beta^5)+(\alpha^4\beta+\\
&\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\\
&\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big]+\theta_2\Big[(\alpha^3+\beta^3)+(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+\\
&(\theta_1+\theta_1\theta_2)\Big[(\alpha^6+\beta^6)+(\alpha^5\beta+\alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5)(1-\alpha\beta)\Big]+\\
&\theta_2\Big[(\alpha^7+\beta^7)+(\alpha^6\beta+\alpha^5\beta^2+\alpha^4\beta^3+\alpha^3\beta^4+\alpha^2\beta^5+\alpha\beta^6)(1-\alpha\beta)\Big]\Bigg\}
\end{aligned}
$$

$$=\frac{1}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\alpha+\beta)^5-$$

$$4\alpha\beta(\alpha+\beta)^3+3(\alpha\beta)^2(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)^3+2(\alpha\beta)^3(\alpha+\beta)\Big]+$$

$$(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]$$

$$+\theta_2\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\alpha+\beta)^6-$$

$$5\alpha\beta(\alpha+\beta)^4+6(\alpha\beta)^2(\alpha+\beta)^2-(\alpha\beta)^2(\alpha+\beta)^4+3(\alpha\beta)^3(\alpha+\beta)^2-$$

$$(\alpha\beta)^3-(\alpha\beta)^4\Big]+\theta_2\Big[(\alpha+\beta)^7-6\alpha\beta(\alpha+\beta)^5-(\alpha\beta)^2(\alpha+\beta)^5+$$

$$10(\alpha\beta)^2(\alpha+\beta)^3+4(\alpha\beta)^3(\alpha+\beta)^3-4(\alpha\beta)^3(\alpha+\beta)-3(\alpha\beta)^4(\alpha+\beta)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2$$

$$-2\phi_1\phi_2^3)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]+\theta_2\Big[(\phi_1^3+$$

$$2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+(\theta_1+\theta_1\theta_2)\Big[(\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+$$

$$\phi_2^3-\phi_2^4)\Big]+\theta_2\Big[(\phi_1^7+6\phi_1^5\phi_2-\phi_1^5\phi_2^2+10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-$$

$$2\phi_1\phi_2^3\Big]+(\theta_1+\theta_1\theta_2)\Big[\phi_1^6+\phi_1^4+5\phi_1^4\phi_2+5\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2^4+$$

$$3\phi_1^2\phi_2+\phi_2^2\Big]+\theta_2\Big[\phi_1^7+\phi_1^3+6\phi_1^5\phi_2+10\phi_1^3\phi_2^2-\phi_1^5\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-$$

$$3\phi_1\phi_2^4+2\phi_1\phi_2-\phi_1\phi_2^2\Big]\Bigg\}$$

Expanding some of the terms and regrouping, we shall obtain

$$
\begin{aligned}
\gamma(5) =& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^5+3\phi_1^3\phi_2+\phi_1\phi_2^2- \\
& \phi_1^3\phi_2^2-\phi_1\phi_2^3\Big]+(\theta_1+\theta_1\theta_2)\Big[\phi_1^6+\phi_1^4+4\phi_1^4\phi_2+2\phi_1^2\phi_2^2+2\phi^2\phi_2- \\
& \phi_1^4\phi_2^2-2\phi_1^2\phi_2^3\Big]+\theta_2\Big[\phi_1^6+\phi_1^2+4\phi_1^4\phi_2+3\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-2\phi_1^2\phi_2^3\Big]\Bigg\}+ \\
& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^3\phi_2+2\phi_1\phi_2^2-\phi_1\phi_2^3\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^4\phi_2+\phi_1^2\phi_2+3\phi_1^2\phi_2^2-\phi_1^2\phi_2^3+\phi_2^2-\phi_2^4\Big]+\theta_2\Big[\phi_1^5\phi_2+ \\
& \phi_1\phi_2+4\phi_1^3\phi_2^2+3\phi_1\phi_2^3-\phi_1^3\phi_2^3-2\phi_1\phi_2^4\Big]\Bigg\} \\
=& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^4+3\phi_1^2\phi_2+\phi_2^2- \\
& \phi_1^2\phi_2^2-\phi_2^3\Big]+(\theta_1+\theta_1\theta_2)\Big[\phi_1^5+\phi_1^3+4\phi_1^3\phi_2+2\phi_1\phi_2^2+2\phi_1\phi_2- \\
& \phi_1^3\phi_2^2-2\phi_1\phi_2^3\Big]+\theta_2\Big[\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2+\phi_2-\phi_2^2+\phi_2^3- \\
& \phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2^4\Big]\Bigg\}+ \\
& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{(1+\theta_1^2+\theta_2^2)\Big[\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big]+ \\
& (\theta_1+\theta_1\theta_2)\Big[\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3\Big]+\theta_2\Big[\phi_1^5+\phi_1+ \\
& 4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3\Big]\Bigg\} \\
=& \phi_1\gamma(4)+\phi_2\gamma(3) \\
=& \phi_1\left(\frac{c_{3,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0)+\phi_2\left(\frac{c_{2,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0) \\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\left[\phi_1\Big(c_{3,21}+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda\Big)+\phi_2\Big(c_{2,21}+\theta_2\kappa\Big)\right]\gamma_{2,2}(0)
\end{aligned}
$$

$$= \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\phi_1\Big((\phi^2+\phi_2)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+$$
$$\theta_2\phi_1\kappa+\theta_2\phi_2\lambda\Big)+\phi_2\Big(\phi_1 c_{1,21}+\phi_1\phi_2+$$
$$\theta_1\phi_2\varpi+\theta_2\kappa\Big)\Bigg]\gamma_{2,2}(0)$$
$$= \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big\{\Big(\phi_1^2+\phi_2\Big)\phi_1 c_{1,21}+\phi_1^3\phi_2+\theta_1\varpi\phi_1^2\phi_2+$$
$$\theta_2\phi_1^2\kappa+\theta_2\phi_1\phi_2\lambda\Big\}+\Big\{\phi_1\phi_2 c_{1,21}+\phi_1\phi_2^2+$$
$$\theta_1\phi_2^2\varpi+\theta_2\phi_2\kappa\Big\}\Bigg]\gamma_{2,2}(0)$$
$$= \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+$$
$$\theta_1\phi_1^2\phi_2\varpi+\theta_1\phi_2^2\varpi+\theta_2\phi_1^2\kappa+\theta_2\phi_2\kappa+\theta_2\phi_1\phi_2\lambda\Bigg]\gamma_{2,2}(0)$$
$$= \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1 c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2 c_{1,21}+$$
$$\theta_1\varpi\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1\phi_2\Big)+\theta_2\kappa\Big(\phi_1^2+\phi_2\Big)\Bigg]\gamma_{2,2}(0)$$
$$= \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[c_{4,21}+\theta_2\lambda\Big(\phi_1\phi_2\Big)+\theta_2\kappa\Big(\phi_1^2+\phi_2\Big)\Bigg]\gamma_{2,2}(0)$$
$$= \frac{c_{4,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)$$

It is obvious that our simplification of the coefficient of s^k from $\gamma(4)$ leads to the Yule-Walker equations. Subsequently, it is now convenient to precede using the Y-W equations.

Thus,

$$
\begin{aligned}
\gamma(6) =& \phi_1\gamma(5) + \phi_2\gamma(4)\\
=& \phi_1\left(\frac{c_{4,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0) + \phi_2\left(\frac{c_{3,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\right)\gamma_{2,2}(0)\\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\phi_1\Big(c_{4,21}+\theta_2\lambda(\phi_1\phi_2)+\theta_2\kappa(\phi_1^2+\phi_2)\Big)+\\
& \phi_2\Big(c_{3,21}+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda\Big)\Bigg]\gamma_{2,2}(0)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\phi_1\Big\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2c_{1,21}+\\
& \theta_1\varpi\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1\phi_2\Big)+\theta_2\kappa\Big(\phi_1^2+\phi_2\Big)\Big\}+\\
& \phi_2\Big\{\Big(\phi_1^2+\phi_2\Big)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+\theta_2\phi_1\kappa+\\
& \theta_2\phi_2\lambda\Big\}\Bigg]\gamma_{2,2}(0)\\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,21}+\\
& \theta_1\varpi\phi_1\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1^2\phi_2\Big)+\theta_2\kappa\Big(\phi_1^3+\\
& \phi_1\phi_2\Big)\Big\}+\Big\{\Big(\phi_1^2+\phi_2\Big)\phi_2c_{1,21}+\phi_1^2\phi_2^2+\theta_1\phi_1\phi_2^2\varpi+\\
& \theta_2\phi_1\phi_2\kappa+\theta_2\phi_2^2\lambda\Big\}\Bigg]\gamma_{2,2}(0)\\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_2c_{1,21}+\phi_1^2\phi_2\Big)+\phi_1^2\phi_2c_{1,21}+\\
& \phi_1^2\phi_2^2+\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)+\theta_2\lambda\Big(\phi_1^2\phi_2+\phi_2^2\Big)+\\
& \theta_2\kappa\Big(\phi_1^3+2\phi_1\phi_2\Big)\Bigg]\gamma_{2,2}(0)\\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[c_{5,21}+\theta_2\lambda\Big(\phi_1^2\phi_2+\phi_2^2\Big)+\theta_2\kappa\Big(\phi_1^3+2\phi_1\phi_2\Big)\Bigg]\gamma_{2,2}(0)\\
=& \frac{c_{5,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)
\end{aligned}
$$

Using similar deductions,

$$\gamma(7) = \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[c_{6,21}+\theta_2\lambda\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)+ \\ \theta_2\kappa\Big(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2\Big)\Bigg]\gamma_{2,2}(0)$$
$$= \frac{c_{6,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)$$

$$\gamma(8) = \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg[c_{7,21}+\theta_2\lambda\Big(\phi_1^4\phi_2+3\phi_1^2\phi_2^2+\phi_2^3\Big)+ \\ \theta_2\kappa\Big(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2\Big)\Bigg]\gamma_{2,2}(0)$$
$$= \frac{c_{7,22}}{1-\phi_2+\theta_1\chi+\theta_2\tau}\gamma_{2,2}(0)$$

Then for $r \geq 4$,

$$c_{r,22} = c_{r,21}+\theta_2\lambda\Bigg[\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\Bigg]+ \\ \theta_2\kappa\Bigg[\sum_{r-2\geq 2s}\binom{(r-2-s)}{s}\phi_1^{r-2-2s}\phi_2^{s}\Bigg] \tag{4.61}$$

Hence for an ARMA(2,2) process,

$$\gamma(k) = \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}c_{(k-1),22}\gamma_{2,2}(0) \\ = \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau}\Bigg\{c_{r,21}+\theta_2\lambda\Bigg[\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s+1}\Bigg]+ \\ \theta_2\kappa\Bigg[\sum_{r-2\geq 2s}\binom{(r-2-s)}{s}\phi_1^{r-2-2s}\phi_2^{s}\Bigg]\Bigg\}\gamma_{2,2}(0), \tag{4.62}$$

for $r \geq 4$, noting that $k-1=r$

The results in Equation (4.61) shows that the ACF of the ARMA(2,2) is a function of a coefficient $c_{r,22}$, which may further be given in terms of a coefficient $c_{r,21}$, a general coefficient for ARMA(2,1). The results then involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$ for which $r-t \geq 2s$, $2 \leq t \leq 3$.

It is also observed that the Y-W relation emerged only after lag 3. This means that there is the need for the computation of individual $\gamma(k)$ for $k \leq 3$.

Another important observation is that three more constants (τ, λ, and κ) have been introduced than the constants in the general expression for ARMA(2,1) process.

ACF of an ARMA(2,3) Process

In this section, the ACF of an ARMA(2,3) process is derived. The autocovariance generating function (acgf) is used to obtain the variance and autocovariances, after which the autocovariances are normalized to obtain the autocorrelation functions.

An ARMA (2,3) process is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3} + Z_t \tag{4.63}$$

By introducing a lag operator, Equation (4.63) can be simplified as

$$(1 - \phi_1 L - \phi_2 L^2)X_t = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)Z_t$$

Further simplification yields

$$X_t = \frac{(1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)}{1 - \phi_1 L - \phi_2 L^2} Z_t \tag{4.64}$$

Assuming the quadratic $1-\phi_1L-\phi_2L^2$ has two different real roots, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, then

$$1-\phi_1L-\phi_2L^2=0$$

can be written as

$$(1-\alpha L)(1-\beta L)=0$$

It can be verified that $(\alpha+\beta)$=ϕ_1 and $\alpha\beta=-\phi_2$

If

$$X_t=\frac{1+\theta_1L+\theta_2L^2+\theta_3L^3}{(1-\alpha L)(1-\beta L)}Z_t$$

then

$$c(s)=\frac{1+\theta_1s+\theta_2s^2+\theta_3s^3}{(1-\alpha s)(1-\beta s)}$$

The autocovariance generating function can thus, be written as

$$c(s)c(s^{-1})=\sigma^2\left[\frac{1+\theta_1s+\theta_2s^2+\theta_3s^3}{(1-\alpha s)(1-\beta s)}\times\frac{1+\theta_1s^{-1}+\theta_2s^{-2}+\theta_3s^{-3}}{(1-\alpha s^{-1})(1-\beta s^{-1})}\right] \tag{4.65}$$

This simplifies to

$$\begin{aligned}c(s)c(s^{-1})=&\frac{1}{(1-\alpha s)(1-\alpha s^{-1})(1-\beta s)(1-\beta s^{-1})}\sigma^2\Big\{\theta_3s^3+(\theta_2+\theta_1\theta_3)s^2+\\&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s+(1+\theta_1^2+\theta_2^2+\theta_3^3)+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\\&+(\theta_2+\theta_1\theta_3)s^{-2}+\theta_3s^{-3}\Big\}\end{aligned}$$

Further simplification yields

$$c(s)c(s^{-1}) = \sigma^2\Bigg[\theta_3 s^3 + (\theta_2+\theta_1\theta_3)s^2 + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)s + (1+\theta_1^2+\theta_2^2+\theta_3^2)+ (\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1} + (\theta_2+\theta_1\theta_3)s^{-2} + \theta_3 s^{-3}\Bigg]\times \sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\left[\sum_{r=0}^{\infty}(\alpha s)^r + \sum_{r=1}^{\infty}(\alpha s^{-1})^r\right]\times \left[\sum_{r=0}^{\infty}(\beta s)^r + \sum_{r=1}^{\infty}(\beta s^{-1})^r\right] \tag{4.66}$$

Equation (4.66) simplifies to

$$c(s)c(s^{-1}) = \sigma^2\left(\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\right)\mathrm{T}(\mathrm{s^r}) \tag{4.67}$$

where $\mathrm{T}(\mathrm{s^r})$ are expression in terms of s^r obtained from Equation (4.66).

At lag 0, we consider terms in Equation (4.66) that results in s^0 and obtain

$$\gamma_{2,3}(0) = \Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big]\mathrm{T}(\mathbf{s^0}) + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big]\mathrm{T}(\mathbf{s^{-1}})+ \Big[(\theta_2+\theta_1\theta_3)s^2\Big]\mathrm{T}(\mathbf{s^{-2}}) + \Big[\theta_3 s^3\Big]\mathrm{T}(\mathbf{s^{-3}}) + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big]\mathrm{T}(\mathbf{s})+ \Big[(\theta_2+\theta_1\theta_3)s^{-2}\Big]\mathrm{T}(\mathbf{s^2}) + \Big[\theta_3 s^{-3}\Big]\mathrm{T}(\mathbf{s^3})$$

Making substitutions for the respective $\mathbf{T}(\mathrm{s^r})$ and simplifying as before gives

$$\gamma_{2,3}(0) = \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(1-\phi_2\Big) + \theta_1\Big(2\phi_1+\theta_1-\theta_1\phi_2\Big)+ \theta_2\Big(2\phi_1\theta_1+2\phi_1^2+2\phi_2-2\phi_2^2+\theta_2-\theta_2\phi_2\Big) + \theta_3\Big(2\phi_1^3+4\phi_1\phi_2- 2\phi_1\phi_2^2+\theta_3-\theta_3\phi_2+2\theta_2\phi_1+2\theta_1\phi_1^2+2\theta_1\phi_2-2\theta_1\phi_2^2\Big)\Bigg\}$$

Let $\chi = 2\phi_1+\theta_1-\theta_1\phi_2$, $\tau = 2\phi_1\theta_1+2\phi_1^2+2\phi_2-2\phi_2^2+\theta_2-\theta_2\phi_2$ and $\mu = 2\phi_1^3+4\phi_1\phi_2-2\phi_1\phi_2^2+\theta_3-\theta_3\phi_2+2\theta_2\phi_1+2\theta_1\phi_1^2+2\theta_1\phi_2-2\theta_1\phi_2^2$

Therefore, the variance function of ARMA(2,3) denoted as $\gamma_{2,3}(0)$ is given as

$$\gamma_{2,3}(0) = \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\sigma^2\Bigg\{1 - \phi_2 + \theta_1\chi + \theta_2\tau + \theta_3\mu\Bigg\} \quad (4.68)$$

It can be observed that the $\gamma(0)$ of the lower processes can be deduced from Equation (4.68)

At lag 1, we consider terms that result in s and obtain

$$\begin{aligned}\gamma(1) =& \Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big]\mathrm{T}(\mathbf{s}) + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big]\mathrm{T}(\mathbf{s^0}) + \\ & \Big[(\theta_2+\theta_1\theta_3)s^2\Big]\mathrm{T}(\mathbf{s^{-1}}) + \Big[\theta_3 s^3\Big]\mathrm{T}(\mathbf{s^{-2}}) + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big]\mathrm{T}(\mathbf{s^2}) + \\ & \Big[(\theta_2+\theta_1\theta_3)s^{-2}\Big]\mathrm{T}(\mathbf{s^3}) + \Big[\theta_3 s^{-3}\Big]\mathrm{T}(\mathbf{s^4})\end{aligned}$$

Making substitutions for the respective $\mathbf{T}(\mathrm{s}^{\mathbf{r}})$ and simplifying gives

$$\begin{aligned}\gamma(1) =& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\phi_1 + \theta_1\Big(1+\theta_1\phi_1+\phi_1^2-\phi_2^2\Big) + \\ & \theta_2\Big(\theta_1+\theta_2\phi_1+\theta_1\phi_1^2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big) + \\ & \theta_3\Big(\theta_3\phi_1+\theta_2+\theta_2\phi_1^2-\theta_2\phi_2^2+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2+\phi_1^4 + \\ & \phi_1^2+\phi_2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2^3\Big)\Bigg\}\end{aligned}$$

Let $\varpi = 1+\theta_1\phi_1+\phi_1^2-\phi_2^2$, $\pi = \theta_1+\theta_2\phi_1+\theta_1\phi_1^2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2$ and

$\xi = \theta_3\phi_1+\theta_2+\theta_2\phi_1^2-\theta_2\phi_2^2+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2+\phi_1^4+\phi_1^2+\phi_2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2^3$

Therefore,

$$\begin{aligned}\gamma(1) =&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Bigg\{\phi_1+\theta_1\varpi+\theta_2\pi+\theta_3\xi\Bigg\}\\ =&\frac{\phi_1+\theta_1\varpi+\theta_2\pi+\theta_3\xi}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)\end{aligned}$$

At lag 2, we consider terms in s^2 and obtain

$$\begin{aligned}\gamma(2) =&\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big]\mathbf{T}(\mathbf{s^2})+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big]\mathbf{T}(\mathbf{s})+\\ &\Big[(\theta_2+\theta_1\theta_3)s^2\Big]\mathbf{T}(\mathbf{s^0})+\Big[\theta_3 s^3\Big]\mathbf{T}(\mathbf{s^{-1}})+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big]\mathbf{T}(\mathbf{s^3})+\\ &\Big[(\theta_2+\theta_1\theta_3)s^{-2}\Big]\mathbf{T}(\mathbf{s^4})+\Big[\theta_3 s^{-3}\Big]\mathbf{T}(\mathbf{s^5})\\ =&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha^2+\beta^2)+\\ &\alpha\beta(1-\alpha\beta)\Big]+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\alpha+\beta)\Big]+(\theta_2+\theta_1\theta_3)\\ &\Big[(1+\alpha\beta)\Big]+\theta_3\Big[(\alpha+\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^3+\beta^3)+\\ &(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\\ &\alpha\beta^3)(1-\alpha\beta)\Big]+\theta_3\Big[(\alpha^5+\beta^5)+(\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\\ &\alpha\beta^4)(1-\alpha\beta)\Big]\Bigg\}\\ =&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha+\beta)^2-\alpha\beta-\\ &(\alpha\beta)^2\Big]+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\alpha+\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(1+\alpha\beta)\Big]+\\ &\theta_3\Big[(\alpha+\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-\\ &(\alpha\beta)^2(\alpha+\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-\\ &(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]+\theta_3\Big[(\alpha+\beta)^5-4\alpha\beta(\alpha+\beta)^3+\\ &3(\alpha\beta)^2(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)^3+2(\alpha\beta)^3(\alpha+\beta)\Big]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+\\
&\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\phi_1\Big]+\Big[(\theta_2+\theta_1\theta_3)(1-\phi_2)\Big]+\theta_3\Big[\phi_1\Big]+\\
&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\phi_1^4+\\
&3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]+\theta_3\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]\Bigg\}\\
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^2+\phi_2-\phi_2^2)\Big]+\\
&\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\Big[(\theta_2+\theta_1\theta_3)(1+\phi_1^4+\\
&3\phi_1^2\phi_2-\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]+\theta_3\Big[(\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\\
&\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]\Bigg\}\\
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(\phi_1^2+\phi_2-\phi_2^2\Big)+\theta_1\Big(\theta_1\phi_1^2+\theta_1\phi_2-\\
&\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2\Big)+\theta_2\Big(1+\phi_1^4+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\\
&\phi_2+\phi_2^2-\phi_2^3+\theta_2\phi_1^2+\theta_2\phi_2-\theta_2\phi_2^2+\theta_1\phi_1^3+\theta_1\phi_1+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2\Big)\\
&+\theta_3\Big(\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_3\phi_1^2+\theta_3\phi_2-\theta_3\phi_2^2+\\
&\theta_2\phi_1^3+\theta_2\phi_1+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\theta_1+\theta_1\phi_1^4+3\theta_1\phi_1^2\phi_2-\theta_1\phi_2+\theta_1\phi_2^2-\\
&\theta_1\phi_1^2\phi_2^2-\theta_1\phi_2^3\Big)\Bigg\}
\end{aligned}
$$

Let $\eta=\theta_1\phi_1^2+\theta_1\phi_2-\theta_1\phi_2^2+\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2$,
$\lambda=1+\phi_1^4+3\phi_1^2\phi_2-\phi_1^2\phi_2^2-\phi_2+\phi_2^2-\phi_2^3+\theta_2\phi_1^2+\theta_2\phi_2-\theta_2\phi_2^2+\theta_1\phi_1^3+\theta_1\phi_1+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2$ and
$\varrho=\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_3\phi_1^2+\theta_3\phi_2-\theta_3\phi_2^2+\theta_2\phi_1^3+\theta_2\phi_1+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\theta_1+\theta_1\phi_1^4+3\theta_1\phi_1^2\phi_2-\theta_1\phi_2+\theta_1\phi_2^2-\theta_1\phi_1^2\phi_2^2-\theta_1\phi_2^3$

Therefore,

$$\begin{aligned}
\gamma(2) =& \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Big\{\phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda+\theta_3\varrho\Big\}\\
=& \frac{\phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda+\theta_3\varrho}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,2}(0)\\
=& \frac{c_{1,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,2}(0)
\end{aligned}$$

where $c_{1,23} = \phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda+\theta_3\varrho$

$$\begin{aligned}
c_{1,23} =& \phi_1^2+\phi_2-\phi_2^2+\theta_1\eta+\theta_2\lambda+\theta_3\varrho\\
=& c_{1,22}+\theta_3\varrho\\
=& c_{1,21}+\theta_2\lambda+\theta_3\varrho
\end{aligned}$$

Similarly, the autocovariance at lag 3 is obtained as

$$\begin{aligned}
\gamma(3) =& \Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big]\mathbf{T}(\mathbf{s^3})+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big]\mathbf{T}(\mathbf{s^2})+\\
& \Big[(\theta_2+\theta_1\theta_3)s^2\Big]\mathbf{T}(\mathbf{s})+\Big[\theta_3 s^3\Big]\mathbf{T}(\mathbf{s^0})+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big]\mathbf{T}(\mathbf{s^4})+\\
& \Big[(\theta_2+\theta_1\theta_3)s^{-2}\Big]\mathbf{T}(\mathbf{s^5})+\Big[\theta_3 s^{-3}\Big]\mathbf{T}(\mathbf{s^6})\\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha^3+\beta^3)+\\
& (\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^2+\beta^2)+\\
& \alpha\beta(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)\Big]+\theta_3\Big[(1+\alpha\beta)\Big]+\\
& (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big]+\\
& (\theta_2+\theta_1\theta_3)\Big[(\alpha^5+\beta^5)+(\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big]+\\
& \theta_3\Big[(\alpha^6+\beta^6)+(\alpha^5\beta+\alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\\
& \alpha\beta^5)(1-\alpha\beta)\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha+\beta)^3-\\
&2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha+\beta)^2-\alpha\beta-\\
&(\alpha\beta)^2\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)\Big]+\theta_3\Big[(1+\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\\
&\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]+\\
&(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^5-4\alpha\beta(\alpha+\beta)^3+3(\alpha\beta)^2(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)^3\\
&+2(\alpha\beta)^3(\alpha+\beta)\Big]+\theta_3\Big[(\alpha+\beta)^6-5\alpha\beta(\alpha+\beta)^4+6(\alpha\beta)^2(\alpha+\beta)^2\\
&-(\alpha\beta)^2(\alpha+\beta)^4+3(\alpha\beta)^3(\alpha+\beta)^2-(\alpha\beta)^3-(\alpha\beta)^4\Big]\Big\}\\
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\\
&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+(\theta_2+\theta_1\theta_3)\Big[\phi_1\Big]+\theta_3\Big[(1-\phi_2)\Big]+\\
&(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3)\Big]+(\theta_2+\theta_1\theta_3)\\
&\Big[(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\theta_3\Big[(\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\\
&\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2^3-\phi_2^4)\Big]\Big\}\\
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Big\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\\
&\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3)\Big]+\\
&\Big[(\theta_2+\theta_1\theta_3)(\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\\
&\theta_3\Big[(1+\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2+\phi_2^3-\phi_2^4))\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2\Big)+\\
&\theta_1\Big(\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2\Big)+\\
&\theta_2\Big(\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_2\phi_1^3+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\\
&\theta_1\phi_1^4+\theta_1\phi_1^2+3\theta_1\phi_1^2\phi_2-\theta_1\phi_1^2\phi_2^2+\theta_1\phi_2-\theta_1\phi_2^3\Big)+\theta_3\Big(1+\phi_1^6+5\phi_1^4\phi_2+\\
&6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2+\phi_2^3-\phi_2^4+\theta_3\phi_1^3+2\theta_3\phi_1\phi_2-\theta_3\phi_1\phi_2^2+\\
&\theta_2\phi_1^4+\theta_2\phi_1^2+3\theta_2\phi_1^2\phi_2-\theta_2\phi_1^2\phi_2^2+\theta_2\phi_2-\theta_2\phi_2^3+\theta_1\phi_1^5+\\
&\theta_1\phi_1+4\theta_1\phi_1^3\phi_2+3\theta_1\phi_1\phi_2^2-\theta_1\phi_1^3\phi_2^2-2\theta_1\phi_1\phi_2^3\Big)\Bigg\}
\end{aligned}
$$

Let $\omega=\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3+\theta_1\phi_1^3+2\theta_1\phi_1\phi_2-\theta_1\phi_1\phi_2^2$, $\kappa=\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3+\theta_2\phi_1^3+2\theta_2\phi_1\phi_2-\theta_2\phi_1\phi_2^2+\theta_1\phi_1^4+\theta_1\phi_1^2+3\theta_1\phi_1^2\phi_2-\theta_1\phi_1^2\phi_2^2+\theta_1\phi_2-\theta_1\phi_2^3$ and $\varsigma=1+\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2+\phi_2^3-\phi_2^4+\theta_3\phi_1^3+2\theta_3\phi_1\phi_2-\theta_3\phi_1\phi_2^2+\theta_2\phi_1^4+\theta_2\phi_1^2+3\theta_2\phi_1^2\phi_2-\theta_2\phi_1^2\phi_2^2+\theta_2\phi_2-\theta_2\phi_2^3+\theta_1\phi_1^5+\theta_1\phi_1+4\theta_1\phi_1^3\phi_2+3\theta_1\phi_1\phi_2^2-\theta_1\phi_1^3\phi_2^2-2\theta_1\phi_1\phi_2^3$

Then,

$$
\begin{aligned}
\gamma(3)=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\sigma^2\Big\{\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2+\theta_1\omega+\theta_2\kappa+\theta_3\varsigma\Big\}\\
=&\frac{\phi_1^3+\phi_1\phi_2-\phi_1\phi_2^2+\phi_1\phi_2+\theta_1\omega+\theta_2\kappa+\theta_3\varsigma}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)\\
=&\frac{\phi_1\Big(\phi_1^2+\phi_2-\phi_2^2\Big)+\phi_1\phi_2+\theta_1\omega+\theta_2\kappa+\theta_3\varsigma}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)\\
=&\frac{c_{2,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)
\end{aligned}
$$

where $c_{2,23} = \phi_1\left(\phi_1^2 + \phi_2 - \phi_2^2\right) + \phi_1\phi_2 + \theta_1\omega + \theta_2\kappa + \theta_3\varsigma$

$$\begin{aligned}
c_{2,23} =& \phi_1\left(\phi_1^2 + \phi_2 - \phi_2^2\right) + \phi_1\phi_2 + \theta_1\omega + \theta_2\kappa + \theta_3\varsigma \\
=& c_{2,22} + \theta_3\varsigma \\
=& c_{2,21} + \theta_2\kappa + \theta_3\varsigma \\
=& \phi_1 c_{1,21} + \phi_1\phi_2 + \theta_1\phi_2\varpi + \theta_2\kappa + \theta_3\varsigma
\end{aligned}$$

At lag 4, we consider terms in s^4 and obtain

$$\begin{aligned}
\gamma(4) =& \Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big]\mathbf{T(s^4)} + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s\Big]\mathbf{T(s^3)} + \\
& \Big[(\theta_2+\theta_1\theta_3)s^2\Big]\mathbf{T(s^2)} + \Big[\theta_3 s^3\Big]\mathbf{T(s^2)} + \Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)s^{-1}\Big]\mathbf{T(s^5)} + \\
& \Big[(\theta_2+\theta_1\theta_3)s^{-2}\Big]\mathbf{T(s^6)} + \Big[\theta_3 s^{-3}\Big]\mathbf{T(s^7)} \\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha^4+\beta^4) + \\
& (\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big] + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^3+\beta^3) + \\
& (\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big] + (\theta_2+\theta_1\theta_3)\Big[(\alpha^2+\beta^2)+\alpha\beta(1-\alpha\beta)\Big] \\
& + \theta_3\Big[(\alpha+\beta)\Big] + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^5+\beta^5) + (\alpha^4\beta+\alpha^3\beta^2+ \\
& \alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big] + (\theta_2+\theta_1\theta_3)\Big[(\alpha^6+\beta^6) + (\alpha^5\beta+\alpha^4\beta^2+ \\
& \alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5)(1-\alpha\beta)\Big] + \theta_3\Big[(\alpha^7+\beta^7) + \\
& (\alpha^6\beta+\alpha^5\beta^2+\alpha^4\beta^3+\alpha^3\beta^4+\alpha^2\beta^5+\alpha\beta^6)(1-\alpha\beta)\Big]\Bigg\}
\end{aligned}$$

$$=\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha+\beta)^4-$$

$$3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]+(\theta_1+\theta_1\theta_2+$$

$$\theta_2\theta_3)\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]+$$

$$(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]+\theta_3\Big[(\alpha+\beta)\Big]+$$

$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha+\beta)^5-4\alpha\beta(\alpha+\beta)^3+3(\alpha\beta)^2(\alpha+\beta)-$$

$$(\alpha\beta)^2(\alpha+\beta)^3+2(\alpha\beta)^3(\alpha+\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^6-$$

$$5\alpha\beta(\alpha+\beta)^4+6(\alpha\beta)^2(\alpha+\beta)^2-(\alpha\beta)^2(\alpha+\beta)^4+$$

$$3(\alpha\beta)^3(\alpha+\beta)^2-(\alpha\beta)^3-(\alpha\beta)^4\Big]+\theta_3\Big[(\alpha+\beta)^7-6\alpha\beta(\alpha+\beta)^5-$$

$$(\alpha\beta)^2(\alpha+\beta)^5+10(\alpha\beta)^2(\alpha+\beta)^3+4(\alpha\beta)^3(\alpha+\beta)^3-$$

$$4(\alpha\beta)^3(\alpha+\beta)-3(\alpha\beta)^4(\alpha+\beta)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-$$

$$\phi_1^2\phi_2^2-\phi_2^3)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+$$

$$(\theta_2+\theta_1\theta_3)\Big[(\phi_1^2+\phi_2-\phi_2^2)\Big]+\theta_3\Big[\phi_1\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^5+$$

$$4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\phi_1^6+5\phi_1^4\phi_2+$$

$$6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2^3-\phi_2^4)\Big]+\theta_3\Big[(\phi_1^7+6\phi_1^5\phi_2-\phi_1^5\phi_2^2+$$

$$10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-$$

$$\phi_1^2\phi_2^2-\phi_2^3)\Big]+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^5+\phi_1^3+4\phi_1^3\phi_2-\phi_1^3\phi_2^2+$$

$$2\phi_1\phi_2^2+2\phi_1\phi_2-2\phi_1\phi_2^3)\Big]+\Big[(\theta_2+\theta_1\theta_3)(\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+$$

$$6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2-\phi_2^2+\phi_2^3-\phi_2^4)\Big]+\theta_3\Big[(\phi_1^7+\phi_1+$$

$$6\phi_1^5\phi_2-\phi_1^5\phi_2^2+10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4)\Big]\Bigg\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\phi_1^2\phi_2^2-\phi_2^3\Big)+$$

$$\theta_1\Big(\phi_1^5+\phi_1^3+4\phi_1^3\phi_2-\phi_1^3\phi_2^2+2\phi_1\phi_2^2+2\phi_1\phi_2-2\phi_1\phi_2^3+\theta_1\phi_1^4+$$

$$3\theta_1\phi_1^2\phi_2+\theta_1\phi_2^2-\theta_1\phi_1^2\phi_2^2-\theta_1\phi_2^3\Big)+\theta_2\Big(\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2$$

$$-3\phi_1^2\phi_2^3+\phi_2-\phi_2^2+\phi_2^3-\phi_2^4+\theta_2\phi_1^4+3\theta_2\phi_1^2\phi_2+\theta_2\phi_2^2-$$

$$\theta_2\phi_1^2\phi_2^2-\theta_2\phi_2^3+\theta_1\phi_1^5+\theta_1\phi_1^3+4\theta_1\phi_1^3\phi_2-\theta_1\phi_1^3\phi_2^2+2\theta_1\phi_1\phi_2^2+$$

$$2\theta_1\phi_1\phi_2-2\theta_1\phi_2^3\Big)+\theta_3\Big(\phi_1^7+\phi_1+6\phi_1^5\phi_2-\phi_1^5\phi_2^2+10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+$$

$$4\phi_1\phi_2^3-3\phi_1\phi_2^4+\theta_3\phi_1^4+3\theta_3\phi_1^2\phi_2+\theta_3\phi_2^2-\theta_3\phi_1^2\phi_2-\theta_3\phi_2^3+\theta_2\phi_1^5+$$

$$\theta_2\phi_1^3+4\theta_2\phi_1^3\phi_2-\theta_2\phi_1^3\phi_2^2+2\theta_2\phi_1\phi_2^2+2\theta_2\phi_1\phi_2-2\theta_2\phi_1\phi_2^3+\theta_1\phi_1^6+$$

$$\theta_1\phi_1^2+5\theta_1\phi_1^4\phi_2+6\theta_1\phi_1^2\phi_2^2-\theta_1\phi_1^4\phi_2^2-3\theta_1\phi_1^2\phi_2^3+\theta_1\phi_2-\theta_1\phi_2^2+$$

$$\theta_1\phi_2^3-\theta_1\phi_2^4\Big)\Bigg\}$$

Let $\nu=\phi_1^5+\phi_1^3+4\phi_1^3\phi_2-\phi_1^3\phi_2^2+2\phi_1\phi_2^2+2\phi_1\phi_2-2\phi_1\phi_2^3+\theta_1\phi_1^4+3\theta_1\phi_1^2\phi_2+\theta_1\phi_2^2-\theta_1\phi_1^2\phi_2^2-\theta_1\phi_2^3$,
$\iota=\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2-\phi_2^2+\phi_2^3-\phi_2^4+\theta_2\phi_1^4+3\theta_2\phi_1^2\phi_2+\theta_2\phi_2^2-\theta_2\phi_1^2\phi_2^2-\theta_2\phi_2^3+\theta_1\phi_1^5+\theta_1\phi_1^3+4\theta_1\phi_1^3\phi_2-\theta_1\phi_1^3\phi_2^2+2\theta_1\phi_1\phi_2^2+2\theta_1\phi_1\phi_2-2\theta_1\phi_2^3$ and
$\upsilon=\phi_1^7+\phi_1+6\phi_1^5\phi_2-\phi_1^5\phi_2^2+10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4+\theta_3\phi_1^4+3\theta_3\phi_1^2\phi_2+\theta_3\phi_2^2-\theta_3\phi_1^2\phi_2-\theta_3\phi_2^3+\theta_2\phi_1^5+\theta_2\phi_1^3+4\theta_2\phi_1^3\phi_2-\theta_2\phi_1^3\phi_2^2+2\theta_2\phi_1\phi_2^2+2\theta_2\phi_1\phi_2-2\theta_2\phi_1\phi_2^3+\theta_1\phi_1^6+\theta_1\phi_1^2+5\theta_1\phi_1^4\phi_2+6\theta_1\phi_1^2\phi_2^2-\theta_1\phi_1^4\phi_2^2-3\theta_1\phi_1^2\phi_2^3+\theta_1\phi_2-\theta_1\phi_2^2+\theta_1\phi_2^3-\theta_1\phi_2^4$

Therefore,

$$\gamma(4) = \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\sigma^2\Bigg\{\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 - \phi_1^2\phi_2^2 - \phi_2^3 + \theta_1\nu + \theta_2\iota + \theta_3\upsilon\Bigg\}$$
$$= \frac{\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 - \phi_1^2\phi_2^2 - \phi_2^3 + \theta_1\nu + \theta_2\iota + \theta_3\upsilon}{1 - \phi_2 + \theta_1\chi + \theta_2\tau + \theta_3\mu}\gamma_{2,3}(0)$$
$$= \frac{\Big(\phi_1^2 + \phi_2\Big)\Big(\phi_1^2 + \phi_2 - \phi_2^2\Big) + \phi_1^2\phi_2 + \theta_1\nu + \theta_2\iota + \theta_3\upsilon}{1 - \phi_2 + \theta_1\chi + \theta_2\tau + \theta_3\mu}\gamma_{2,3}(0)$$
$$= \frac{c_{3,23}}{1 - \phi_2 + \theta_1\chi + \theta_2\tau + \theta_3\mu}\gamma_{2,3}(0)$$

where $c_{3,23} = \Big(\phi_1^2 + \phi_2\Big)\Big(\phi_1^2 + \phi_2 - \phi_2^2\Big) + \phi_1^2\phi_2 + \theta_1\nu + \theta_2\iota + \theta_3\upsilon$

From ARMA(2,2), it can be verified that $\theta_2\iota = \theta_2\Big(\phi_1\kappa + \phi_2\lambda\Big)$

$$\begin{aligned}
c_{3,23} =& \Big(\phi_1^2 + \phi_2\Big)\Big(\phi_1^2 + \phi_2 - \phi_2^2\Big) + \phi_1^2\phi_2 + \theta_1\nu + \theta_2\iota + \theta_3\upsilon \\
=& c_{3,22} + \theta_3\upsilon \\
=& c_{3,21} + \theta_2\iota + \theta_3\upsilon \\
=& \Big(\phi_1^2 + \phi_2\Big)c_{1,21} + \phi_1^2\phi_2 + \theta_1\phi_1\phi_2\varpi + \theta_2\iota + \theta_3\upsilon
\end{aligned}$$

The autocovariance at lag 5 is obtained as

$$\begin{aligned}
\gamma(5) =& \Big[(1 + \theta_1^2 + \theta_2^2 + \theta_3^2)\Big]\mathrm{T}(\mathbf{s^5}) + \Big[(\theta_1 + \theta_1\theta_2 + \theta_2\theta_3)s\Big]\mathrm{T}(\mathbf{s^4}) + \\
& \Big[(\theta_2 + \theta_1\theta_3)s^2\Big]\mathrm{T}(\mathbf{s^3}) + \Big[\theta_3 s^3\Big]\mathrm{T}(\mathbf{s^2}) + \Big[(\theta_1 + \theta_1\theta_2 + \theta_2\theta_3)s^{-1}\Big]\mathrm{T}(\mathbf{s^6}) + \\
& \Big[(\theta_2 + \theta_1\theta_3)s^{-2}\Big]\mathrm{T}(\mathbf{s^7}) + \Big[\theta_3 s^{-3}\Big]\mathrm{T}(\mathbf{s^8})
\end{aligned}$$

This simplifies to

$$
\begin{aligned}
\gamma(5) =& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha^5+\beta^5)+\\
&(\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\\
&\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\\
&\Big[(\alpha^3+\beta^3)+(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+\theta_3\Big[(\alpha^2+\beta^2)+\alpha\beta\\
&(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^6+\beta^6)+(\alpha^5\beta+\alpha^4\beta^2+\\
&\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5)(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha^7+\beta^7)+\\
&(\alpha^6\beta+\alpha^5\beta^2+\alpha^4\beta^3+\alpha^3\beta^4+\alpha^2\beta^5+\alpha\beta^6)(1-\alpha\beta)\Big]+\\
&\theta_3\Big[(\alpha^8+\beta^8)+(\alpha^7\beta+\alpha^6\beta^2+\alpha^5\beta^3+\alpha^4\beta^4+\alpha^3\beta^5+\\
&\alpha^2\beta^6+\alpha\beta^7)(1-\alpha\beta)\Big]\Bigg\}\\
=& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha^5+\beta^5)+\\
&(\alpha^4\beta+\alpha^3\beta^2+\alpha^2\beta^3+\alpha\beta^4)(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\\
&\Big[(\alpha^4+\beta^4)+(\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3)(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\\
&\Big[(\alpha^3+\beta^3)+(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)\Big]+\theta_3\Big[(\alpha^2+\beta^2)+\alpha\beta\\
&(1-\alpha\beta)\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha^6+\beta^6)+(\alpha^5\beta+\alpha^4\beta^2+\\
&\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5)(1-\alpha\beta)\Big]+(\theta_2+\theta_1\theta_3)\Big[(\alpha^7+\beta^7)+\\
&(\alpha^6\beta+\alpha^5\beta^2+\alpha^4\beta^3+\alpha^3\beta^4+\alpha^2\beta^5+\alpha\beta^6)(1-\alpha\beta)\Big]+\\
&\theta_3\Big[(\alpha^8+\beta^8)+(\alpha^7\beta+\alpha^6\beta^2+\alpha^5\beta^3+\alpha^4\beta^4+\alpha^3\beta^5+\\
&\alpha^2\beta^6+\alpha\beta^7)(1-\alpha\beta)\Big]\Bigg\}
\end{aligned}
$$

Further simplifications yield

$$\gamma(5) = \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\alpha+\beta)^5 -$$
$$4\alpha\beta(\alpha+\beta)^3 + 3(\alpha\beta)^2(\alpha+\beta) - (\alpha\beta)^2(\alpha+\beta)^3 + 2(\alpha\beta)^3(\alpha+\beta)\Big] +$$
$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\alpha+\beta)^4 - 3\alpha\beta(\alpha+\beta)^2 + (\alpha\beta)^2 - (\alpha\beta)^2$$
$$\{(\alpha+\beta)^2+\alpha\beta\}\Big] + (\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^3 - 2\alpha\beta(\alpha+\beta) -$$
$$(\alpha\beta)^2(\alpha+\beta)\Big] + \theta_3\Big[(\alpha+\beta)^2 - \alpha\beta - (\alpha\beta)^2\Big] + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)$$
$$\Big[(\alpha+\beta)^6 - 5\alpha\beta(\alpha+\beta)^4 + 6(\alpha\beta)^2(\alpha+\beta)^2 - (\alpha\beta)^2(\alpha+\beta)^4 +$$
$$3(\alpha\beta)^3(\alpha+\beta)^2 - (\alpha\beta)^3 - (\alpha\beta)^4\Big] + (\theta_2+\theta_1\theta_3)\Big[(\alpha+\beta)^7 -$$
$$6\alpha\beta(\alpha+\beta)^5 - (\alpha\beta)^2(\alpha+\beta)^5 + 10(\alpha\beta)^2(\alpha+\beta)^3 + 4(\alpha\beta)^3(\alpha+\beta)^3 -$$
$$4(\alpha\beta)^3(\alpha+\beta) - 3(\alpha\beta)^4(\alpha+\beta)\Big] + \theta_3\Big[(\alpha+\beta)^8 - 7\alpha\beta(\alpha+\beta)^6 -$$
$$\alpha^2\beta^2(\alpha+\beta)^6 + 15\alpha^2\beta^2(\alpha+\beta)^4 - 10\alpha^3\beta^3(\alpha+\beta)^2 + 5\alpha^3\beta^3(\alpha+\beta)^4 -$$
$$6\alpha^4\beta^4(\alpha+\beta)^2 + \alpha^4\beta^4 + \alpha^5\beta^5\Big]\Big\}$$

$$= \frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2 - \phi_1^2\Big]}\Big\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[(\phi_1^5 + 4\phi_1^3\phi_2 +$$
$$3\phi_1\phi_2^2 - \phi_1^3\phi_2^2 - 2\phi_1\phi_2^3)\Big] + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 -$$
$$\phi_1^2\phi_2^2 - \phi_2^3)\Big] + (\theta_2+\theta_1\theta_3)\Big[(\phi_1^3 + 2\phi_1\phi_2 - \phi_1\phi_2^2)\Big] + \theta_3\Big[(\phi_1^2 + \phi_2 -$$
$$\phi_2^2)\Big] + (\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[(\phi_1^6 + 5\phi_1^4\phi_2 + 6\phi_1^2\phi_2^2 - \phi_1^4\phi_2^2 - 3\phi_1^2\phi_2^3 +$$
$$\phi_2^3 - \phi_2^4)\Big] + (\theta_2+\theta_1\theta_3)\Big[(\phi_1^7 + 6\phi_1^5\phi_2 - \phi_1^5\phi_2^2 + 10\phi_1^3\phi_2^2 - 4\phi_1^3\phi_2^3 +$$
$$4\phi_1\phi_2^3 - 3\phi_1\phi_2^4)\Big] + \theta_3\Big[(\phi_1^8 + 7\phi_1^6\phi_2 - \phi_1^6\phi_2^2 + 15\phi_1^4\phi_2^2 + 10\phi_1^2\phi_2^3 -$$
$$5\phi_1^4\phi_2^3 - 6\phi_1^2\phi_2^4 + \phi_2^4 - \phi_2^5)\Big]\Big\}$$

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^5+4\phi_1^3\phi_2+3\phi_1\phi_2^2-$$
$$\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^6+\phi_1^4+5\phi_1^4\phi_2+3\phi_1^2\phi_2+$$
$$\phi_2^2+5\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2^4)\Big]+\Big[(\theta_2+\theta_1\theta_3)(\phi_1^7+\phi_1^3+6\phi_1^5\phi_2-$$
$$\phi_1^5\phi_2^2+10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+$$
$$\theta_3\Big[(\phi_1^8+\phi_1^2+7\phi_1^6\phi_2-\phi_1^6\phi_2^2+15\phi_1^4\phi_2^2+10\phi_1^2\phi_2^3-5\phi_1^4\phi_2^3-6\phi_1^2\phi_2^4+$$
$$\phi_2-\phi_2^2+\phi_2^4-\phi_2^5)\Big]\Bigg\}$$

Expanding some of the terms and regrouping, we shall obtain

$$=\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1^5+3\phi_1^3\phi_2+\phi_1\phi_2^2-$$
$$\phi_1^3\phi_2^2-\phi_1\phi_2^3\Big]+(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^6+\phi_1^4+4\phi_1^4\phi_2-\phi_1^4\phi_2^2+$$
$$2\phi_1^2\phi_2^2+2\phi_1^2\phi_2-2\phi_1^2\phi_2^3\Big]+\Big[(\theta_2+\theta_1\theta_3)(\phi_1^7+\phi_1^3+5\phi_1^5\phi_2+6\phi_1^3\phi_2^2-$$
$$\phi_1^5\phi_2^2-3\phi_1^3\phi_2^3+\phi_1\phi_2-\phi_1\phi_2^2+\phi_1\phi_2^3-\phi_1\phi_2^4)\Big]+\theta_3\Big[(\phi_1^8+\phi_1^2+6\phi_1^6\phi_2-$$
$$\phi_1^6\phi_2^2+10\phi_1^4\phi_2^2-4\phi_1^4\phi_2^3+4\phi_1^2\phi_2^3-3\phi_1^2\phi_2^4)\Big]\Bigg\}+$$
$$\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{(1+\theta_1^2+\theta_2^2+\theta_3^2)\Big[\phi_1^3\phi_2+2\phi_1\phi_2^2-\phi_1\phi_2^3\Big]+$$
$$(\theta_1+\theta_1\theta_2+\theta_2\theta_3)\Big[\phi_1^4\phi_2+\phi_1^2\phi_2+3\phi_1^2\phi_2^2-\phi_1^2\phi_2^3+\phi_2^2-\phi_2^4\Big]+$$
$$(\theta_2+\theta_1\theta_3)\Big[\phi_1^5\phi_2+\phi_1\phi_2+4\phi_1^3\phi_2^2+3\phi_1\phi_2^3-\phi_1^3\phi_2^3-2\phi_1\phi_2^4\Big]+$$
$$\theta_3\Big[\phi_2+\phi_1^6\phi_2+5\phi_1^4\phi_2^2+6\phi_1^2\phi_2^3-\phi_1^4\phi_2^3-3\phi_1^2\phi_2^4-\phi_2^2+\phi_2^4-\phi_2^5\Big]\Bigg\}$$

$$
\begin{aligned}
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_1\Bigg\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2-\\
&\phi_1^2\phi_2^2-\phi_2^3)\Big]+\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^5+\phi_1^3+4\phi_1^3\phi_2-\phi_1^3\phi_2^2+\\
&2\phi_1\phi_2^2+2\phi_1\phi_2-2\phi_1\phi_2^3)\Big]+\Big[(\theta_2+\theta_1\theta_3)(\phi_1^6+\phi_1^2+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\\
&\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3+\phi_2-\phi_2^2+\phi_2^3-\phi_2^4)\Big]+\theta_3\Big[(\phi_1^7+\phi_1+6\phi_1^5\phi_2-\phi_1^5\phi_2^2+\\
&10\phi_1^3\phi_2^2-4\phi_1^3\phi_2^3+4\phi_1\phi_2^3-3\phi_1\phi_2^4)\Big]\Bigg\}+\\
&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\phi_2\Bigg\{\Big[(1+\theta_1^2+\theta_2^2+\theta_3^2)(\phi_1^3+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]+\\
&\Big[(\theta_1+\theta_1\theta_2+\theta_2\theta_3)(\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\phi_1^2\phi_2^2+\phi_2-\phi_2^3)\Big]+\\
&\Big[(\theta_2+\theta_1\theta_3)(\phi_1^5+\phi_1+4\phi_1^3\phi_2+3\phi_1\phi_2^2-\phi_1^3\phi_2^2-2\phi_1\phi_2^3)\Big]+\\
&\theta_3\Big[(1+\phi_1^6+5\phi_1^4\phi_2+6\phi_1^2\phi_2^2-\phi_1^4\phi_2^2-3\phi_1^2\phi_2^3-\phi_2+\phi_2^3-\phi_2^4))\Big]\Bigg\}\\
=&\phi_1\gamma(4)+\phi_2\gamma(3)\\
=&\phi_1\left(\frac{c_{3,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\right)\gamma_{2,3}(0)+\\
&\phi_2\left(\frac{c_{2,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\right)\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\phi_1\Big(c_{3,22}+\theta_3\nu\Big)+\phi_2\Big(c_{2,22}+\theta_3\varsigma\Big)\Bigg]\gamma_{2,3}(0)\\
&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\phi_1\Big(c_{3,21}+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda+\theta_3\nu\Big)+\\
&\phi_2\Big(c_{2,21}+\theta_2\kappa+\theta_3\varsigma\Big)\Bigg]\gamma_{2,3}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\phi_1\Big((\phi^2+\phi_2)c_{1,21}+\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+\\
&\theta_2\phi_1\kappa+\theta_2\phi_2\lambda+\theta_3\nu\Big)+\phi_2\Big(\phi_1c_{1,21}+\phi_1\phi_2+\theta_1\phi_2\varpi+\theta_2\kappa+\theta_3\varsigma\Big)\Bigg]\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\Big\{\Big(\phi_1^2+\phi_2\Big)\phi_1c_{1,21}+\phi_1^3\phi_2+\theta_1\varpi\phi_1^2\phi_2+\\
&\theta_2\phi_1^2\kappa+\theta_2\phi_1\phi_2\lambda+\theta_3\phi_1\nu\Big\}+\Big\{\phi_1\phi_2c_{1,21}+\phi_1\phi_2^2+\theta_1\phi_2^2\varpi+\theta_2\phi_2\kappa+\\
&\theta_3\phi_2\varsigma\Big\}\Bigg]\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2c_{1,21}+\\
&\theta_1\phi_1^2\phi_2\varpi+\theta_1\phi_2^2\varpi+\theta_2\phi_1^2\kappa+\theta_2\phi_2\kappa+\theta_2\phi_1\phi_2\lambda+\theta_3\phi_1\nu+\theta_3\phi_2\varsigma\Bigg]\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1c_{1,21}+\phi_1\phi_2\Big)+\phi_1\phi_2c_{1,21}+\\
&\theta_1\varpi\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1\phi_2\Big)+\theta_2\kappa\Big(\phi_1^2+\phi_2\Big)+\theta_3\Big(\phi_1\nu+\phi_2\varsigma\Big)\Bigg]\gamma_{2,3}(0)\\
=&\frac{c_{4,22}+\theta_3\Big(\phi_1\nu+\phi_2\varsigma\Big)}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)\\
=&\frac{c_{4,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)
\end{aligned}
$$

Subsequently,

$$
\begin{aligned}
\gamma(6)=&\phi_1\gamma(5)+\phi_2\gamma(4)\\
=&\phi_1\left(\frac{c_{4,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\right)\gamma_{2,3}(0)+\\
&\phi_2\left(\frac{c_{3,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\right)\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\phi_1\Big(c_{4,21}+\theta_2\lambda(\phi_1\phi_2)+\theta_2\kappa(\phi_1^2+\phi_2)+\\
&\theta_3(\phi_1\nu+\phi_2\varsigma)\Big)+\phi_2\Big(c_{3,21}+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda+\theta_3\nu\Big)\Bigg]\gamma_{2,3}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\phi_1\Big\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1c_{1,21}+\phi_1\phi_2\Big)+\\
&\phi_1\phi_2c_{1,21}+\theta_1\varpi\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1\phi_2\Big)+\theta_2\kappa\Big(\phi_1^2+\phi_2\Big)+\\
&\theta_3\Big(\phi_1\nu+\phi_2\varsigma\Big)\Big\}+\phi_2\Big\{\Big(\phi_1^2+\phi_2\Big)c_{1,21}+\\
&\phi_1^2\phi_2+\theta_1\phi_1\phi_2\varpi+\theta_2\phi_1\kappa+\theta_2\phi_2\lambda+\theta_3\nu\Big\}\Bigg]\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\Big\{\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_1^2\phi_2\Big)+\\
&\phi_1^2\phi_2c_{1,21}+\theta_1\varpi\phi_1\phi_2\Big(\phi_1^2+\phi_2\Big)+\theta_2\lambda\Big(\phi_1^2\phi_2\Big)+\theta_2\kappa\Big(\phi_1^3+\phi_1\phi_2\Big)+\\
&\theta_3\Big(\phi_1^2\nu+\phi_1\phi_2\varsigma\Big)\Big\}+\Big\{\Big(\phi_1^2+\phi_2\Big)\phi_2c_{1,21}+\phi_1^2\phi_2^2+\theta_1\phi_1\phi_2^2\varpi+\\
&\theta_2\phi_1\phi_2\kappa+\theta_2\phi_2^2\lambda+\theta_3\phi_2\varsigma\Big\}\Bigg]\gamma_{2,3}(0)\\
=&\frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg[\Big(\phi_1^2+\phi_2\Big)\Big(\phi_1^2c_{1,21}+\phi_2c_{1,21}+\phi_1^2\phi_2\Big)+\\
&\phi_1^2\phi_2c_{1,21}+\phi_1^2\phi_2^2+\theta_1\varpi\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)+\theta_2\lambda\Big(\phi_1^2\phi_2+\phi_2^2\Big)+\\
&\theta_2\kappa\Big(\phi_1^3+2\phi_1\phi_2\Big)+\theta_3\Big(\phi_1\phi_2\Big)+\theta_3\nu\Big(\phi_1^2+\phi_2\Big)\Bigg]\gamma_{2,3}(0)\\
=&\frac{c_{5,22}+\theta_3\Big(\phi_1\phi_2\Big)+\theta_3\nu\Big(\phi_1^2+\phi_2\Big)}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)\\
=&\frac{c_{5,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0)
\end{aligned}
$$

Subsequently,

$$
\begin{aligned}
c_{6,23}=&c_{6,21}+\theta_2\lambda\Big(\phi_1^3\phi_2+2\phi_1\phi_2^2\Big)+\theta_2\kappa\Big(\phi_1^4+3\phi_1^2\phi_2+\phi_2^2\Big)+\\
&\theta_3\varsigma\Big(\phi_1^2\phi_2+\phi_2^2\Big)+\theta_3\nu\Big(\phi_1^3+2\phi_1\phi_2\Big)\\
=&c_{6,22}++\theta_3\varsigma\Big(\phi_1^2\phi_2+\phi_2^2\Big)+\theta_3\nu\Big(\phi_1^3+2\phi_1\phi_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
c_{7,23} =& c_{7,21} + \theta_2\lambda\Big(\phi_1^4\phi_2 + 3\phi_1^2\phi_2^2 + \phi_2^3\Big) + \theta_2\kappa\Big(\phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2\Big) + \\
& \theta_3\varsigma\Big(\phi_1^3\phi_2 + 2\phi_1\phi_2^2\Big) + \theta_3\nu\Big(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2\Big) \\
=& c_{7,22} + \theta_3\varsigma\Big(\phi_1^3\phi_2 + 2\phi_1\phi_2^2\Big) + \theta_3\nu\Big(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2\Big)
\end{aligned}
$$

Then for $r \geq 5$,

$$
\begin{aligned}
c_{r,23} =& c_{r,22} + \theta_3\varsigma\left[\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-4-2s}\phi_2^{s+1}\right] + \\
& \theta_3\nu\left[\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s}\right]
\end{aligned}
\tag{4.69}
$$

Hence for an ARMA(2,3) process, the general autocovariance function is given as

$$
\begin{aligned}
\gamma(k) =& \frac{c_{r,23}}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\gamma_{2,3}(0) \\
=& \frac{1}{1-\phi_2+\theta_1\chi+\theta_2\tau+\theta_3\mu}\Bigg\{c_{r,22} + \theta_3\varsigma\left[\sum_{r-4\geq 2s}\binom{(r-4-s)}{s}\phi_1^{r-4-2s}\phi_2^{s+1}\right] + \\
& \theta_3\nu\left[\sum_{r-3\geq 2s}\binom{(r-3-s)}{s}\phi_1^{r-3-2s}\phi_2^{s}\right]\Bigg\}\gamma_{2,3}(0),
\end{aligned}
\tag{4.70}
$$

for $r \geq 5$, noting that $k-1=r$

The results in Equation (4.70) shows that the ACF of the ARMA(2,3) is a function of a coefficient $c_{r,23}$, which may further be given in terms of a coefficient $c_{r,22}$, a general coefficient for ARMA(2,2). The results then involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$ for which $r-t \geq 2s$, $3 \leq t \leq 4$.

It is also observed that the Y-W relation emerged only after lag 4. This means that there is the need for the computation of individual $\gamma(k)$ for $k \leq 4$.

Another important observation is that three more constants (μ, ς, and ν) have been introduced than the constants in the general expression for ARMA(2,2) process.

The processes so far for deriving the ARMA$(2, q)$, $q \leq 3$ have shown some clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. For example, for any $\gamma_{2,3}(k)$ of the ARMA(2,3) process, it is possible to deduce the expression for $\gamma_{2,2}(k)$, $\gamma_{2,1}(k)$ and $\gamma_{2,0}(k)$ for a given value of k. The pattern can similarly be extended further down to the $\gamma_{1,q}(k)$ of the ARMA$(1, q)$ process. Following this pattern, the general autocovariance function for the ARMA$(2, q)$ may be obtained.

ACF of an ARMA(2,q) Process

In this section, we seek to extend the acgf to obtain a generalized ACF of an ARMA$(2, q)$ process. Our motivation is that if the approach has worked for ARMA(2,0), ARMA(2,1), ARMA(2,2) and ARMA(2,3), then it should work for ARMA$(2, q)$.

The ARMA$(2, q)$ process is given by

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \theta_1 Z_{t-1} + \cdots + \theta_{q-1} Z_{t-(q-1)} + \theta_q Z_{t-q} + Z_t$$

which is equivalent to

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \sum_{j=0}^{q} \theta_j Z_{t-j} \tag{4.71}$$

Introducing a lag operator, the ARMA(2,q) process can be written as

$$X_t(1 - \phi_1 L - \phi_2 L^2) = \sum_{j=0}^{q} \theta_j L^j Z_t \tag{4.72}$$

Noting that $\theta_0 = 1$

Equation (4.72) can be simplified as

$$X_t = \frac{\sum_{j=0}^{q} \theta_j L^j}{1 - \phi_1 L - \phi_2 L^2} Z_t \tag{4.73}$$

Assuming the quadratic $1 - \phi_1 L - \phi_2 L^2$ has two different real roots, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$,

then $1 - \phi_1 L - \phi_2 L^2$ can be written as c

It can be verified that $(\alpha + \beta)$=ϕ_1 and $\alpha\beta = -\phi_2$

Equation (4.73) can thus be written as

$$X_t = \frac{\sum_{j=0}^{q} \theta_j L^j}{(1 - \alpha L)(1 - \beta L)} Z_t \tag{4.74}$$

From autocovariance generating functions,

$$c(s)c(s^{-1}) = \sigma^2 \frac{\Theta(s)\Theta(s^{-1})}{\Phi(s)\Phi(s^{-1})} \tag{4.75}$$

Thus, the autocovariance generating function of an ARMA(2,q) process is given as

$$c(s)c(s^{-1}) = \sigma^2 \frac{\sum_{j=0}^{q} \theta_j s^j \sum_{j=0}^{q} \theta_j s^{-j}}{(1 - \alpha s)(1 - \alpha s^{-1})(1 - \beta s)(1 - \beta s^{-1})} \tag{4.76}$$

Equation (4.76) can be simplified as

$$c(s)c(s^{-1}) = \sigma^2 \sum_{r=0}^{\infty} (\alpha s)^r \cdot \sum_{r=0}^{\infty} (\alpha s^{-1})^r \cdot \sum_{r=0}^{\infty} (\beta s)^r \cdot \sum_{r=0}^{\infty} (\beta s^{-1})^r \sum_{j=0}^{q} \theta_j s^j \sum_{j=0}^{q} \theta_j s^{-j} \tag{4.77}$$

From Equation (4.76),

$$\sum_{r=0}^{\infty} (\alpha s)^r \cdot \sum_{r=0}^{\infty} (\alpha s^{-1})^r \cdot \sum_{r=0}^{\infty} (\beta s)^r \cdot \sum_{r=0}^{\infty} (\beta s^{-1})^r = \left[\sum_{r=0}^{\infty} \alpha^{2r} \sum_{r=0}^{\infty} \beta^{2r} \right]$$
$$\left[\sum_{r=0}^{\infty} (\alpha s)^r + \sum_{r=1}^{\infty} (\alpha s^{-1})^r \right] \left[\sum_{r=0}^{\infty} (\beta s)^r + \sum_{r=1}^{\infty} (\beta s^{-1})^r \right]$$

and

$$\sum_{j=0}^{q}\theta_j s^j \sum_{j=0}^{q}\theta_j s^{-j} = \sum_{j=0}^{q}\theta_j^2 + \Big[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Big]s + \Big[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Big]s^2 +$$
$$\Big[\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Big]s^3 + \Big[\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Big]s^4 + \cdots \Big[\sum_{j=0}^{q-5}\theta_j\theta_{j+5}\Big]s^5 + \cdots +$$
$$\Big[\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\Big]s^{q-2} + \Big[\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Big]s^{q-1} + \Big[\sum_{j=0}^{0}\theta_j\theta_{j+q}\Big]s^{q} +$$
$$\Big[\sum_{j=0}^{q-1}\theta_j\theta_{j+1}\Big]s^{-1} + \Big[\sum_{j=0}^{q-2}\theta_j\theta_{j+2}\Big]s^{-2} + \Big[\sum_{j=0}^{q-3}\theta_j\theta_{j+3}\Big]s^{-3} +$$
$$\Big[\sum_{j=0}^{q-4}\theta_j\theta_{j+4}\Big]s^{-4} + \cdots + \Big[\sum_{j=0}^{2}\theta_j\theta_{j+(q-2)}\Big]s^{-q+2} +$$
$$\Big[\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}\Big]s^{-q+1} + \Big[\sum_{j=0}^{0}\theta_j\theta_{q}\Big]s^{-q}$$

At lag 0, the variance function is obtained by considering in only s^0 as follows

$$\gamma(0) = \Bigg[\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r} + 2\Big(\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r} + \alpha^2\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^2\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^3\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^3\sum_{r=0}^{\infty}\beta^{2r} + \cdots\alpha^{q-2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-2}\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^{q-1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-1}\sum_{r=0}^{\infty}\beta^{2r} + \alpha^{q}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q}\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^{q+1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+1}\sum_{r=0}^{\infty}\beta^{2r} + \alpha^{q+2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+2}\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^{q+3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+3}\sum_{r=0}^{\infty}\beta^{2r} + \cdots + \Big)\Bigg]\sum_{j=0}^{q}\theta_j^2 +$$
$$2\Bigg[\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r} + \alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^2\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^2\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^3\sum_{r=0}^{\infty}\beta^{2r} + \cdots + \alpha^{q-2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-1}\sum_{r=0}^{\infty}\beta^{2r} +$$
$$\alpha^{q-1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q}\sum_{r=0}^{\infty}\beta^{2r} + \alpha^{q}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+1}\sum_{r=0}^{\infty}\beta^{2r} +$$

$$\alpha^{q+1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+2}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q+2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+3}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{q+3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+4}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{2}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+$$

$$\alpha^{q-1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-2}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-1}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{q+1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q+2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+1}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\left.\alpha^{q+3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+2}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q+4}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+3}\sum_{r=0}^{\infty}\beta^{2r}+\cdots\right]\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+\cdots$$

$$2\left[\alpha^{q-2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q-3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{2}\sum_{r=0}^{\infty}\beta^{2r}+\right.$$

$$\alpha^{q-4}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{3}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+\alpha^{3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-4}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-3}\sum_{r=0}^{\infty}\beta^{2r}+\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-2}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+$$

$$\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-1}\sum_{r=0}^{\infty}\beta^{2r}+\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+1}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+2}\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\alpha^{4}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+3}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+\alpha^{q-1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\sum_{r=0}^{\infty}\beta^{2r}+$$

$$\left.\alpha^{q}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q+1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{2}\sum_{r=0}^{\infty}\beta^{2r}+\cdots\right]\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}+$$

$$2\left[\alpha^{q-1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{q-2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{2}\sum_{r=0}^{\infty}\beta^{2r}+\right.$$

$$\alpha^{q-3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{3}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+\alpha^{3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot$$

$$\beta^{q-3}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-2}\sum_{r=0}^{\infty}\beta^{2r}+\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q-1}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+$$

$$\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q}\sum_{r=0}^{\infty}\beta^{2r}+\alpha\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+1}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+2}\sum_{r=0}^{\infty}\beta^{2r}+$$
$$\alpha^{3}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+3}\sum_{r=0}^{\infty}\beta^{2r}+\alpha^{4}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{q+4}\sum_{r=0}^{\infty}\beta^{2r}+\cdots+$$
$$\alpha^{q}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\sum_{r=0}^{\infty}\beta^{2r}\alpha^{q+1}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta\sum_{r=0}^{\infty}\beta^{2r}+$$
$$\left.\alpha^{q+2}\sum_{r=0}^{\infty}\alpha^{2r}\cdot\beta^{2}\sum_{r=0}^{\infty}\beta^{2r}+\cdots\right]\sum_{j=0}^{0}\theta_j\theta_{j+q}$$

The expression simplifies as

$$\gamma(0)=\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\Bigg\{\Big[1+2\alpha\beta\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q}\theta_j^2+$$
$$2\Big[(\alpha+\beta)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+2\Big[\alpha\beta+(\alpha^2+\beta^2)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-2}\theta_j\theta_{j+2}+$$
$$2\Big[(\alpha^2\beta+\alpha\beta^2)+(\alpha^3+\beta^3)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-3}\theta_j\theta_{j+3}+\cdots$$
$$2\Big[(\alpha^{q-2}\beta+\alpha^{q-3}\beta^2+\cdots+\alpha^2\beta^{q-3}+\alpha\beta^{q-2})+(\alpha^{q-1}+$$
$$\beta^{q-1})\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}+2\Big[(\alpha^{q-1}\beta+\alpha^{q-2}\beta^2+\cdots+$$
$$\alpha^2\beta^{q-2}+\alpha\beta^{q-1})+(\alpha^q+\beta^q)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg\}$$
$$=\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\Bigg\{\Big[(1+\alpha\beta)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q}\theta_j^2+$$
$$2\Big[(\alpha+\beta)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+2\Big[\alpha\beta+(\alpha^2+\beta^2)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-2}\theta_j\theta_{j+2}+$$
$$2\Big[(\alpha^2\beta+\alpha\beta^2)+(\alpha^3+\beta^3)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{q-3}\theta_j\theta_{j+3}+\cdots$$
$$2\Big[(\alpha^{q-2}\beta+\alpha^{q-3}\beta^2+\cdots+\alpha^2\beta^{q-3}+\alpha\beta^{q-2})+$$
$$(\alpha^{q-1}+\beta^{q-1})\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}+2\Big[(\alpha^{q-1}\beta+\alpha^{q-2}\beta^2+\cdots+$$
$$\alpha^2\beta^{q-2}+\alpha\beta^{q-1})+(\alpha^q+\beta^q)\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Big]\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg\}$$

$$
\begin{aligned}
=&\sum_{r=0}^{\infty}\alpha^{2r}\sum_{r=0}^{\infty}\beta^{2r}\sum_{r=0}^{\infty}(\alpha\beta)^{r}\Bigg\{\Big[(1+\alpha\beta)\Big]\sum_{j=0}^{q}\theta_j^2+2\Big[(\alpha+\beta)\Big]\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+\\
&2\Big[\alpha\beta(1-\alpha\beta)+(\alpha^2+\beta^2)\Big]\sum_{j=0}^{q-2}\theta_j\theta_{j+2}+2\Big[(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)+\\
&(\alpha^3+\beta^3)\Big]\sum_{j=0}^{q-3}\theta_j\theta_{j+3}+\cdots+2\Big[(\alpha^{q-2}\beta+\alpha^{q-3}\beta^2+\cdots+\alpha^2\beta^{q-3}+\\
&\alpha\beta^{q-2})(1-\alpha\beta)+(\alpha^{q-1}+\beta^{q-1})\Big]\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}+2\Big[(\alpha^{q-1}\beta+\\
&\alpha^{q-2}\beta^2+\cdots+\alpha^2\beta^{q-2}+\alpha\beta^{q-1})(1-\alpha\beta)+(\alpha^q+\beta^q)\Big]\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg\}\\
=&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{\Big[(1+\alpha\beta)\Big]\sum_{j=0}^{q}\theta_j^2+2\Big[(\alpha+\beta)\Big]\sum_{j=0}^{q-1}\theta_j\theta_{j+1}+\\
&2\Big[\alpha\beta(1-\alpha\beta)+(\alpha^2+\beta^2)\Big]\sum_{j=0}^{q-2}\theta_j\theta_{j+2}+2\Big[(\alpha^2\beta+\alpha\beta^2)(1-\alpha\beta)+\\
&(\alpha^3+\beta^3)\Big]\sum_{j=0}^{q-3}\theta_j\theta_{j+3}+\cdots+2\Big[(\alpha^{q-2}\beta+\alpha^{q-3}\beta^2+\cdots+\alpha^2\beta^{q-3}+\\
&\alpha\beta^{q-2})(1-\alpha\beta)+(\alpha^{q-1}+\beta^{q-1})\Big]\sum_{j=0}^{1}\theta_j\theta_{j+(q-1)}+2\Big[(\alpha^{q-1}\beta+\\
&\alpha^{q-2}\beta^2+\cdots+\alpha^2\beta^{q-2}+\alpha\beta^{q-1})(1-\alpha\beta)+(\alpha^q+\beta^q)\Big]\sum_{j=0}^{0}\theta_j\theta_{j+q}\Bigg\}
\end{aligned}
$$

Therefore, the variance function of an ARMA$(2, q)$ in terms of the reciprocal of the roots α and β, of the quadratic $Q(L) = 0$ is given as

$$
\begin{aligned}
\gamma(0) =&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q}\theta_j^2+\\
&2\sum_{n=1}^{q}\Big[(1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^{r}+(\alpha^n+\beta^n)\Big]\sum_{j=0}^{q-n}\theta_j\theta_{j+n}\Bigg\}
\end{aligned}
\tag{4.78}
$$

The obvious restriction on this function is that whenever a limit in a summand exceeds the upper limit, the term goes to zero.

Using similar deductions, it can be shown that

$$\gamma(1) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q-1}\theta_j\theta_{j+1} + \sum_{n=0}^{q}\left[(\alpha^{n+1}+\beta^{n+1}) + (1-\alpha\beta)\sum_{r=1}^{n}\alpha^{n+1-r}\beta^r\right]\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=1}^{q-1}\left[(\alpha^n+\beta^n) + (1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r\right]\sum_{j=0}^{q-(n+1)}\theta_j\theta_{j+(n+1)}\Bigg\}$$

and

$$\gamma(2) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q-2}\theta_j\theta_{j+2} + (\alpha+\beta)\sum_{j=0}^{q-1}\theta_j\theta_{j+1} \sum_{n=0}^{q}\left[(\alpha^{n+2}+\beta^{n+2}) + (1-\alpha\beta)\sum_{r=1}^{n}\alpha^{n+2-r}\beta^r\right]\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=1}^{q-2}\left[(\alpha^n+\beta^n) + (1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r\right]\sum_{j=0}^{q-(n+2)}\theta_j\theta_{j+(n+2)}\Bigg\}$$

Subsequently, for $h = 1, 2, \cdots, q-1$,

$$\gamma(h) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q-h}\theta_j\theta_{j+h} + \sum_{n=1}^{h-1}\left[(\alpha^n+\beta^n) + (1-\alpha\beta)\sum_{r=1}^{h-2}\alpha^{h+n-2-r}\beta^r\right]\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=0}^{q}\left[(\alpha^{n+h}+\beta^{n+h}) + (1-\alpha\beta)\sum_{r=1}^{n}\alpha^{n+h-r}\beta^r\right]\sum_{j=0}^{q-n}\theta_j\theta_{j+n} + \sum_{n=1}^{q-h}\left[(\alpha^n+\beta^n) + (1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r\right]\sum_{j=0}^{q-(n+h)}\theta_j\theta_{j+(n+h)}\Bigg\},$$

At lag q, we consider terms in s^q. After similar derivation,

$$\gamma(q) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \sum_{n=0}^{q-1} \Bigg\{ \Big[(\alpha^{q-n} + \beta^{q-n}) + (1-\alpha\beta) \sum_{r=1}^{q-(n+1)} \alpha^{q-n-r}\beta^r\Big] \sum_{j=0}^{q-n} \theta_j\theta_{j+n} + (1+\alpha\beta)\theta_q + \Big[(\alpha^{q+(n+1)} + \beta^{q+(n+1)}) + (1-\alpha\beta)\sum_{r=1}^{q+n} \alpha^{q+1+n-r}\beta^r\Big] \sum_{j=0}^{q-(n+1)} \theta_j\theta_{j+n+1} \Bigg\}$$

At lag $(q+1)$, terms in s^{q+1} gives

$$\gamma(q+1) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \sum_{n=0}^{q-1} \Bigg\{ \Big[(\alpha^{q+1-n} + \beta^{q+1-n}) + (1-\alpha\beta) \sum_{r=1}^{q+1-(n+1)} \alpha^{q+1-n-r}\beta^r\Big] \sum_{j=0}^{q+1-(n+1)} \theta_j\theta_{j+n} + \Big[(\alpha^{q+1+(n+1)} + \beta^{q+1+(n+1)}) + (1-\alpha\beta)\sum_{r=1}^{q+1+n} \alpha^{q+1+n-r-1}\beta^r\Big] \sum_{j=0}^{q+1-(n+1)} \theta_j\theta_{j+n+1} \Bigg\}$$

Using similar deductions,

$$\gamma(q+2) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \sum_{n=0}^{q-1} \Bigg\{ \Big[(\alpha^{q+2-n} + \beta^{q+2-n}) + (1-\alpha\beta) \sum_{r=1}^{q+2-(n+1)} \alpha^{q+2-n-r}\beta^r\Big] \sum_{j=0}^{q+2-(n+1)} \theta_j\theta_{j+n} + \Big[(\alpha^{q+2+(n+1)} + \beta^{q+2+(n+1)}) + (1-\alpha\beta)\sum_{r=1}^{q+2+n} \alpha^{q+2+n-r-1}\beta^r\Big] \sum_{j=0}^{q+2-(n+1)} \theta_j\theta_{j+n+1} \Bigg\}$$

Subsequently,

$$\begin{aligned}\gamma(q+h) =&\frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\sum_{n=0}^{q-1}\Bigg\{\Bigg[(\alpha^{q+h-n}+\beta^{q+h-n})+\\&(1-\alpha\beta)\sum_{r=1}^{q+h-(n+1)}\alpha^{q+h-n-r}\beta^r\Bigg]\sum_{j=0}^{q+h-(n+1)}\theta_j\theta_{j+n}+\\&\Bigg[(\alpha^{q+h(n+1)}+\beta^{q+h(n+1)})+(1-\alpha\beta)\sum_{r=1}^{q+h+n}\alpha^{q+h+n-r-1}\beta^r\Bigg]\\&\sum_{j=0}^{q+h-(n+1)}\theta_j\theta_{j+n+1}\Bigg\}\end{aligned}$$

The autocovariance of an ARMA(2,q) process can be summarized as

$$\begin{aligned}\gamma(0) =&\frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q}\theta_j^2+\\&2\sum_{n=1}^{q}\Bigg[(1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r+(\alpha^n+\beta^n)\Bigg]\sum_{j=0}^{q-n}\theta_j\theta_{j+n}\Bigg\}\end{aligned}$$

for all $h = 1, 2, \cdots, q-1$

$$\begin{aligned}\gamma(h) =&\frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{q-h}\theta_j\theta_{j+h}+\\&\sum_{n=1}^{h-1}\Bigg[(\alpha^n+\beta^n)+(1-\alpha\beta)\sum_{r=1}^{h-2}\alpha^{h+n-2-r}\beta^r\Bigg]\sum_{j=0}^{q-n}\theta_j\theta_{j+n}+\\&\sum_{n=0}^{q}\Bigg[(\alpha^{n+h}+\beta^{n+h})+(1-\alpha\beta)\sum_{r=1}^{n}\alpha^{n+h-r}\beta^r\Bigg]\sum_{j=0}^{q-n}\theta_j\theta_{j+n}+\\&\sum_{n=1}^{q-h}\Bigg[(\alpha^n+\beta^n)+(1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r\Bigg]\sum_{j=0}^{q-(n+h)}\theta_j\theta_{j+(n+h)}\Bigg\}\end{aligned} \tag{4.79}$$

$$\gamma(q) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \sum_{n=0}^{q-1} \Bigg\{ \Big[(\alpha^{q-n} + \beta^{q-n}) + (1-\alpha\beta) \sum_{r=1}^{q-(n+1)} \alpha^{q-n-r}\beta^r \Big] \sum_{j=0}^{q-n} \theta_j\theta_{j+n} + (1+\alpha\beta)\theta_q + \Big[(\alpha^{q+(n+1)} + \beta^{q+(n+1)}) + (1-\alpha\beta) \sum_{r=1}^{q+n} \alpha^{q+1+n-r}\beta^r \Big] \sum_{j=0}^{q-(n+1)} \theta_j\theta_{j+n+1} \Bigg\} \tag{4.80}$$

for all $h \geq 1$

$$\gamma(q+h) = \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \sum_{n=0}^{q-1} \Bigg\{ \Big[(\alpha^{q+h-n} + \beta^{q+h-n}) + (1-\alpha\beta) \sum_{r=1}^{q+h-(n+1)} \alpha^{q+h-n-r}\beta^r \Big] \sum_{j=0}^{q+h-(n+1)} \theta_j\theta_{j+n} + \Big[(\alpha^{q+h(n+1)} + \beta^{q+h(n+1)}) + (1-\alpha\beta) \sum_{r=1}^{q+h+n} \alpha^{q+h+n-r-1}\beta^r \Big] \sum_{j=0}^{q+h-(n+1)} \theta_j\theta_{j+n+1} \Bigg\} \tag{4.81}$$

For each autocovariance function, the usual restriction is that whenever a limit exceeds an upper limit, the term in the summand equals to zero.

It will be demonstrated later in the chapter that the general expression for the autocovariance function of the ARMA$(2, q)$ may further be expressed in terms of the parameters of the process as obtained for the lower orders in the earlier part of this section. It should be noted in the meantime that $\gamma(k)$ of the ARMA$(2, q)$ process is a step-function for $k = 0$, $1 \leq k < q$, $k = q$ and $k > q$.

The next section examines much higher order ARMA(p, q) processes particularly for $p = 3$.

The ACF of an ARMA(3,0) Process

This section examines the first two autocorrelation functions of an ARMA(3,0) process. The autocovariance generating function is used to obtain the variance and autocovariance at lag 1, after which the autocovariances are normalized to obtain the autocorrelation functions. An ARMA (3,0) process is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t \tag{4.82}$$

By introducing a lag operator, Equation (4.82) can be simplified as

$$(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3) X_t = Z_t$$

Further simplification yields

$$X_t = \frac{1}{1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3} Z_t \tag{4.83}$$

Assuming the polynomial $1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3$ has three different real roots, $\frac{1}{\alpha_1}$, $\frac{1}{\alpha_2}$ and $\frac{1}{\alpha_3}$, then

$$1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3$$

can be written as

$$P(L) = (1 - \alpha_1 L)(1 - \alpha_2 L)(1 - \alpha_3 L)$$

It can be verified that $(\alpha_1 + \alpha_2 + \alpha_3)$=$\phi_1$, $(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) = -\phi_2$ and $\alpha_1\alpha_2\alpha_3 = \phi_3$

If

$$X_t = \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)(1 - \alpha_3 L)} Z_t$$

then

$$c(s) = \frac{1}{(1 - \alpha_1 s)(1 - \alpha_2 s)(1 - \alpha_3 s)}$$

The autocovariance generating function can thus be written as

$$c(s)c(s^{-1}) = \sigma^2 \left[\frac{1}{(1-\alpha_1 s)(1-\alpha_2 s)(1-\alpha_3 s)} \times \frac{1}{(1-\alpha_1 s^{-1})(1-\alpha_2 s^{-1})(1-\alpha_3 s^{-1})} \right] \tag{4.84}$$

Equation (4.84) simplifies to

$$\begin{aligned} c(s)c(s^{-1}) =& \sigma^2 \sum_{r=0}^{\infty} (\alpha_1 s)^r \cdot \sum_{r=0}^{\infty} (\alpha_1 s^{-1})^r \cdot \sum_{r=0}^{\infty} (\alpha_2 s)^r \cdot \sum_{r=0}^{\infty} (\alpha_2 s^{-1})^r \cdot \\ & \sum_{r=0}^{\infty} (\alpha_3 s)^r \cdot \sum_{r=0}^{\infty} (\alpha_3 s^{-1})^r \\ =& \sigma^2 \left[\sum_{r=0}^{\infty} \alpha_1^{2r} \left(\sum_{r=0}^{\infty} (\alpha_1 s)^r + \sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \right) \right] \times \\ & \left[\sum_{r=0}^{\infty} \alpha_2^{2r} \left(\sum_{r=0}^{\infty} (\alpha_2 s)^r + \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \right) \right] \times \\ & \left[\sum_{r=0}^{\infty} \alpha_3^{2r} \left(\sum_{r=0}^{\infty} (\alpha_3 s)^r + \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \right) \right] \\ =& \sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \left[\left(\sum_{r=0}^{\infty} (\alpha_1 s)^r + \right. \right. \\ & \left. \sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \right) \left(\sum_{r=0}^{\infty} (\alpha_2 s)^r + \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \right) \\ & \left. \left(\sum_{r=0}^{\infty} (\alpha_3 s)^r + \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \right) \right] \end{aligned}$$

which further simplifies to

$$
\begin{aligned}
c(s)c(s^{-1}) =&\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \Bigg\{ \Bigg[\sum_{r=0}^{\infty} (\alpha_1 s)^r \sum_{r=0}^{\infty} (\alpha_2 s)^r \sum_{r=0}^{\infty} (\alpha_3 s)^r \Bigg] + \\
&\Bigg[\sum_{r=0}^{\infty} (\alpha_1 s)^r \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \sum_{r=0}^{\infty} (\alpha_3 s)^r \Bigg] + \\
&\Bigg[\sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \sum_{r=0}^{\infty} (\alpha_2 s)^r \sum_{r=0}^{\infty} (\alpha_3 s)^r \Bigg] + \\
&\Bigg[\sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \sum_{r=0}^{\infty} (\alpha_3 s)^r \Bigg] + \\
&\Bigg[\sum_{r=0}^{\infty} (\alpha_1 s)^r \sum_{r=0}^{\infty} (\alpha_2 s)^r \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \Bigg] + \\
&\Bigg[\sum_{r=0}^{\infty} (\alpha_1 s)^r \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \Bigg] + \\
&\Bigg[\sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \sum_{r=0}^{\infty} (\alpha_2 s)^r \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \Bigg] + \\
&\Bigg[\sum_{r=1}^{\infty} (\alpha_1 s^{-1})^r \sum_{r=1}^{\infty} (\alpha_2 s^{-1})^r \sum_{r=1}^{\infty} (\alpha_3 s^{-1})^r \Bigg] \Bigg\}
\end{aligned}
\tag{4.85}
$$

To obtain the autocovariance at a particular lag, say k, we consider terms in s^k in each of the expressions in Equation (4.86), and sum all of them.

At lag 0, terms in s^0 gives the variance function as follows:

$$\begin{aligned}
\gamma(0) =&\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \Bigg\{ \Big[1\Big] + \Big[1(\alpha_2\alpha_3 + \alpha_2^2\alpha_3^2 + \alpha_2^3\alpha_3^3 + \cdots) + \\
&\alpha_1 s(\alpha_2 + \alpha_2^2\alpha_3 + \alpha_2^3\alpha_3^2 + \cdots)s^{-1} + \alpha_1^2 s^2(\alpha_2^2 + \alpha_2^3\alpha_3 + \cdots)s^{-2} + \\
&\alpha_1^3 s^3(\alpha_2^3 + \alpha_2^4\alpha_3 + \cdots)s^{-3} + \cdots\Big] + \Big[1(\alpha_1\alpha_3 + \alpha_1^2\alpha_3^2 + \\
&\alpha_1^3\alpha_3^3 + \cdots) + \alpha_2 s(\alpha_1 + \alpha_1^2\alpha_3 + \quad \alpha_1^3\alpha_3^2 + \cdots)s^{-1} + \alpha_2^2 s^2(\alpha_1^2 + \\
&\alpha_1^3\alpha_3 + \alpha_1^4\alpha_3^2 + \cdots)s^{-2} + \alpha_2^3 s^3(\alpha_1^3 + \alpha_1^4\alpha_3 + \cdots)s^{-3} + \cdots\Big] + \\
&\Big[\alpha_3^2 s^2(\alpha_1\alpha_2)s^{-2} + \alpha_3^3 s^3(\alpha_1\alpha_2^2 + \alpha_1^2\alpha_2)s^{-3} + \alpha_3^4 s^4(\alpha_1\alpha_2^3 + \alpha_1^2\alpha_2^2 + \\
&\alpha_1^3\alpha_2)s^{-4} + \cdots\Big] + \Big[1(\alpha_2\alpha_3 + \alpha_2^2\alpha_3^2 + \alpha_2^3\alpha_3^3 + \alpha_2^4\alpha_3^4 + \cdots) + \\
&\alpha_1 s(\alpha_3 + \alpha_2\alpha_3^2 + \alpha_2^2\alpha_3^3 + \alpha_2^3\alpha_3^4 + \cdots)s^{-1} + \alpha_1^2 s^2(\alpha_3^2 + \alpha_2\alpha_3^3 + \\
&\alpha_2^2\alpha_3^4 + \cdots)s^{-2} + \alpha_1^3 s^3(\alpha_3^3 + \alpha_2\alpha_3^4 + \alpha_2^2\alpha_3^5 + \cdots)s^{-3} + \cdots\Big] + \\
&\Big[\alpha_1^2 s^2(\alpha_2\alpha_3)s^{-2} + \alpha_1^3(\alpha_2\alpha_3^2 + \alpha_2^2\alpha_3)s^{-3} + \alpha_1^4 s^4(\alpha_2\alpha_3^3 + \alpha_2^2\alpha_3^2 + \\
&\alpha_2^3\alpha_3) + \alpha_1^5 s^5(\alpha_2\alpha_3^4 + \alpha_2^2\alpha_3^3 + \alpha_2^3\alpha_3^2 + \alpha_2^4\alpha_3)s^{-4} + \cdots\Big] + \\
&\Big[\alpha_2^2 s^2(\alpha_1\alpha_3)s^{-2} + \alpha_2^3 s^3(\alpha_1\alpha_3^2 + \alpha_1^2\alpha_3) + \alpha_2^4 s^4(\alpha_1\alpha_3^3 + \alpha_1^2\alpha_3^2 + \\
&\alpha_1^3\alpha_3)s^{-4} + \cdots\Big]\Bigg\} \\
=&\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \Bigg\{ \Big[1\Big] + \Big[\alpha_1\alpha_2 \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r + \\
&\alpha_2\alpha_3 \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r\Big] + \Big[\alpha_1\alpha_3 \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r + \\
&\alpha_1\alpha_2 \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r\Big] + \Big[\alpha_1\alpha_2\alpha_3^2 \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r\Big] \\
&+ \Big[\alpha_2\alpha_3 \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r + \alpha_1\alpha_3 \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r\Big] + \\
&\Big[\alpha_1^2\alpha_2\alpha_3 \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r\Big] + \Big[\alpha_1\alpha_2^2\alpha_3 \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r\Big]\Bigg\}
\end{aligned}$$

$$\gamma(0) =\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r \Bigg\{ 1+ \Big[\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3\Big] - \alpha_1\alpha_2\alpha_3\Big[\alpha_1 + \alpha_2 + \alpha_3\Big] - \Big[(\alpha_1\alpha_2\alpha_3)^2\Big]\Bigg\}$$

$$= \frac{\sigma^2}{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)} \Bigg\{ 1+ \Big[\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3\Big] - \alpha_1\alpha_2\alpha_3\Big[\alpha_1 + \alpha_2 + \alpha_3\Big] - \Big[(\alpha_1\alpha_2\alpha_3)^2\Big]\Bigg\}$$

The denominator in the expression above simplifies to

$$\Big[1 - (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) + \alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) - (\alpha_1\alpha_2\alpha_3)^2\Big] \cdot \Big[1 - \{(\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\} + \{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^2 - 2\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)\} - (\alpha_1\alpha_2\alpha_3)^2\Big]$$

Substituting $\phi_1 = (\alpha_1 + \alpha_2 + \alpha_3)$, $\phi_2 = -(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$ and $\phi_3 = \alpha_1\alpha_2\alpha_3$, the variance of the ARMA(3,0) process denoted by $\gamma_{3,0}(0)$ is given by

$$\begin{aligned} \gamma(0) &= \sigma^2 \frac{1 - \phi_2 - \phi_1\phi_3 - \phi_3^2}{\Big[1 + \phi_2 + \phi_1\phi_3 - \phi_3^3\Big]\Big[1 - \phi_1^2 - 2\phi_2 + \phi_2^2 - 2\phi_1\phi_3 - \phi_3^2\Big]} \\ &= \sigma^2 \frac{1 - \phi_2 - \phi_3(\phi_1 - \phi_3)}{\Big[1 + \phi_2 + \phi_3(\phi_1 - \phi_3^2)\Big]\Big[(1-\phi_2)^2 - \phi_1^2 - \phi_3(2\phi_1 + \phi_3)\Big]} \end{aligned} \tag{4.86}$$

It is clear from Equation (4.87) that if $\phi_3 = 0$, we obtain $\gamma_{2,0}(0)$, the variance function of the ARMA(2,0) process. Additionally, if $\phi_2 = \phi_3 = 0$, we obtain $\gamma_{1,0}(0)$, the variance function of the ARMA(1,0) process. It is noted in this case that the MA component vanishes.

At lag 1, terms in s gives the autocovariance as

$$\begin{aligned}
\gamma(1) =&\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \Bigg\{ \Big[1(\alpha_2 + \alpha_3)s + (1)\alpha_1 s\Big] + \Big[1(\alpha_2\alpha_3^2 + \alpha_2^2\alpha_3^3 \\
&+ \alpha_2^2\alpha_3^4 + \cdots)s + \alpha_1 s(\alpha_2\alpha_3 + \alpha_2^2\alpha_3^2 + \alpha_2^3\alpha_3^3 + \cdots) + \alpha_1^2 s^2(\alpha_2 + \alpha_2^2\alpha_3 + \\
&\alpha_2^3\alpha_3^2 + \cdots)s^{-1} + \alpha_1^3 s^3(\alpha_2^2 + \alpha_2^3\alpha_3 + \alpha_2^4\alpha_3^2 + \cdots)s^{-2}\Big] + \Big[1(\alpha_1\alpha_3^2 + \\
&\alpha_1^2\alpha_3^3 + \alpha_1^3\alpha_3^4 + \cdots) + \alpha_2 s(\alpha_1\alpha_3 + \alpha_1^2\alpha_3^2 + \alpha_1^3\alpha_3^3 + \cdots) + \alpha_2 s^2(\alpha_1 + \\
&\alpha_1^2\alpha_3 + \alpha_1^3\alpha_3^2 + \cdots)s^{-1} + \alpha_2^3 s^3(\alpha_1^2 + \alpha_1^3\alpha_3 + \alpha_1^4\alpha_3^2 + \cdots)s^{-3}\Big] + \\
&\Big[\alpha_3^3 s^3(\alpha_1\alpha_2)s^{-2} + \alpha_3^4 s^4(\alpha_1\alpha_2^2 + \alpha_1^2\alpha_2)s^{-3} + \alpha_3^5 s^5(\alpha_1\alpha_2^3 + \alpha_1^2\alpha_2^2 + \\
&\alpha_1^3\alpha_2)s^{-4} + \alpha_3^6 s^6(\alpha_1\alpha_2^4 + \alpha_1^2\alpha_2^3 + \alpha_1^3\alpha_2^2 + \alpha_1^4\alpha_2)s^{-5} + \cdots\Big] + \\
&\Big[1(\alpha_2^2\alpha_3 + \alpha_2^3\alpha_3^2 + \alpha_2^4\alpha_3^3 + \cdots)s + \alpha_1 s(\alpha_2\alpha_3 + \alpha_2^2\alpha_3^2 + \alpha_2^3\alpha_3^3 + \cdots) + \\
&\alpha_2^2\alpha_3^2 + \alpha_2^3\alpha_3)s^{-4} + \alpha_1^6 s^6(\alpha_2\alpha_3^4 + \alpha_2^2\alpha_3^3 + \alpha_2^3\alpha_3^2 + \alpha_2^4\alpha_3)s^{-5}\Big] + \cdots + \\
&\Big[\alpha_2^3 s^3(\alpha_1\alpha_3)s^{-2} + \alpha_2^4 s^4(\alpha_1\alpha_3^2 + \alpha_1^2\alpha_3)s^{-3} + \alpha_2^5 s^5(\alpha_1\alpha_3^3 + \alpha_1^2\alpha_3^2 + \\
&\alpha_1^3\alpha_3)s^{-4} + \alpha_2^6 s^6(\alpha_1\alpha_3^4 + \alpha_1^2\alpha_3^3 + \alpha_1^3\alpha_3^2 + \alpha_1^4\alpha_3)\Big] + \cdots \Bigg\} \\
=&\sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \Bigg\{ \Big[\alpha_1 + \alpha_2 + \alpha_3\Big] + \Big[\alpha_1^2\alpha_2^2\alpha_3 \sum_{r=0}^{\infty}(\alpha_1\alpha_2)^r \\
&\sum_{r=0}^{\infty}(\alpha_2\alpha_3)^r\Big] + \Big[\alpha_1\alpha_3^2 \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r + \alpha_1\alpha_2\alpha_3 \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r\Big] + \\
&\Big[\alpha_1\alpha_2^2 \sum_{r=0}^{\infty}(\alpha_1\alpha_2)^r \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r\Big] + \Big[\alpha_1\alpha_2\alpha_3^2 \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r \sum_{r=0}^{\infty}(\alpha_2\alpha_3)^r\Big] + \\
&\Big[\alpha_1\alpha_2\alpha_3 \sum_{r=0}^{\infty}(\alpha_2\alpha_3)^r \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r\Big] + \Big[\alpha_2^2\alpha_3 \sum_{r=0}^{\infty}(\alpha_2\alpha_3)^r\Big] + \\
&\Big[\alpha_1^3\alpha_2\alpha_3 \sum_{r=0}^{\infty}(\alpha_1\alpha_3)^r \sum_{r=0}^{\infty}(\alpha_1\alpha_2)^r\Big] + \\
&\Big[\alpha_1\alpha_2^3\alpha_3 \sum_{r=0}^{\infty}(\alpha_2\alpha_3)^r \sum_{r=0}^{\infty}(\alpha_1\alpha_2)^r\Big] \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\gamma(1) =& \sigma^2 \sum_{r=0}^{\infty} \alpha_1^{2r} \sum_{r=0}^{\infty} \alpha_2^{2r} \sum_{r=0}^{\infty} \alpha_3^{2r} \sum_{r=0}^{\infty} (\alpha_1\alpha_2)^r \sum_{r=0}^{\infty} (\alpha_1\alpha_3)^r \sum_{r=0}^{\infty} (\alpha_2\alpha_3)^r \Bigg\{ \Big[\alpha_1 + \\
& \alpha_2 + \alpha_3 \Big] - \Big[\alpha_1\alpha_2\alpha_3(\alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2) \Big] \Bigg\} \\
=& \frac{\sigma^2}{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)} \Bigg\{ \Big[\alpha_1 + \\
& \alpha_2 + \alpha_3 \Big] - \Big[\alpha_1\alpha_2\alpha_3(\alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2) \Big] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) =& \sigma^2 \frac{\phi_1 + \phi_2\phi_3}{\Big[1 + \phi_2 + \phi_1\phi_3 - \phi_3^3\Big] \Big[1 - \phi_1^2 - 2\phi_2 + \phi_2^2 - 2\phi_1\phi_3 - \phi_3^2\Big]} \\
=& \frac{\phi_1 + \phi_2\phi_3}{\Big[1 - \phi_2 - \phi_3(\phi_1 - \phi_3)\Big]} \gamma_{3,0}(0)
\end{aligned} \tag{4.87}
$$

It is possible to deduce the expression for $\gamma_{2,0}(1)$ and $\gamma_{1,0}(1)$ from the $\gamma_{3,0}(1)$ given in Equation (4.88) by putting $\phi_3 = 0$ and $\phi_2 = \phi_3 = 0$, respectively.

The processes so far for deriving the autocovariance at lag k of the ARMA(3,0) have shown a clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. For example, for any $\gamma_{3,0}(k)$ of the ARMA(3,0) process, it is possible to deduce the expression for $\gamma_{2,0}(k)$ and $\gamma_{1,0}(k)$ from the $\gamma_{3,0}(k)$ by putting relevant parameters to zero. It is clear therefore that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained. There is also a clear connection between autocovariance at lags of consecutive orders of the process.

Using similar deductions, the variances of an ARMA$(3, 1)$ is also obtained as

$$
\gamma_{3,1}(0) = \sigma^2 \frac{(1+\theta_1^2)\Big[1 - \phi_2 - \phi_3(\phi_1 - \phi_3)\Big] + 2\theta_1(\phi_1 + \phi_2\phi_3)}{\Big[1 + \phi_2 + \phi_3(\phi_1 - \phi_3^2)\Big]\Big[(1-\phi_2)^2 - \phi_1^2 - \phi_3(2\phi_1 + \phi_3)\Big]} \tag{4.88}
$$

Validation of Derived Expressions

For the general ARMA$(2, q)$ process, for example, the autocovariance function is already given in terms of the reciprocal of the roots of the quadratic equation in terms of the lag operator L even though specific expressions for some values of q have been shown in terms of the actual parameters. In this section, is to demonstrate that it is possible to derive any specific autocovariance function from the given expression in terms of the roots. Particularly, we validate that ARMA(1,4) and ARMA(2,3) processes can be obtained from the step functions in Equations (4.23-4.26) and (4.78 - 4.81).

Deduction of the ACFs of an ARMA(1,4) process

For an ARMA(1,4) process, $q = 4$. From our derivations of the ACFs of an ARMA(1,q) process, the variance of an ARMA(1,4) process can be obtained as

$$\begin{aligned}
\gamma(0) =& \sigma^2\left\{\sum_{j=0}^{4}\theta_j + 2\sum_{n=1}^{4}\sum_{j=0}^{4-n}\phi^n\theta_j\theta_{j+n}\right\}\frac{1}{1-\phi^2} \\
=& \sigma^2\left\{\sum_{j=0}^{4}\theta_j^2 + 2\sum_{j=0}^{4-1}\phi\theta_j\theta_{j+1} + 2\sum_{j=0}^{4-2}\phi^2\theta_j\theta_{j+2} + 2\sum_{j=0}^{4-3}\phi\theta_j\theta_{j+3}\right. \\
&\left. + 2\sum_{j=0}^{4-4}\phi\theta_j\theta_{j+4}\right\}\frac{1}{1-\phi^2} \\
=& \sigma^2\left\{\sum_{j=0}^{4}\theta_j^2 + 2\sum_{j=0}^{3}\phi\theta_j\theta_{j+1} + 2\sum_{j=0}^{2}\phi^2\theta_j\theta_{j+2} + 2\sum_{j=0}^{1}\phi^3\theta_j\theta_{j+3}\right. \\
&\left. + 2\sum_{j=0}^{0}\phi^4\theta_j\theta_{j+4}\right\}\frac{1}{1-\phi^2}
\end{aligned}$$

Therefore, the variance function of the ARMA(1,4) process denoted as $\gamma_{1,4}(0)$

is given as

$$\gamma_{1,4}(0) = \sigma^2 \left\{ \sum_{j=0}^{4} \theta_j^2 + 2\phi \sum_{j=0}^{3} \theta_j\theta_{j+1} + 2\phi^2 \sum_{j=0}^{2} \theta_j\theta_{j+2} + 2\phi^3 \sum_{j=0}^{1} \theta_j\theta_{j+3} + 2\phi^4\theta_4 \right\} \frac{1}{1-\phi^2} \tag{4.89}$$

To obtain $\gamma(1)$, we use the equation in h where $h = 1, 2, \cdots, q-1$.

$$\begin{aligned}
\gamma(1) =& \sigma^2 \left\{ \sum_{n=0}^{1-1} \sum_{j=0}^{4-(n+1)} \phi^{1-1-n}\theta_j\theta_{j+(n+1)} + \sum_{n=0}^{4} \sum_{j=0}^{4-n} \phi^{1+n}\theta_j\theta_{j+n} + \right. \\
& \left. \sum_{n=0}^{4-1-1} \sum_{j=0}^{4-(1+1+n)} \phi^{1+n}\theta_j\theta_{j+(1+n+1)} \right\} \frac{1}{1-\phi^2} \\
=& \sigma^2 \left\{ \sum_{j=0}^{3} \theta_j\theta_{j+1} + \sum_{j=0}^{4} \phi\theta_j^2 + \sum_{j=0}^{3} \phi^2\theta_j\theta_{j+1} + \sum_{j=0}^{2} \phi^3\theta_j\theta_{j+2} + \right. \\
& \sum_{j=0}^{1} \phi^4\theta_j\theta_{j+3} + \sum_{j=0}^{0} \phi^5\theta_j\theta_{j+4} + \sum_{j=0}^{2} \phi\theta_j\theta_{j+2} + \sum_{j=0}^{1} \phi^2\theta_j\theta_{j+3} + \\
& \left. \sum_{j=0}^{0} \phi^3\theta_j\theta_{j+4} \right\} \frac{1}{1-\phi^2} \\
=& \sigma^2 \left\{ \sum_{j=0}^{3} \theta_j\theta_{j+1} + \phi \sum_{j=0}^{4} \theta_j^2 + \phi^2 \sum_{j=0}^{3} \theta_j\theta_{j+1} + \phi^3 \sum_{j=0}^{2} \theta_j\theta_{j+2} + \right. \\
& \phi^4 \sum_{j=0}^{1} \theta_j\theta_{j+3} + \phi^5 \sum_{j=0}^{0} \theta_j\theta_{j+4} + \phi \sum_{j=0}^{2} \theta_j\theta_{j+2} + \phi^2 \sum_{j=0}^{1} \theta_j\theta_{j+3} + \\
& \left. \phi^3 \sum_{j=0}^{0} \theta_j\theta_{j+4} \right\} \frac{1}{1-\phi^2}
\end{aligned}$$

Using the same deduction for lag h where $h = 1, 2, \cdots, q-1$, $\gamma(2)$ and $\gamma(3)$

will be obtained respectively as

$$\begin{aligned}\gamma(2) =&\sigma^2\Bigg\{\phi\sum_{j=0}^{3}\theta_j\theta_{j+1}+\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^2\sum_{j=0}^{4}\theta_j^2+\phi^3\sum_{j=0}^{3}\theta_j\theta_{j+1}+\\&\phi^4\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^5\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^6\sum_{j=0}^{0}\theta_j\theta_{j+4}+\phi\sum_{j=0}^{1}\theta_j\theta_{j+3}+\\&\phi^2\sum_{j=0}^{0}\theta_j\theta_{j+4}\Bigg\}\frac{1}{1-\phi^2}\\=&\sigma^2\Bigg\{\phi\sum_{j=0}^{3}\theta_j\theta_{j+1}+\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^2\sum_{j=0}^{4}\theta_j^2+\phi^3\sum_{j=0}^{3}\theta_j\theta_{j+1}+\\&\phi^4\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^5\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^6\theta_4+\phi\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^2\theta_4\Bigg\}\frac{1}{1-\phi^2}\end{aligned}$$

and

$$\begin{aligned}\gamma(3) =&\sigma^2\Bigg\{\phi^2\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi\sum_{j=0}^{2}\theta_j\theta_{j+2}+\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^3\sum_{j=0}^{4}\theta_j\theta_j^2+\\&\phi^4\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi^5\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^6\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^7\sum_{j=0}^{0}\theta_j\theta_{j+4}+\\&\phi\sum_{j=0}^{0}\theta_j\theta_{j+4}\Bigg\}\frac{1}{1-\phi^2}\\=&\sigma^2\Bigg\{\phi^2\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi\sum_{j=0}^{2}\theta_j\theta_{j+2}+\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^3\sum_{j=0}^{4}\theta_j\theta_j^2+\\&\phi^4\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi^5\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^6\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^7\theta_4+\phi\theta_4\Bigg\}\frac{1}{1-\phi^2}\end{aligned}$$

The autocovariances at lag 4 will be given as

$$\begin{aligned}\gamma(4) =&\sigma^2\Bigg\{\phi^3\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi^2\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^4\sum_{j=0}^{4}\theta_j\theta_j^2+\\&\phi^5\sum_{j=0}^{3}\theta_j\theta_{j+1}+\phi^6\sum_{j=0}^{2}\theta_j\theta_{j+2}+\phi^7\sum_{j=0}^{1}\theta_j\theta_{j+3}+\phi^8\theta_4+\theta_4\Bigg\}\frac{1}{1-\phi^2}\end{aligned}$$

Subsequently, the autocovariances after lag 4 will be given as

$$\gamma(4+h) = \phi^h \gamma(4), \qquad h \geq 1$$

The ACF at each lag of the process is obtained by dividing the respective autocovarince function by the variance.

Deduction of the ACFs of an ARMA(2,3)

For an ARMA(2,3) process, $q = 3$. From our derivations of the ACFs of an ARMA(2,q) process, the variance of an ARMA(2,3) process can be obtained as

$$\begin{aligned}
\gamma(0) =& \frac{1}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)} \Bigg\{ (1+\alpha\beta) \sum_{j=0}^{3} \theta_j^2 + \\
& 2 \sum_{n=1}^{3} \left[(1-\alpha\beta) \sum_{r=1}^{n-1} \alpha^{n-r}\beta^r + (\alpha^n + \beta^n) \right] \sum_{j=0}^{3-n} \theta_j \theta_{j+n} \Bigg\} \\
=& \frac{\sigma^2}{(1-\alpha\beta)\left[(1+\alpha\beta)^2 - (\alpha+\beta)^2\right]} \Bigg\{ \left[(1+\alpha\beta) \sum_{j=0}^{3} \theta_j^2 \right] + \\
& 2\left[(\alpha+\beta) \sum_{j=0}^{2} \theta_j \theta_{j+1} \right] + 2\left[(\alpha^2+\beta^2) + \alpha\beta(1-\alpha\beta) \right] \sum_{j=0}^{1} \theta_j \theta_{j+2} + \\
& 2\theta_3 \left[(\alpha^3+\beta^3) + (\alpha^2\beta + \alpha\beta^2)(1-\alpha\beta) \right] \Bigg\} \\
=& \frac{\sigma^2}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]} \Bigg\{ \left[1-\phi_2 \right] \sum_{j=0}^{3} \theta_j^2 + \left[2\phi_1 \right] \sum_{j=0}^{2} \theta_j \theta_{j+1} + \\
& \left[2(\phi_1^2 + \phi_2 - \phi_2^2) \right] \sum_{j=0}^{1} \theta_j \theta_{j+2} + 2\theta_3 \left[\phi_1^3 + \phi_1 + 2\phi_1\phi_2 - \phi_1\phi_2^2 \right] \Bigg\}
\end{aligned}$$

This expression is the same as $\gamma_{2,3}(0)$ obtained earlier in Equation (4.68).

The autocovariance at lag 1 will be obtained as

$$
\begin{aligned}
\gamma(1) =& \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)}\Bigg\{(1+\alpha\beta)\sum_{j=0}^{3-1}\theta_j\theta_{j+h}+\\
&\sum_{n=1}^{1-1}\Big[(\alpha^n+\beta^n)+(1-\alpha\beta)\sum_{r=1}^{1-2}\alpha^{1+n-2-r}\beta^r\Big]\sum_{j=0}^{q-n}\theta_j\theta_{j+n}+\\
&\sum_{n=0}^{q}\Big[(\alpha^{n+1}+\beta^{n+1})+(1-\alpha\beta)\sum_{r=1}^{n}\alpha^{n+1-r}\beta^r\Big]\sum_{j=0}^{3-n}\theta_j\theta_{j+n}+\\
&\sum_{n=1}^{3-1}\Big[(\alpha^n+\beta^n)+(1-\alpha\beta)\sum_{r=1}^{n-1}\alpha^{n-r}\beta^r\Big]\sum_{j=0}^{3-(n+1)}\theta_j\theta_{j+(n+1)}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) =& \frac{\sigma^2}{(1-\alpha\beta)\Big[(1+\alpha\beta)^2-(\alpha+\beta)^2\Big]}\Bigg\{\Big[(\alpha+\beta)\sum_{j=0}^{3}\theta_j^2\Big]+\\
&\Big[(1+\alpha\beta)\sum_{j=0}^{2}\theta_j\theta_{j+1}\Big]+\Big[(\alpha+\beta)\sum_{j=0}^{1}\theta_j\theta_{j+2}\Big]+\theta_3\Big[(\alpha+\beta)^2-\\
&\alpha\beta-(\alpha\beta)^2\Big]+\Big[(\alpha+\beta)^2-\alpha\beta-(\alpha\beta)^2\Big]\sum_{j=0}^{2}\theta_j\theta_{j+1}+\\
&\Big[(\alpha+\beta)^3-2\alpha\beta(\alpha+\beta)-(\alpha\beta)^2(\alpha+\beta)\Big]\sum_{j=0}^{1}\theta_j\theta_{j+2}+\theta_3\\
&\Big[(\alpha+\beta)^4-3\alpha\beta(\alpha+\beta)^2+(\alpha\beta)^2-(\alpha\beta)^2\{(\alpha+\beta)^2+\alpha\beta\}\Big]\Bigg\}\\
=&\frac{1}{(1+\phi_2)\Big[(1-\phi_2)^2-\phi_1^2\Big]}\Bigg\{\Big[\phi_1\sum_{j=0}^{3}\theta_j^2\Big]+\Big[1+\phi_1^2-\phi_2^2\Big]\sum_{j=0}^{2}\theta_j\theta_{j+1}+\\
&\Big[(\phi_1^3+\phi_1+2\phi_1\phi_2-\phi_1\phi_2^2)\Big]\sum_{j=0}^{1}\theta_j\theta_{j+2}+\theta_3\Big[\phi_1^4+\phi_1^2+3\phi_1^2\phi_2-\\
&\phi_1^2\phi_2^2+\phi_2-\phi_2^3\Big]\Bigg\}
\end{aligned}
$$

It can be verified subsequently that $\gamma_{2,3}(k)$ is the same as what is obtained earlier under ARMA(2,3) process.

Application to Pandemic Data

Having studied the general autocorrelation function of the ARMA(p, q) process, this section now uses real data to examine the performance of the derived procedure. In order to relate the derived functions with the literature, the results obtained is compared with existing functions. The data used covers the Covid-19 cases for Ghana, Nigeria and South Africa. It is only in these countries that cases are found to be stationary in nature with respect to the series, and are therefore suitable for the implementations of the results.

Table 1 is a summary of the selected ARMA processes that were observed to characterize the daily new Covid-19 cases in the selected countries around the globe. The thoretical ACFs were obtained from the parameters of each respective model. It can be seen from Table 1 that all parameter values for the various ARMA processes are all statistically significant.

Table 1: **Summary of appropriate ARMA models of the daily new Covid-19 cases in some selected countries**

Country	Process	Parameter	Coeff.	SE Coeff.	$p-$value
Ghana	ARMA(1,4)	ϕ_1	0.9560	0.0137	0.000
		θ_1	-0.9326	0.0380	0.001
		θ_2	0.0248	0.0520	0.011
		θ_3	0.0250	0.0539	0.028
		θ_4	0.1373	0.0358	0.003
Nigeria	ARMA(1,2)	ϕ_1	0.9798	0.0077	0.000
		θ_1	-0.8707	0.0371	0.003
		θ_2	0.1318	0.0338	0.012
South Africa	ARMA(2,2)	ϕ_1	1.2075	0.1044	0.034
		ϕ_2	-0.2210	0.1024	0.014
		θ_1	-0.5621	0.1020	0.009
		θ_2	-0.1051	0.0594	0.004

Source: Researcher's computation (2023)

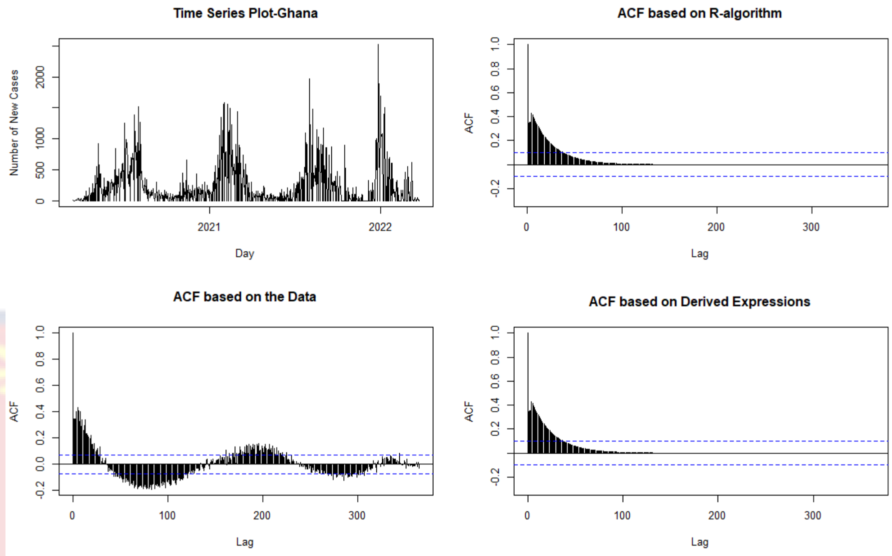

Figure 1: Time series and ACF plots of daily Covid-19 cases for Ghana

Figure 1 shows the time series plot of the daily corona virus cases reported in Ghana, together with its emperical(sample) ACF. The data was subjected to the ”ARIMAfit” function in R, and ARMA(1,4) was selected to best fit the data. Based on the model and its associated parameters, the ACF based on the McLeod algorithm presented in the R statistical software, and the theoretical ACF based on the derived expressions are obtained, and the results are included to Figure 1. It can be observed from the figure that the sample ACF shows a sinusoidal pattern, which implies that the autocorrelations of the daily new covid-19 cases in Ghana demonstrate some wave-forms. It can again be observed that the times between successive waves are not even, showing that the waves are not necessarily periodic. Notably, it is evident from Figure 1 that the autocorrelations die out as the lag gets larger, an indication that in the distant future, incidence of cases would not be influenced significantly by previous cases. The ACF based on McLeod’s algorithm and that obtained from the derived expressions do not exhibit any significant difference. From the theoretical ACF, the ACF of the daily covid-19 cases in Ghana attenuates exponentially, and cuts off just after lag 100. Although an ARMA(1,4) process is best chosen to characterize the series, it is interesting to observe that while the emperical (sample) ACF shows sinusoidal waves which eventually dies out at higher lags, the theoretical ACF shows a damper exponential pattern.

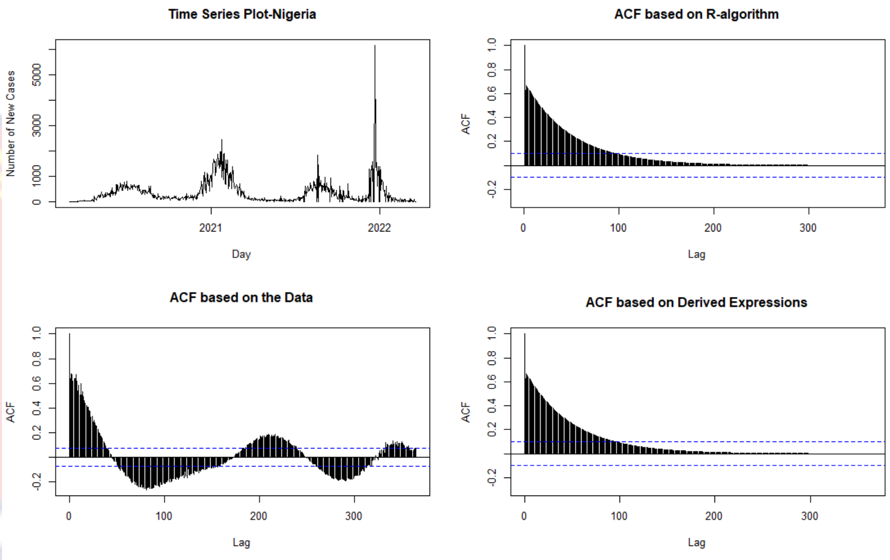

Figure 2: Time series and ACF plots of daily Covid-19 cases for Nigeria

Figure 2 shows the time series plot, sample ACF, ACF based on the McLeod algorithm presented in the R statistical software, and the theoretical ACF based on our derived expressions. The sample ACF shows a sinusoidal pattern, which implies that the autocorrelations of the daily new covid-19 cases in Nigeria depicted some wave-forms. It can be seen again that the times between successive waves are not even, showing clearly that the waves are not necessarily periodic. Notably, it is evident from Figure 2 that the autocorrelations die out as the lag gets larger. The ACF based on McLeod's algorithm and that obtained from our derived expressions do not exhibit any significant difference. From the theoretical ACF, the ACF of the daily covid-19 cases in Nigeria attenuates exponentially, and cuts off at about lag 300. Although an ARMA(1,2) process is best chosen to characterize the series, it is interesting to observe that while the emperical (sample) ACF shows cosine waves which eventually dies out at higher lags, the theoretical ACF shows a prolonged damper exponential pattern.

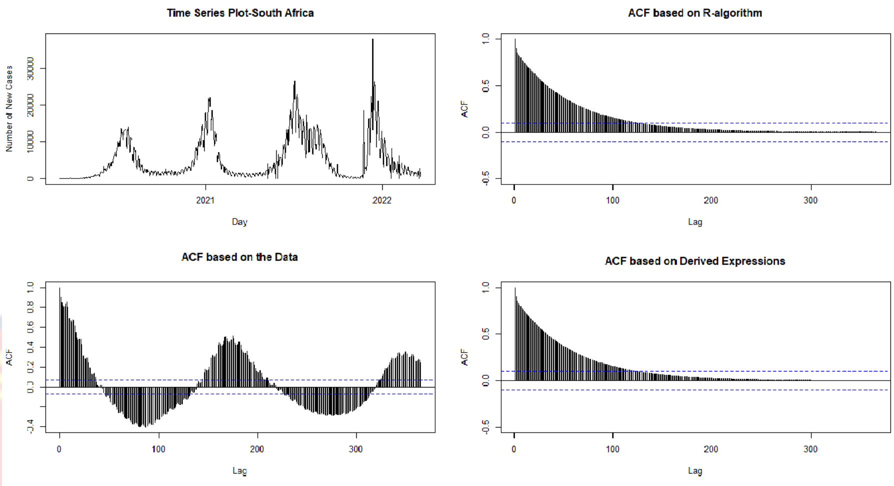

Figure 3: Time series and ACF plots of daily Covid-19 cases for South Africa

Figure 3 shows the time series plot of the daily corona virus cases reported in South Africa, together with its emperical ACF. The data is subjected to the ”ARIMAfit” function in R, and ARMA(2,2) is selected to best fit the data. Based on the model and its associated parameters, the ACF based on the McLeod algorithm presented in the R statistical software, and the theoretical ACF based on the derived expressions are obtained, and the results are added to Figure 3. It can be observed from the figure that the sample ACF shows a sinusoidal pattern, which implies that the autocorrelations of the daily new covid-19 cases in South Africa also demonstrate some wave-forms. The times between successive waves does not appear to be even, showing clearly that the waves are not necessarily periodic. Additionally, it is evident from the emperical ACF of Figure 3 that the autocorrelations do not die out as the lag gets larger, an indication that in the distant future, incidence of cases would still be influenced significantly by previous cases. The theoritical ACF based on McLeod’s algorithm and that obtained from the derived expressions do not exhibit any significant difference. From the theoretical ACFs, the ACF of the daily covid-19 cases in South Africa attenuates exponentially, and cuts off at about lag 300 in the derived ACF but appears to cut off well after lag 300 in the one based on R. Although an ARMA(2,2) process is best chosen to characterize the series, it is interesting to observe that while the sample ACF showed sinusoidal waves which eventually dies out at higher lags, the theoretical ACF shows a prolonged damper exponential pattern.

Discussion of Results

The derivations made in this chapter have shown into much detail that the ACF of an ARMA(p, q) process is predominantly influenced by the Moving Average order. More clearly from the generalization of the ARMA$(1, q)$ process, it is seen that ACF at lag q, which is the order of the MA component, and that beyond lag q is given by the relation $\gamma(q + h) = \phi^h \gamma(q)$, for all $h \geq 1$. The relation shows that the ACFs after lag q is expected to decrease by a rate of ϕ^h. The ACFs from lag 1 to $(q - 1)$ for $q \geq 2$, do not quite relate to the ACFs at lag q and beyond. This supports the reason why the ACFs of an ARMA$(1, q)$ that precede lag q are determined separately. Similarly, the presentation of the ACFs of a generalized ARMA$(2, q)$ process shows that the ACFs at lags after q is related to the ACF at lag q, while those that precede lag q are obtained separately. This pattern is expected to be a dominant feature of the ARMA$(3, q)$ and the general ARMA(p, q) process. The study has therefore obtained explicit expressions for a general ACF of the ARMA(p, q) process, a result that is not existent in the literature.

The literature (Chen et al., 2011; Eshel, 2003) point out the relationship among the autocovariances and autocorrelations obtained from the Yule-Walker simplifications respectively as $\gamma(k) = \phi_1 \gamma(k - 1) + \cdots + \phi_p \gamma(k - p)$ and $\rho(k) = \phi_1 \rho(k - 1) + \cdots + \phi_p \rho(k - p)$ for $k \geq p$, where p is the order of the Autoregressive part. Notably, it has been seen that the ACF of ARMA(p, q) processes for cases where $p \geq 2$ does not follow the Y-W recursive formula in general. The derivations pinpoint that the Y-W recursive formula does not hold for the ACFs at certain lags of some ARMA$(2, q)$ processes. Noticeably, for an ARMA(2,1) process the Y-W recursive formula holds for lag $k \geq 2$, which is consistent with the literature. However, for an ARMA(2,2) and (2,3) processes, the Yule-Walker recursive formula holds for lags $k \geq 4$ and $k \geq 5$, respectively, a result which is also not observed in the literature. The derivations also show that for ARMA processes where $p \geq 2$, the Autoregressive component needed to be expressed in a factorized form. The consequence of this is that the initial ARMA model parameters are transformed into several new roots. To be able to relate the ACFs of the ARMA(p, q) process for $p \geq 2$ to the ACFs of an ARMA$(1, q)$ process, the new roots should be converted back to the original parameters.

The slow decay of the theoretical ACFs in Figures 1, 2 and 3 show that the corona virus cases in Ghana, Nigeria and South Africa are expected to continue for a long time, but will eventually die out. Comparatively, it is expected that daily new Covid-19 cases in Ghana which follows an ARMA(1,4) process cuts off faster than that of Nigeria and South Africa, which follows ARMA(1,2) and ARMA(2,2), respectively, since there is a longer memory in the ACF of the two countries than that of Ghana. This agrees with Montgomery et al. (2015) that generally, the higher the order of the Moving Average in an ARMA(p, q) process, the shorter the memory. Although there was a clear difference between the emperical ACF and the theoretical, the waves which eventually diminishes in the emperical ACF shows that as the number of lags increases, incidence of future cases could only be sporadic, and would not follow any discernible pattern. This is in line with the literature that the theoritical ACFs are the limiting values of the emperical ACFs. In other words, the emperical ACF mimics the theoretical ACF for cases where the data points are extremely large. There does not appear visible differences in the performance of the derived and the existing theoretical ACF. This is visible in all the three datasets applied in the study. The only difference identified is the running time of the codes based on the derived ACF. It is observed that for higher order ARMA processes, the derived ACF could be quite slow.

Chapter Summary

This chapter has presented extensions and generalizations of the ACFs of stationary ARMA(p, q) processes. Since the ACFs of lower order processes which includes the ARMA(1,0), ARMA(0,1), and ARMA(1,1) are well-documented in the literature, this chapter begins with a study of the ACFs of ARMA(1,2) and ARMA(1,3), after which a generalization is made for ARMA(1,q) process.

The ARMA(1,2) and ARMA(1,3) reveals a clear pattern among autocovariance at consecutive lags of the respective process. It is observed that for any $\gamma(k)$ of a given ARMA$(1, q)$ process,

$$\gamma(k) = \phi^{k-q}\gamma(q) \quad \text{for } k \geq q+1$$

The pattern also suggests that separate ACFs should be obtained for individual lags prior to the order q. The chapter also reveals that for an ARMA$(1, q)$ process,

$$\rho^2(q+1) = \rho(q) \times \rho(q+2)$$

Additionally, the ACFs of ARMA(2,0), ARMA(2,1), ARMA(2,2) and ARMA(2,3) are derived, after which a generalization is made for ARMA(2,q).

The results show that the ACF of higher order ARMA(p, q) may be expressed as a function of a certain coefficient $c_{r,pq}$, which may further be given in terms of a lower order coefficient $c_{r,(pq-1)}$, a general coefficient for ARMA($p, q-1$). The results then involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$ for which $r - t \geq 2s$.

It is also observed that the Y-W relation emerged after lag $q + 2$ for processes higher than ARMA(2,1). This means that there is the need for the computation of individual $\gamma(k)$ for $k \leq (q + 2)$.

In an attempt to obtain explicit expressions for specific ARMA processes, there is an inclusion of additional constants in the ACFs to those in the ACF of the lower order process. For example, three more constants are introduced in the ACF of the ARMA(2,3) than the constants in the general ACF expression for ARMA(2,2) process.

The processes for deriving the ARMA$(2, q)$, $q \leq 3$ shows some clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. For instance, for any $\gamma_{2,3}(k)$ of the ARMA(2,3) process, it is possible to deduce the expression for $\gamma_{2,2}(k)$, $\gamma_{2,1}(k)$ and $\gamma_{2,0}(k)$ for a given value of k. The pattern can similarly be extended further down to the $\gamma_{1,q}(k)$ of the ARMA$(1, q)$ process. It is further noted that $\gamma(k)$ of the ARMA$(2, q)$ process is a step-function for $k = 0$, $1 \leq k < q$, $k = q$ and $k > q$.

The chapter has also explored the strategy adopted for deriving ACF for ARMA(3,0) process. The processes for deriving the autocovariance at lag k of the ARMA(3,0) have shown a clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. The chapter establishes that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained, since there is also a clear connection between autocovariance at lags of consecutive orders of the process.

The daily new Covid-19 cases for Ghana, Nigeria and South Africa are found to be stationary among Covid-19 cases in several countries explored around the world. These three datasets are therefore used for illustration and have brought out some pertinent observations. The data for each country is subjected to the "ARIMAfit" function in R, and ARMA(1,4), ARMA(1,2), and ARMA(2,2) are selected to best fit the data for Ghana, Nigeria, and South Africa, respectively. Based on the models and their associated parameters, the ACF based on the McLeod algorithm presented in the R statistical software, and the theoretical ACF based on the derived expressions are obtained for Covid-19 cases in each country. In each case, it is observed that the sample ACFs show an imperfect sinusoidal pattern that has no specific periodicity. This implies that the autocorrelations of the daily new Covid-19 cases in all the selected countries demonstrate some wave-forms that may not be significant. Thus, the times between successive waves are not even, showing that the waves are not necessarily periodic. The theoretical ACFs in each case tails off slowly as the lags increased. The slow decay of the theoretical ACFs show that the corona virus cases are expected to continue for a long time, but will eventually die off. Additionally, the observations show that incidence of future cases could only be sporadic, and would not follow any discernible pattern. Comparatively, it is expected that the corona virus cases in Ghana would cut off faster than those of Nigeria and South Africa.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter presents an overview of the entire work and conclusions based on the discussion of the results in Chapter Four. Based on that, recommendations will be given.

Summary

The nature and importance of the autocorrelation function (ACF) of a time series process cannot be over-emphasized. This is because the ACF reveals the inherent characteristics of a time series data that may not be visible from the original time series plot. There are also clear differences between the emperical ACF and the theoretical ACF, since the latter is based on only the parameter estimates from the sample. These reasons, among others, provide adequate motivation for continued studies on the ACF of a time series in order to generate deeper understanding on the concept. The study has examined with a presentation of the three main approaches to obtaining the theoretical ACF of lower order ARMA(p, q) processes existing in the literature. The three approaches are the Yule-Walker approach, the comparison of moving average weights, and the autocovariance generating function method. The review of the literature on the theoretical ACFs of higher order ARMA(p, q) process has shown that the process is an arduous one and lacks analytical clarity. This challenge is particularly the motivation for considering a derivation of an alternative theoretical ACF through the autocovariance generating function. The new attempt will deepen understanding of identifying inherent characteristics of a time series that follows a linear process.

As an application to the covid-19 pandemic, a brief review on how previous investigations have used mathematical, statistical, and deep machine learning procedures to model and forecast covid-19 cases across the world is made. Due to the changing waves of the pandemic, the study applies results to examine the real characteristics of the parameters of the covid-19 pandemic around some parts of the globe.

This study has presented the basis for the use of the acvgf in obtaining the ACFs of stationary ARMA(p, q) processes. Specifically the ACF of an AR(1) process has been examined in addition to that of lower order MA(q) process. To aid in the implementation of the formulas that will be derived, the chapter has studied the characteristics of the data selected to aid in this regard.

The study has presented extensions and generalizations of the ACFs of stationary ARMA(p, q) processes. Examination of the general ARMA(1,q) process reveals a clear pattern among autocovariance at consecutive lags of the respective process. It is observed that for any $\gamma(k)$ of a given ARMA($1, q$) process,

$$\gamma(k) = \phi^{k-q}\gamma(q) \quad \text{for } k \geq q+1$$

The pattern also suggests that separate ACFs should be obtained for individual lags prior to the order q. It is also shown that for ARMA($1, q$) process,

$$\rho^2(q+1) = \rho(q) \times \rho(q+2)$$

Additionally, a generalization is made for the ACF of ARMA(2,q). Using this generalization, specific ACF of ARMA(2,q) processes have been deduced for, e.g., the ARMA(2,0) and ARMA(2,3). The results show that the ACF of higher order ARMA(p, q) may be obtained explicitly and expressed as a function of a certain coefficient $c_{r,pq}$, which may further be given in terms of a lower order coefficient $c_{r,(pq-1)}$, a general coefficient for ARMA($p, q-1$). The results then involves computation of combinatorial values of the form $\binom{(r-t-s)}{s}$

for which $r - t \geq 2s$.

It is also observed that the Y-W relation emerges after lag $(q + 2)$ for processes higher than ARMA(2,1). This means that there is the need for the computation of individual $\gamma(k)$ for $k \leq (q + 2)$.

The processes for deriving the ARMA$(2, q)$, $q \leq 3$ shows some clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. For instance, for any $\gamma_{2,3}(k)$ of the ARMA(2,3) process, it is possible to deduce the expression for $\gamma_{2,2}(k)$, $\gamma_{2,1}(k)$ and $\gamma_{2,0}(k)$ for a given value of k. The pattern can similarly be extended further down to the $\gamma_{1,q}(k)$ of the ARMA$(1, q)$ process. It is further noted that $\gamma(k)$ of the ARMA(p, q) process is a step-function for $k = 0$, $1 \leq k < q$, $k = q$ and $k > q$.

The technique of autocovariance generating function has been used to explore the ACF of processes as far as ARMA(3,0) process. The processes for deriving the autocovariance at lag k of the ARMA(3,0) have shown a clear pattern among the autocovariances at consecutive lags of the respective process as well as between particular lags of consecutive orders of the process. It is established that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained, since there is a clear connection between autocovariance at lags of consecutive orders of the process.

The daily new Covid-19 cases for Ghana, Nigeria and South Africa are found to be stationary among Covid-19 cases in several countries explored around the world. These three datasets are therefore used for illustration and have brought out some pertinent observations. The data for each country is subjected to the "ARIMAfit" function in R, and ARMA(1,4), ARMA(1,2), and ARMA(2,2) are selected to best fit the data for Ghana, Nigeria, and South Africa, respectively. Based on the models and their associated parameters, the ACF based on the McLeod algorithm presented in the R statistical software, and the theoretical ACF based on the derived expressions are obtained for Covid-19 cases in each country. In each case, it is observed that the sample ACFs show an imperfect sinusoidal pattern that has no specific periodicity. This implies that the autocorrelations of the daily new Covid-19 cases in all the selected countries demonstrate some wave-forms that may not be significant. Thus, the times between successive waves are not even, showing that the waves are not necessarily periodic. The theoretical ACFs in each case tails off slowly as the lags increased. The slow decay of the theoretical ACFs show that the corona virus cases are expected to continue for a long time, but will eventually die off. Additionally, the observations show that incidence of future cases could only be sporadic, and would not follow any discernible pattern. Comparatively, it is expected that the pandemic cases in Ghana would cut off faster than those of Nigeria and South Africa.

Conclusions

The thesis is an additional attempt at adding to the already existing literature, an alternative approach for obtaining a generalized autocorrelation function of higher order ARMA processes. The autocovariance generating function was used to achieve this ideal. Although the approach is widely known, its usage in obtaining the ACFs of higher stationary time series processes is scarce, while the few have had it difficult to generalize the ACFs after breaking the autocovariance generating function into partial fractions.

To obtain the ACFs of a higher order ARMA process using acvgf, one has to have a solid understanding of power series. The study has therefore identified how the AR(p) part of the ARMA(p, q) process is converted into an infinite power series to enhance easier derivations. Additionally, the study determined that for ARMA processes where $p \geq 2$, the AR(p) process will have to be broken into new roots so that the polynomial expression in terms of the original coefficients can be written in factorized form.

The study has presented coherent derivations of generalized ARMA$(1, q)$ and ARMA$(2, q)$ processes as well as the variance and the first autocorrelation of ARMA(3,0) process. The derivations of the general ACF expressions for ARMA$(1, q)$ and ARMA$(2, q)$ has shown that the ACF of an ARMA(p, q) relies heavily on the order of the Moving Average. It has been shown that the ACF of an ARMA(p, q) process at lag q, the order of the Moving Average part, is the reference point for the ACFs of the lags after it. In addition, it has been seen that to link the ACF of an ARMA$(2, q)$ process to the ACF of ARMA$(1, q)$ process, one will have to convert the new roots of the Autoregressive part back to the original parameters of the ARMA$(2, q)$ process, and set ϕ_2, the second parameter of the Autoregressive part to zero. The study establishes that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained, since there is a clear connection between autocovariance at lags of consecutive orders of the process.

The study has also examined the behaviour of the ACF of the Corona virus pandemic cases in some locations around the globe where incidence is stationary and found that in general, the pandemic would eventually die out, though there could be sporadic cases.

It is established that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained, since there is a clear connection between autocovariance at lags of consecutive orders of the process.

Recommendations

The study has shown that for a general ARMA(p, q) process, the autocovariance at any lag k can be obtained as there is a clear connection between autocovariance at lags of consecutive orders of the process. The results therefore provide useful relations that may be utilized as diagnostic tests for determining whether a given data follows a specified process.

The results of the generalized ACF of the ARMA(p, q) process shows that the theoretical ACF presents the long term behaviour of the series rather than the actual behaviour based on the sample values. The application therefore shows that the corona virus pandemic is expected to die out eventually, in the studied locations with stable cases. It should however be reiterated that there could also be sporadic incidence that are not informed by previous cases. Measures should therefore be put in place to contain such eventualities. In addition, actual causes of these possible sporadic cases may be investigated.

It has also been reported that the algorithm based on the derived ACF could by slow, particularly for higher order ARMA processes. The causes of this slow convergence may be a subject for further investigation.

Future studies can consider ACFs of ARMA(p, q) processes beyond conditions that has guided the work in this study. Specifically, the ACF could be studied also under conditions of real and repeated roots, and complex roots.

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