UNIVERSITY OF CAPE COAST

FIXED POINTS AND SOME QUALITATIVE PROPERTIES OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS, NEUTRAL FUNCTIONAL DIFFERENCE EQUATIONS, AND DYNAMIC EQUATIONS ON TIME SCALE

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BY

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APRIL, 2013

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this University or elsewhere.

Candidate's Name:..

Signature: .... Date: ....

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Name:..

Signature: .... Date: ....

Co-Supervisor's Name:..

Signature: .... Date: ....

ABSTRACT

This thesis is concerned with the qualitative properties of solutions of neutral functional differential equations, neutral functional difference equations and dynamic equations on time scale. Some of the equations are of the first and second order whereas some are systems of equations. All these equations are delay equations with constant or variable delays.

Fixed point theory is used extensively in this thesis to investigate the qualitative properties of solutions of neutral delay equations. In particular, the Krasnoselskii's fixed point theorem, the Krasnoselskii-Burton fixed point theorem and the Banach's fixed point theorem are used in the thesis. We invert the equations and the results of the inversions are used to define suitable mappings which are then used to discuss the qualitative properties of solutions to certain classes of neutral functional equations considered.

Sufficient conditions are established to discuss the qualitative properties such as periodicity, positivity, and stability of the classes of neutral equations of our focus.

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DEDICATION

To my family

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

The attempt to solve physical problems led gradually to mathematical models involving an equation in which a function and its derivatives play important roles. However, the theoretical development of this new branch of mathematics - Ordinary Differential Equations - has its origins rooted in a small number of mathematical problems. These problems and their solutions led to an independent discipline with the solution of such equations an end in itself.

According to Kline (1972), the history of Ordinary Differential Equations (ODEs) goes all the way back to the XVII century when two great scientists Isaac Newton and Gottfried Leibniz introduced calculus which came to place from the concept of functions.

Delay differential equations, differential integral equations and functional differential equations have been studied for at least 200 years (see Schmitt (1911)). Some of the early work originated from problems in geometry and number theory.

Volterra (1909), (1928) discussed the integrodifferential equations that model viscoelasticity. In (1931), he wrote a fundamental book on the role of hereditary effects on models for the interaction of species. The subject gained much momentum (especially in the Soviet Union) after 1940 due to the consideration of meaningful models of engineering systems and control. It is probably true that most engineers were well aware of the fact that hereditary effects occur in physical systems, but this effect was often ignored because there was insufficient theory to discuss such models in detail.

During the last 50 years, the theory of functional differential equations has been developed extensively and has become part of the vocabulary of researchers dealing with specific applications such as viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flow, neural networks, combustion, interaction of species, microbiology, learning models, epidemiology, physiology, as well as many others (see Kolmanovski and Myshkis (1999)).

Differential equations are essential tools in scientific modeling of physical problems which found their relevance in almost every sphere of human endeavour from Agricultural Sciences, Engineering, Medical Science, Physical Sciences to Social Sciences. Among the earlier work on differential equations, the works of Euler and Lagrange stand out. They first worked on the theory of small oscillations and consequently also, the theory of linear system of ordinary differential equations.

In the construction of mathematical models of physical systems it is usually assumed that all of the independent variables, such as time and space are continuous. This assumption normally leads to a realistic and justified approximation of the real variables of the systems. However, we regularly encounter systems for which this continuous variable assumption cannot be made.

Systems in which one or more variables are inherently discrete are in areas such as population growth (Smith (1971)), digital communication networks (McClamroch (1980)) and delayed feedback oscillation as in laser emission pulsation (Rabinovich (1980)). Due to their discrete character, these systems must be modelled by the use of difference equations. This lead to the development of the basic theory of linear difference equations in the eighteenth century by de Moivre, Euler, Lagrange, Laplace, and others.

Delayed differential or difference equations sometimes are also called differential or difference equations with deviating arguments. However nowadays, this later title is seldom used; instead, the terminology of functional differential or difference equations is mostly utilized. Functional differential or difference equations are classified as retarded, neutral, or advance type. Such a classification, first introduced by Myshkis (1951) in his monograph, lay the foundation for a general theory of linear delayed systems.

The concept of time scale analysis is a fairly new idea. It was introduced in 1988 by the German mathematician Stefan Hilger. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different in nature from their continuous counterparts. The study of dynamic equations on time scale reveals such discrepancies, and helps avoid proving results twice, once for differential equations and once for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary nonempty closed subset of the reals. By choosing the time scale to be the set of real numbers, the general result yields a result concerning an ordinary differential equation. On the other hand, by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result.

In the study of the theory of differential and difference equations, we most often encounter equations in which the conventional methods, such as the Laplace transform method and the power series solutions, can be used to solve the differential or difference equations analytically, that is, the solutions can be written out using formulas.

However, in most applications in biology, chemistry, and physics modelled by differential or difference equations where analytical solutions may be unavailable, people are interested in the questions related to the so-called qualitative properties, such as:

(1) Will the system have at least one solution ?

(2) Will the system have at most one solution?

(3) Can certain behaviour of the system be controlled or stabilized?

(4) Will the system exhibit some periodicity?

and

(5) Will the system have positive solutions?

If these questions can be answered without solving the differential or difference equations, especially when analytical solutions are unavailable, we can still get a very good understanding of the system.

The first person to carry out a major investigation in the line of the qualitative theory and hence the development of the qualitative theory of differential and difference equations was Henry Poincare (see Boyer (1968)). This qualitative theory is now the most actively developing area of the theory of differential and difference equations, with most important applications in diverse areas such as Engineering, Economics, Physical and Biological sciences. It is well known that mathematical formulations of many physical problems often result in differential or difference equations that are non-linear. Much has been done on the theory and method of dealing with the linear differential and difference equations in Mathematics but just little of general nature is known about non-linear differential and difference equations. By non-linear differential or difference equations, we are referring to equations where the terms involving the unknown function are not linear in the unknown function. In general, the study of non-linear differential and difference equations are restricted to a variety of special cases and the method of solution usually involves one or more of a limited number of different methods. There are several important differences between linear differential equations and non-linear differential equations as well as linear difference equations and non-linear difference equations. For instance, for the linear ordinary differential and difference equations, it is possible to derive a closed-form expressions for the solutions of the equations whereas this is not possible in general for the non-linear differential and difference equations. As a consequence, it is desirable to be able to make predictions about the behaviour (qualitative analysis) of non-linear ordinary differential and difference equations even in the absence of the closed-form expressions for the solution of the equations.

The analysis of non-linear ordinary differential and difference equations makes use of a wide variety of approaches and mathematical tools than does the analysis of linear differential and difference equations. The main reason for this variety is that no tool or methodology in non-linear differential and difference equations analysis is universally applicable to handle them in a fruitful manner. Close to half a century now, great efforts have been devoted to the study of qualitative theory of non-linear differential and difference equations, to be precise non-linear neutral functional differential and difference equations. During these periods, new methods and outstanding results have appeared. These were extensively summarized in the monograph of Reisig, Sansone and Conti (1974). The major directions which must be emphasized in this context, consist in the investigation of solutions of non-linear differential and difference equations involving boundedness, stability, periodicity and positivity of solutions.

Some of the techniques used in the investigation of these qualitative properties of solutions include the Lyapunov's Direct Method which involves the construction of a suitable positive definite function whose derivative is negative definite. The frequency domain method is another method employed in the investigation. This method involves the study of location of the characteristic polynomial roots in the complex plane. We can also mention the topological degree method which demand the verification of continuity properties of a certain operator and the proof of existence of a particular a-priori bound. Moreover, fixed point theorems are also used to establish qualitative properties. Each of the first three methods has its limitations, for instance, the limitation of the Lyapunov's Second Method is on the non-unique way of constructing a suitable Lyapunov function; the frequency domain method though overcomes the problem of constructing Lyapunov's functions, it is narrower in scope than the Lyapunov's Second Method (see Rouche, Habets, and Laloy, (1977)). The Topological Degree Methods on the other hand are mainly used in proving existence of periodic solutions.

This thesis is concerned with the following qualitative properties of solutions:

- positivity,
- periodicity,
- stability;

for neutral functional differential and difference equations.

A neutral functional differential equation is one in which the derivatives of the past history are involved, as well as those of the present state of the system. Similarly, a neutral functional difference equation is one in which the difference of the past history are involved, as well as those of the present state of the system.

The following classes of equations are considered;

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d}{dt}c(t,x(t-\tau(t)))+f(t,h(x(t)),g(x(t-\tau(t)))), \tag{1.1}$$

$$x'(t)=-a(t)h(x(t))+c(t)x'(t-g(t))+q\big(t,x(t),x(t-g(t)\big), \tag{1.2}$$

$$x'(t) = -a(t)x^3(t)+c(t)x'(g(t))g'(t)+q(t,x^3(g(t))), \tag{1.3}$$

$$x(n+1)=A(n)x(n)+C(n)\Delta x(n-\tau(n))+g(n,x(n-\tau(n))), \tag{1.4}$$

$$\Delta x(n)=-\sum_{j=1}^{N}a_j(n)x(n-\tau_j(n)), \tag{1.5}$$

$$\Delta\Big(x(n)-h(x(n-L_1))\Big)=g(x(n))-g(x(n-L_2)),\ \ n\in\mathbb{Z}, \tag{1.6}$$

$$\begin{aligned} x^{\Delta}(t) &= -a(t)h(x(\sigma(t))+(Q(t,x(t),x(t-g(t)))))^{\Delta} \\ &\quad + G\big(t,x(t),x(t-g(t))\big),t\in\mathbb{T}, \end{aligned} \tag{1.7}$$

and

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(t - g(t)) + \int_{t-r(t)}^{t} k(t,s)h(x(s))\,\Delta s,\; t \in \mathbb{T}. \tag{1.8}$$

Equations of the form (1.1)-(1.8) are not only of theoretical importance but also of practical importance. For example, equation (1.1) has applications in problems dealing with the study of two or more simple oscillatory systems with some interconnections between them (see Cooke and Krumme (1968)), and in modelling physical problems such as vibration of masses attached to an elastic bar (see Hale (1977)). Also, neutral equations such as Equations (1.2) and (1.3) arise in blood cell models (see for instance Beretta, Solimano, and Takeuchi (1996), Wazewska-Czyzewska and Lasota (1976), and Xu and Li (1998)) and food-limited population models (see for instance Chen (2005), Chen and Shi (2005), Fan and Wang (2000)).

Definition 1.1.1. Let

$$x'(t) = f(t, x(t), x(t-\tau(t)), x'(t-\tau(t))),\; t \geq t_0 \tag{1.9}$$

with an assumed initial function $x(t) = \psi(t), t \in [m_{t_0}, t_0]$, with $\psi \in C([m_{t_0}, t_0], \mathbb{R})$,

$$[m_{t_0}, t_0] = \{u \leq t_0 \;:\; u = t - \tau(t), t \geq t_0\}.$$

The solution $x(t)$ of (1.9) is said to be periodic if $x(t+T) = x(t)$ for $T > 0$, and for all $t \in [m_{t_0}, \infty)$. T is called the period of x.

Definition 1.1.2. The solution $\phi(t)$ of (1.9) is said to be stable if for any $t_0 \geq 0$ and any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that $|\psi - \phi(t_0)| \leq \delta$ implies $|x(t, t_0, \psi) - \phi(t)| \leq \varepsilon$ for $t \geq t_0$.

Definition 1.1.3. The solution $\phi(t)$ of (1.9) is said to be asymptotically stable if it is stable and in addition , for any $t_0 \geq 0$, there exists an $r(t_0) > 0$ such that $|\psi - \phi(t_0)| \leq r(t_0)$ implies $\lim_{t\to\infty} |x(t, t_0, \psi) - \phi(t)| = 0$.

Definition 1.1.4. The solution $\phi(t)$ of (1.9) is said to be positive if $\phi(t) > 0$ for all $t \in [m_{t_0}, \infty)$.

Definition 1.1.5. Let

$$x(n+1) = f(n, x(n), x(n-\tau(n)), \Delta x(n-\tau(n))),\; n \geq n_0 \tag{1.10}$$

with an assumed initial function $x(n) = \psi(n), n \in [m_{n_0}, n_0] \cap \mathbb{Z}$, with $\psi \in D([m_{n_0}, n_0] \cap \mathbb{Z}, \mathbb{R})$,

$$[m_{n_0}, n_0] \cap \mathbb{Z} = \{u \leq n_0 \,:\, u = n - \tau(n), n \geq n_0\},$$

where D is the set of bounded sequences on the interval $[m_{n_0}, n_0] \cap \mathbb{Z}$.

The solution $x(n)$ of (1.10) is said to be periodic if $x(n+N) = x(n)$ for some positive integer N, and for all $n \in [m_{n_0}, \infty) \cap \mathbb{Z}$. N is called the period of x.

Definition 1.1.6. The solution $\phi(n)$ of (1.10) is said to be stable if for any $n_0 \geq 0$ and any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, n_0) > 0$, such that $|\psi - \phi(n_0)| \leq \delta$ implies $|x(n, n_0, \psi) - \phi(n)| \leq \varepsilon$ for $n \geq n_0$.

Definition 1.1.7. The solution $\phi(n)$ of (1.10) is said to be asymptotically stable if it is stable and in addition , for any $n_0 \geq 0$, there exists an $r(n_0) > 0$ such that $|\psi - \phi(n_0)| \leq r(n_0)$ implies $\lim_{n\to\infty} |x(n, n_0, \psi) - \phi(n)| = 0$.

Definition 1.1.8. The solution $\phi(n)$ of (1.10) is said to be positive if $\phi(n) > 0$ for all $n \in [m_{n_0}, \infty) \cap \mathbb{Z}$.

We now examine what we mean by a statement that a solution of a differential or difference equation is stable. Note that a differential or difference equation is typically used to model the movement of a certain physical system or experiment. In running a system or experiment, one needs to deal with some initial measurements, such as putting one gallon of water initially for some experiment, which inevitably involves some errors in measurements or approximations. If the behaviour of a system or experiment is stable, then a small change in initial data will result in a small change in the behaviour for future time. Thus, by a statement that a solution ϕ of a differential equation is stable we mean that other solutions with initial data close to the solution ϕ will remain close to ϕ for future time. For example, for a stable system, if ϕ is the solution corresponding to one gallon of water initially, and x is a solution with its initial value close to one gallon of water, say for example, 1.005 gallons of water, then ϕ and x should be close for the future time, or $|x - \phi|$ should be small for future time.

Fixed point theory will be our approach to the study of qualitative properties in this thesis. Over the years vast outflow of research and publications has resulted from the use of fixed point methods to the study of qualitative properties. This work is mainly motivated by Burton and Furumochi (2001a), which has been appreciated by most researchers in qualitative theory of functional differential and difference equations. Generally, to solve a problem with fixed point theory is to find:

(a) a set S consisting of points which would be acceptable solutions;

(b) a mapping $P : S \to S$ with the property that a fixed point solves the problem;

(c) a fixed point theorem stating that this mapping on this set will have a fixed point.

If the functions in a differential or difference equation of interest all satisfy a local Lipschitz condition, then contraction mappings will usually be our first choice in our qualitative properties investigations. However, if the functions are not Lipschitz then we will turn to fixed point theorems of Krasnoselskii type.

1.2 Research Aims and Objectives

The following are the proposed objectives of this thesis:

- To establish sufficient criteria for the existence of positive periodic solutions of the neutral functional second order differential equation

$$\begin{aligned} & \frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) \\ = & \frac{d}{dt}c(t,x(t-\tau(t))) + f(t,h(x(t)),g(x(t-\tau(t)))). \end{aligned}$$

- To obtain sufficient conditions for the existence of periodic solutions of the neutral functional differential equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-g(t)) + q\big(t,x(t),x(t-g(t))\big),$$

and of the system of differential equations

$$\frac{d}{dt}x(t) = A(t)x(t-\tau).$$

- To prove that the zero solution of the neutral functional differential equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-g(t)) + q\big(t, x(t), x(t-g(t))\big),$$

is asymptotically stable.

- To prove the existence and positivity of periodic solutions of the neutral functional differential equation

$$x'(t) \quad = \quad -a(t)x^3(t) + c(t)x'(g(t))g'(t) + q(t, x^3(g(t))).$$

- To prove that the zero solution of the neutral delay difference equation

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)),$$

is asymptotically stable.

- To obtain sufficient conditions for the existence of positive periodic solutions for the system of neutral functional difference equations

$$x(n+1) = A(n)x(n) + C(n)\Delta x(n-\tau(n)) + g(n, x(n-\tau(n))),$$

where $A(n) = diag[a_1(n), a_2(n), ..., a_k(n)]$, and $C(n) = diag[a_1(n), a_2(n),$ $..., a_k(n)]$.

- To establish sufficient conditions for the existence of periodic solutions of the neutral nonlinear dynamic equation

$$\begin{aligned} x^{\Delta}(t) \quad &= \quad -a(t)h(x(\sigma(t)) + (Q(t, x(t), x(t-g(t)))))^{\Delta} \\ &\quad + G\big(t, x(t), x(t-g(t))\big), t \in \mathbb{T}, \end{aligned}$$

on time scale $\mathbb{T}$ and to also prove that the zero solution of the neutral dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(t-g(t)) + \int_{t-r(t)}^{t} k(t,s)h(x(s))\,\Delta s,\; t \in \mathbb{T},$$

is asymptotically stable on the time scale $\mathbb{T}$.

1.3 Outline of the Thesis

In the following Chapter, we review some relevant literature for our study and also give a brief review of the relevant mathematical concepts as well as provide an overview of the tools used in the discussion of the qualitative properties of solutions considered in this thesis. Relevant literature for our investigation is reviewed in the first section of this Chapter. In the second section of the Chapter, some basic results on delay differential equations are given. In the third section of the same Chapter, we state some fixed point theorems as well as provide some details on how fixed point theorems are used to study qualitative properties. In the fourth section of the same Chapter, we give an overview of some basic concepts in difference calculus and in the fifth section of the Chapter we give some basic concepts in measure theory. Sections six and seven contain some basic results on time scale calculus and some definitions related to qualitative studies of differential and difference equations respectively.

Sufficient conditions for the existence of periodic solutions for totally nonlinear neutral functional differential equations of the first order are established in the third Chapter.

In Chapter four, criteria for the zero solution of totally nonlinear neutral differential equations of the first order to be asymptotically stable are obtained.

Conditions for the existence and positivity of periodic solutions of nonlinear neutral functional differential equations are established in the fifth Chapter.

In Chapter six, we establish sufficient criteria for the existence of positive periodic solutions for neutral functional second order differential equations.

The existence of periodic solutions for a system of differential equations with constant delay is proved in Chapter seven.

In Chapter eight of this thesis, we obtain sufficient conditions for the existence of positive solutions for a system of periodic neutral delay difference equations. We also consider neutral functional difference equations with asymptotically constant or Periodic Solutions.

We establish sufficient conditions for the zero solution for a certain class of neutral delay difference equations with variable delays to be asymptotically stable in Chapter nine. Moreover, criteria for solutions of totally nonlinear neutral difference equations to be periodic are also established.

In Chapter ten, we prove the existence of periodic solutions of totally nonlinear neutral dynamic equations on time scale. Furthermore, sufficient criteria for the asymptotic stability of the zero solution of neutral Volterra dynamic equations are established.

The last Chapter of this thesis contains the conclusion and suggestions for further studies.

CHAPTER TWO

LITERATURE REVIEW AND MATHEMATICAL BACKGROUND

2.1 Literature Review

The study of qualitative behaviour of solutions of differential equations started in the latter part of the nineteenth century and became a subject of intense research since 1940 whiles the qualitative theory of delay difference equations has attracted many researchers since 1988. The first direct reference as far as we know toward this approach is the work of Poincare (1899). Ever since this work appeared, there has been an intensified interest among researchers to explore its richness. There is a substantial amount of literature dealing with numerous qualitative behaviour of solutions of differential and difference equations. These have been summarized in the monographs by Coddington, and Levinson (1955), Hahn (1963), Halanay (1966), Krasovskii (1963), Rouche and Mawhin (1980), Burton (2006) and Agarwal, Bohner, Grace, and O'Regan (2005).

Lyapunov (1892) proposed a fundamental method for studying the problem of stability, boundedness and the existence of periodic solution of functional differential and difference equations by constructing functions known as Lyapunov functions in the modern parlance. This function is often represented as $V(t,x)$ defined in some region or the whole state phase that contains the unperturbed solution $x = 0$ for all $t > 0$ and which together with its derivative $V'(t,x)$ satisfy some sign definiteness. This method is by far the most general method for dealing with stability, boundedness, and the existence of periodic solution of functional differential and difference equations. Yet, numerous difficulties with the theory and application to specific problems persist and it does seem that new methods are needed to address those difficulties. There is, of course, the problem of constructing appropriate Lyapunov functional and also problems with the types of conditions which are typically imposed on the functions in the differential or difference equations. These conditions are virtually always precise pointwise requirements. Real-world problems with all their uncertainties call for conditions which are averages.

Burton and Furumochi (2001a) discovered that a number of the difficulties with the use of Lyapunov's direct method vanish if fixed point theory is used instead. They pointed out that, not only do the fixed point conditions emerge as averages, but in one step the existence, uniqueness, and boundedness of solutions of problems which have challenged investigators for decades are proved. They continued this work in Burton and Furumochi (2001b), where delay equations which may be unstable when the delay is zero were considered. In particular, asymptotic stability results were proved by Schauder's and Banach's fixed point theorems and Schaefer's fixed point theorem was also used to prove that there is a periodic solution when a periodic forcing function is added to that equation.

Neutral equations have been studied for a long time, and with good reason. On the intuitive level, every parent, every gardener, and every stock broker has observed growth spurts; present growth rate is closely tied to recent growth rate. And this is the very essence of neutral equations. Gopalsamy (1992), Gopalsamy and Zhang (1988), Kuang (1993a) and Kuang (1993b) devote much space to population problems heuristically modeled as neutral equations. On the other hand, starting from first principles of physics, Driver (1984), studies a two-body problem in terms of neutral equations.

The study of periodicity of solutions of second order differential equations have gained the attention of many researchers in recent times. For instance, Zeng (1997) studied the existence of almost periodic solutions of the equation

$$x''(t) - x(t) + x^3(t) = f(t), \tag{2.1}$$

where f is an almost periodic function. Also, Li and Shen (1997), obtained sufficient conditions for the existence of periodic solutions of the equation

$$x''(t) + Cx'(t) + g(t, x(t)) = e(t). \tag{2.2}$$

Moreover, Wang (1999) investigated the same problem for the following equation

$$x''(t) - x(t) \pm x^m(t - r) = f(t), \tag{2.3}$$

by Schauder's fixed point theorem.

The first major work on existence of positive periodic solutions of delay differential equations of the second order was carried out by Liu and Ge (2003). In that work, Liu and Ge obtained sufficient conditions for the existence of positive periodic solutions of the equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = \lambda f(t, x(t - \tau(t))) + r(t), \tag{2.4}$$

by employing a fixed point theorem in cones. The existence of positive periodic solutions for second order neutral delay differential equations of the form of (1.1) has not been investigated till now.

Equations of form similar to Equation (1.2) have gained the attention of many researchers recently. For instance, Burton and Furumochi (2001a) proved that the zero solution of

$$x'(t) = -a(t)x(t) + b(t)x(t - g(t)), \tag{2.5}$$

is asymptotically stable. Raffoul (2003) proved that the neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t))\big). \tag{2.6}$$

has periodic solutions. Raffoul (2004b) also obtained sufficient conditions for the zero solution of (2.6) to be asymptotically stable. Moreover, Djoudi and Khemis (2006) proved that the zero solution of the neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x(t - g(t))x'(t - g(t)) + b(t)x^2(t - g(t)), \tag{2.7}$$

is asymptotically stable. Each of the equations (2.5)- (2.7) contains the term $-a(t)x(t)$ which is linear in $x(t)$. When this term is replaced with the highly nonlinear term $-a(t)h(x(t))$ we obtain (1.2). There are no corresponding results on stability and periodicity for (1.2) since the existing results for the equations (2.5)- (2.7) do not hold for (1.2).

Burton (2002) proved that the zero solution of the equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - \tau(t)), \tag{2.8}$$

which is a special form of (1.3) is asymptotically stable. Deham and Djoudi (2008) also proved that the solutions of the equation

$$x'(t) = -a(t)x^3(t) + G(t, x^3(t - \tau(t))), \tag{2.9}$$

are periodic. Deham and Djoudi (2010) also proved that the solutions of the neutral equation

$$x'(t) = -a(t)x^3(t) + c(t)x'(t - \tau(t)) + G(t, x^3(t - \tau(t))), \tag{2.10}$$

are periodic. Results on the positive periodic solutions of equations of the form of (1.3) are not available.

Without question, the study of periodic systems in general and Floquet theory in particular have been central to the differential equations theorist for some time. Chicone (1999), Freedman (1971), Johnson (1980), Pandiyan and Sinha (1994), and Papanicolaou and Kravvavitis (1998) have extensively explored these topics for ordinary differential equations. Floquet theory is a branch of the theory of ordinary differential equations relating to the class of solutions to linear differential equations of the form $\dot{x}(t) = A(t)x(t)$, with $A(t)$ a piecewise continuous periodic function with period T. Not surprisingly, Floquet theory has wide ranging effects, including extensions from time varying linear systems to time varying nonlinear systems of differential equations of the form $x'(t) = f(t, x(t))$, where $f(t, x)$ is smooth and $\omega-$ periodic in t. The paper by Shi (1993) ensures the global existence of solutions and proves that this system is topologically equivalent to an autonomous system $y'(t) = g(y(t))$

via an $\omega-$periodic transformation of variables. The theory has also been extended by Weikard (2000) to nonautonomous linear systems of the form $z' = A(x)z$, where $A : \mathbb{C} \to \mathbb{C}^{n\times n}$ is an $\omega-$ periodic function in the complex variable x, whose solutions are meromorphic.

Recently, Raffoul and Yankson (2010) obtained sufficient conditions under which the scalar version of (1.4) has positive periodic solutions.

Raffoul (2006) also considered the equation

$$\Delta x(n) = a(n)x(n-\tau). \tag{2.11}$$

In particular, sufficient conditions for the zero solution to be asymptotically stable were obtained. Periodicity of solutions was also proved. The delay τ in (2.11) is constant thus Equation (1.5) is a generalized form of (2.11) with variable delays.

Differential and difference equations which have the property that every constant function is a solution and and every solution approaches a constant was first introduced by Cooke and Yorke (1973). In that paper they presented three models that described the growth of a population. They used Lyapunov functionals to arrive at their results. Recently, Raffoul (2011) showed that the nonlinear difference equations of the form $\Delta x(t) = g(x(t)) - g(x(t-L))$ converges to a pre-determined constant.

In the past 10 years, there has been interest in obtaining results for equations on time scales in which the general "delta" derivative x^Δ appears. For instance, Adivar and Raffoul (2009) obtained by means of fixed point theory sufficient conditions for the existence of periodic solutions of the totally nonlinear dynamic equation

$$x^\Delta(t) = -a(t)h(x(t)) + G(t, x(\delta(t))), \;\; t \in \mathbb{T}. \tag{2.12}$$

Equation (1.7) is a generalized neutral version of (2.12). Several monographs and survey papers contain detailed treatment of these types of equations, however, they do not discuss, in detail, the equally (or more) important case of equations that feature the delta integral, rather than the delta derivative. This is possibly due to the basic theory of integral equations on time scales lagging behind that of delta derivative equations on time scales. It is difficult to find any recent papers on the subject, except Wong and Soh (2005), Wong and Boey (2004) and Kulik and Tisdell (2008) where the theory of Fredholm-type and Volterra-type equations on time scales are discussed. None of the above mentioned papers discusses neutral Volterra equations on Time scales of the form of (1.8).

2.2 Delay Differential Equations

2.2.1 Introduction

When modeling a system using a differential equation where the fundamental assumption is that the time rate at time t, given as $x'(t)$, depends only on the current status at time t, given as $f(t,x(t))$ results in the differential equation

$$x'(t) = f(t,x(t)),\ x(t_0) = x_0,\ t \geq t_0,\ x(t) \in \mathbb{R}^n. \tag{2.13}$$

Moreover, the initial condition is given in the form $x(t_0) = x_0$. In applications, this assumption and the initial condition should be improved so we can model the situations more accurately and therefore derive better results.

One improvement of (2.13) is to assume that the time rate depends not only on the current status, but also on the status in the past; that is, the past history will contribute to the future development, or, there is a time-delay effect. For example, for a university, its current population will affect its population growth, however, its population in the past may also affect its population growth. In fact, in his study of predator-prey models, Volterra (1928) had investigated the equation

$$\begin{aligned} x'(t) &= x(t)[a - by(t) - \int_{-r}^{0} F_1(s)y(t+s)ds], \\ y'(t) &= y(t)[-a + cx(t) + \int_{-r}^{0} F_2(s)x(t+s)ds], \end{aligned} \tag{2.14}$$

where x and y are the number of preys and predators, respectively, and all constants and functions are nonnegative and r is a positive constant. In $\int_{-r}^{0} F_1(s)y(t+s)ds$, the variable s varies in the interval $[-r,0]$, thus $y(t+s)$ is a function defined on the interval $[t-r,t]$. This says that for equation (2.14), the time rate at t, $[x'(t),y'(t)]^T$, depends not only on the status of $x(t)$ and $y(t)$ at t, but also on the past status of $x(t+s)$ and $y(t+s)$ defined on the interval $[t-r,t]$. That is , the history on the interval $[t-r,t]$ will affect the growth rates of the preys and predators at time t.

Other physical procedures that possess such time-delay properties include blood moving through arteries, relaxation of materials with memory from bending and signals traveling through mediums. Differential equations incorporating delay effect, or using information from the past, are called delay differential equations. They include finite delay differential equations and infinite delay differential equations.

Consider the delay differential equation below with $x \in \mathbb{R}^n$,

$$x'(t) = f(t,x(t),x(t-\tau)),\ \tau > 0, \tag{2.15}$$

with

$$x(t) = \phi_0(t),\ \ t_0 - \tau \le t \le t_0. \tag{2.16}$$

Here $\phi_0 : \mathbb{R} \to \mathbb{R}^n$ is a known function, usually taken to be continuous. $\phi_0(t)$ is called the initial function for (2.15), t_0 the initial instant and $[t_0-\tau,t_0]$ the initial set.

Definition 2.2.1.1 A function $x : [t_0-\tau,t_0+T] \to \mathbb{R}^n$, where $T > 0$ is a constant, is said to be a solution of (2.15) and (2.16) on $[t_0-\tau,t_0+T]$ if $x(t_0) = \phi_0$, $x(t)$ is differentiable on $[t_0,t_0+T]$, and satisfies (2.15) for $t \in [t_0,t_0+T]$.

Definition 2.2.1.2 A function $f(t,x)$ on a domain $D \subset \mathbb{R}\times\mathbb{R}^n$ is said to satisfy a local Lipschitz condition with respect to x on D if for any $(t_1,x_1) \in D$, there exists a domain D_1 such that $(t_1,x_1) \in D_1 \subset D$ and that $f(t,x)$ satisfies a Lipschitz condition with respect to x on D_1. That is, there exists a positive constant k_1 such that

$$\mid f(t,x) - f(t,y) \mid \le k_1 \mid x-y \mid \ \text{ for } (t,x),\ (t,y) \in D_1.$$

Theorem 2.2.1.3[Driver (1977)] If $f(t,x,y)$ is continuous with respect to t and y and locally Lipschitz with respect to x in some neighbourhood of $(t_0,\phi_0(t_0))$ and ϕ_0 is continuous with respect to t in some neighbourhood of t_0, then there exists a unique solution to (2.15) – (2.16) in a neighbourhood of $(t_0,\phi_0(t_0))$.

2.2.2 Systems with Bounded Delays - General Framework

In section 2.2.1 we considered the existence and uniqueness of solutions for some delay differential equations of specific forms. To consider arbitrary delay differential equations, we need a more general mathematical framework in which to work. This is the subject of the current section.

For $\tau > 0$, let $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n)$ be the space of continuous functions mapping $[-\tau, 0]$ into $\mathbb{R}^n$. Let $\phi \in \mathcal{C}$. We will take the norm on this space to be $\| \phi \|_\tau = \sup_{-\tau \le \theta \le 0} \| \phi(\theta) \|$, where $\| . \|$ is the usual Euclidean norm on $\mathbb{R}^n$. With this norm, $\mathcal{C}$ is a Banach space. Further, for $D \subseteq \mathbb{R}^n$ let $\mathcal{C}_D = C([-\tau, 0], D)$ be the set of continuous functions mapping $[-\tau, 0]$ into D.

Definition 2.2.2.1 If x is a function defined at least on $[t-\tau, t] \to \mathbb{R}^n$ then we define a new function $x_t : [-\tau, 0] \to \mathbb{R}^n$ by

$$x_t(\theta) = x(t+\theta), \;\; -\tau \le \theta \le 0. \tag{2.17}$$

Clearly, if x is continuous on $[t-\tau, t]$, then x_t is continuous on $[-\tau, 0]$. In the following, unless otherwise stated, we will take $J \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}^n$ to be open sets.

Definition 2.2.2.2 If $F : J \times \mathcal{C}_D \to \mathbb{R}^n$ is a given functional, we call the relation

$$x'(t) = F(t, x_t) \tag{2.18}$$

a delay differential equation on $J \times \mathcal{C}_D$.

It must be noted that equation (2.18) includes the following.

(a) Ordinary differential equations (if $\tau = 0$): $x'(t) = F(t, x(t))$.

(b) Differential equations with constant delays:

$$\begin{aligned} x'(t) &= f(t, x(t-\tau_1), \ldots, x(t-\tau_m)) \\ &= f(t, x_t(-\tau_1), \ldots, x_t(-\tau_m)) \\ &= F(t, x_t). \end{aligned}$$

Here $\tau_j \ge 0$ is constant and $\tau = \max_{1 \le j \le m} \tau_j$.

(c) Differential equations with bounded, variable delays:

$$\begin{aligned} x'(t) &= f(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))) \\ &= f(t, x(-\tau_1(t)), \ldots, x(-\tau_m(t))) \\ &= F(t, x_t). \end{aligned}$$

Here $0 \leq \tau_j \leq \tau,\ j = 1, \ldots, m, t \in J$.

(d) Differential equations with a distribution of delays:

$$\begin{aligned} x'(t) &= \int_{-\tau}^{0} f(t, \theta, x(t+\theta))d\theta \\ &= \int_{-\tau}^{0} f(t, \theta, x_t(\theta))d\theta \\ &= F(t, x_t). \end{aligned}$$

We now give a more precise definition of a solution of a delay differential equation.

Definition 2.2.2.3 Let $F : J \times \mathcal{C}_D \to \mathbb{R}^n$. A function $x(t)$ is said to be a solution of (2.18) on $[t_0 - \tau, \beta)$ if there are $t_0 \in \mathbb{R}$ and $\beta > t_0$ such that

(i) $x \in C([t_0 - \tau, \beta), D)$

(ii) $[t_0, \beta) \subset J$

(iii) $x(t)$ satisfies (2.18) for $t \in [t_0, \beta)$.

For a given $t_0 \in \mathbb{R}$ and $\phi_0 \in \mathcal{C}_D$, the initial value problem associated with the delay differential equation (2.18) is

$$\begin{cases} x'(t) = F(t, x_t),\ t \geq t_0 \\ x_{t_0} = \phi_0 \end{cases} \tag{2.19}$$

or

$$\begin{cases} x'(t) = F(t, x_t),\ t \geq t_0 \\ x_{t_0} = \phi_0(t - t_0),\ \ t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.20}$$

The following lemmas will be useful when discussing the properties of solutions.

Lemma 2.2.2.4 If x is continuous on $[t_0-\tau, t_0+\gamma]$ then x_t is a continuous function of t for $t \in [t_0, t_0+\gamma]$.

Proof. Since x is continuous on $[t_0-\tau, t_0+\gamma]$ it is uniformly continuous. Thus for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x(t)-x(s)\| < \varepsilon$ if $s,t \in [t_0-\tau, t_0+\gamma]$ and $|t-s| < \delta$. Consequently, for $s,t \in [t_0-\tau, t_0+\gamma]$ with $|t-s| < \delta$, we have $\|x(t+\theta)-x(s+\theta)\| < \varepsilon$ for all $\theta \in [-\tau, 0]$.

Lemma 2.2.2.5[Driver (1977)] Let $F : J \times \mathcal{C}_D \to \mathbb{R}^n$ be continuous and let $t_0 \in J$ and $\phi_0 \in \mathcal{C}_D$ be given. Then x is a solution of the initial value problem (2.20) on $[t_0-\tau, \beta)$ if and only if $[t_0, \beta) \subset J, x \in C([t_0-\tau, \beta), D)$ and x satisfies

$$\begin{cases} x_{t_0} = \phi_0 \\ x(t) = \phi_0(0) + \int_{t_0}^{t} F(s, x_s)ds, \; t_0 \le t \le \beta. \end{cases} \tag{2.21}$$

Definition 2.2.2.6 Let $F : J \times \mathcal{C}_D \to \mathbb{R}^n$ and let $U \subset J \times \mathcal{C}_D$. We say that F is Lipschitz on U if there exists $K \ge 0$ such that

$$\|F(t,\varphi) - F(t,\psi)\| \le K\|\varphi - \psi\|_\tau,$$

whenever (t,φ) and $(t,\psi) \in U$.

Lemma 2.2.2.7(Generalized Gronwall's inequality) Let c and k be given nonnegative continuous functions on an interval $J = [t_0, \beta)$ and let c be differentiable on J. Then if $v : J \to [0,\infty)$ is continuous and

$$v(t) \le c(t) + \int_{t_0}^{t} k(s)v(s)ds$$

then

$$v(t) \le c(t_0)e^{\int_{t_0}^{t} k(s)ds} + \int_{t_0}^{t} c'(s)e^{\int_{s}^{t} k(u)du}ds.$$

Proof. Let $R(t) = \int_{t_0}^{t} k(s)v(s)ds$. Then

$$R'(t) = k(t)v(t) \le k(t)c(t) + k(t)\int_{t_0}^{t} k(s)v(s)ds.$$

Thus $R'(t) - k(t)R(t) \le k(t)c(t)$. Multiplying through by the integrating factor $e^{-\int_{t_0}^{t} k(s)ds}$ yields

$$\left[e^{-\int_{t_0}^{t} k(s)ds} R(t)\right]' \le k(t)c(t)e^{-\int_{t_0}^{t} k(s)ds}.$$

Integrating from t_0 to t gives

$$e^{-\int_{t_0}^{t} k(s)ds} R(t) - R(t_0) \le \int_{t_0}^{t} k(s)c(s)e^{-\int_{t_0}^{s} k(u)du} ds.$$

Noting that $R(t_0) = 0$ and integrating by parts on the right hand side gives

$$e^{-\int_{t_0}^{t} k(s)ds} R(t) \le c(t_0) - c(t)e^{-\int_{t_0}^{t} k(u)du} + \int_{t_0}^{t} c'(s)e^{-\int_{t_0}^{s} k(u)du} ds.$$

Thus

$$R(t) \le -c(t) + c(t_0)e^{\int_{t_0}^{t} k(s)ds} + \int_{t_0}^{t} c'(s)e^{\int_{s}^{t} k(u)du} ds.$$

Using $v(t) \le c(t) + R(t)$, we obtain the result.

Lemma 2.2.2.8[Reid's Lemma, Driver (1977)] Let C be a given constant and k a given nonnegative continuous function on an interval J. Let $t_0 \in J$. Then if $v : J \to [0, \infty)$ is continuous and

$$v(t) \le C + \left| \int_{t_0}^{t} k(s)v(s)ds \right| \tag{2.22}$$

for all $t \in J$, it follows that

$$v(t) \le Ce^{\left| \int_{t_0}^{t} k(s)ds \right|}$$

for all $t \in J$.

Proof. Suppose $t \ge t_0$ and $t \in J$. Then (2.22) becomes

$$v(t) \le C + \int_{t_0}^{t} k(s)v(s)ds,$$

or

$$k(t)v(t) - k(t)[C + \int_{t_0}^{t} k(s)v(s)ds] \le 0.$$

Let $Q(t) = C + \int_{t_0}^{t} k(s)v(s)ds$, then $Q'(t) - k(t)Q(t) \leq 0$. Multiplying through by the integrating factor $e^{-\int_{t_0}^{t} k(s)ds}$ we obtain

$$\frac{d}{dt}\left[Q(t)e^{-\int_{t_0}^{t} k(s)ds}\right] \leq 0.$$

Integrating from t_0 to t and noting that $Q(t_0) = C$, yields

$$Q(t)e^{-\int_{t_0}^{t} k(s)ds} - C \leq 0$$

or

$$Q(t) \leq Ce^{\int_{t_0}^{t} k(s)ds}.$$

Substituting this into (2.22) yields

$$v(t) \leq Q(t) \leq Ce^{\int_{t_0}^{t} k(s)ds}.$$

The proof for $t < t_0$ is similar.

Theorem 2.2.2.9(Uniqueness) Let $F : [t_0, \alpha) \times \mathcal{C}_D \to \mathbb{R}^n$ be continuous and locally Lipschitz on its domain. Then, given any $\phi_0 \in \mathcal{C}_D$ and $\beta \in (t_0, \alpha]$, there is at most one solution of the initial value problem (2.20) on $[t_0 - \tau, \beta)$.

Proof. Suppose (for contradiction) that for some $\beta \in (t_0, \alpha]$ there are two solutions $x(t)$ and $y(t)$ mapping $[t_0 - \tau, \beta)$ into D with $x \neq y$. Let $t_1 = \inf\{t \in (t_0, \beta) \; : x(t) \neq y(t)\}$. Then $t_0 < t_1 < \beta$ and $x(t) = y(t)$ for $t_0 - \tau \leq t \leq t_1$. Since $(t_1, x_1) \in [t_0, \beta) \times \mathcal{C}_D$ and F is locally Lipschitz, there exist numbers $a > 0$ and $b > 0$ such that the set $U = [t_1, t_1 + a] \times \{\psi \in \mathcal{C} : \|\psi - x_{t_1}\|_r \leq b\}$ is contained in $[t_0, \alpha) \times \mathcal{C}_D$ and F is Lipschitz on U (with Lipschitz constant K). By Lemma 2.2.2.4 there exists $\delta \in (0, a]$ such that $(t, x_t) \in U$ and $(t, y_t) \in U$ for $t_1 \leq t < t_1 + \delta$. Thus for $t_1 \leq t < t_1 + \delta$,

$$\begin{aligned} \|x - y\| &= \left\| \int_{t_0}^{t} [F(s, x_s) - F(s, y_s)]ds \right\| \\ &\leq \int_{t_1}^{t} K\|x_s - y_s\|_\tau ds. \end{aligned}$$

Now since the right hand side is an increasing function of t and since $\|x(t) - y(t)\| = 0$ for $t_1 - \tau \leq t \leq t_1$,

$$\|x_t - y_t\|_\tau \leq \int_{t_1}^{t} K\|x_s - y_s\|_\tau ds$$

for $t_1 \leq t < t_1 + \delta$. From this and the Generalized Gronwall's Lemma it follows that $x(t) = y(t)$ on $[t_1, t_1 + \delta)$ contradicting the definition of t_1.

Theorem 2.2.2.10(Local Existence) Let $F : [t_0, \alpha) \times \mathcal{C}_D \to \mathbb{R}^n$ be continuous and locally Liptschitz. Then, for each $\phi_0 \in \mathcal{C}_D$, the initial value problem (2.20) has a unique solution on $[t_0 - \tau, t_0 + \Delta)$ for some $\Delta > 0$.

Proof. Choose any $a > 0$ and $b > 0$ sufficiently small so that

$$U = [t_0, t_0 + a] \times \{\psi \in \mathcal{C} : \|\psi - \phi_0\|_\tau \leq b\}$$

is a subset of $[t_0, \alpha) \times \mathcal{C}_D$ and F is Lipschitz on U, with Lipschitz constant K. Define a continuous function $\tilde{\chi}$ on $[t_0 - \tau, t_0 + a] \to \mathbb{R}^n$ by

$$\tilde{\chi} = \begin{cases} \phi_0(t - t_0), t_0 - \tau \leq t \leq t_0 \\ \\ \phi_0(0), t_0 < t \leq t_0 + a. \end{cases}$$

Then $F(t, \tilde{\chi}_t)$ depends continuously on t, and hence $\|F(t, \tilde{\chi}_t)\| \leq B_1$ on $[t_0, t_0 + a]$ for some constant B_1. Define $B = Kb + B_1$. Choose $a_1 \in (0, a]$ such that $\|\tilde{\chi}_t - \phi_0\|_\tau = \{\tilde{\chi}_t - \tilde{\chi}_{t_0}\|_\tau \leq b$ for $t_0 \leq t \leq t_0 + a_1$. Choose $\Delta > 0$ such that $\Delta \leq \min\{a_1, b/B\}$. Let S be the set of all continuous functions $\chi : [t_0 - \tau, t_0 + \Delta] \to \mathbb{R}^n$ such that $\chi(t) = \phi_0(t - t_0)$ for $t_0 - \tau \leq t \leq t_0$ and $\|\chi(t) - \phi_0(0)\| \leq b$ for $t_0 \leq t \leq t_0 + \Delta$. Note that if $\chi \in S$ and $t \in [t_0, t_0 + \Delta]$, then $\|\chi_t - \tilde{\chi}_t\|_\tau \leq b$ so that

$$\begin{aligned} \|F(t, \chi_t)\| &\leq \|F(t, \chi_t) - F(t, \tilde{\chi}_t)\| + \|F(t, \tilde{\chi}_t)\| \\ &\leq K\|\chi_t - \tilde{\chi}_t\|_\tau + B_1 \\ &\leq B. \end{aligned}$$

For each $\chi \in S$ define a function $T\chi$ on $[t_0 - \tau, t_0 + \Delta]$ by

$$(T\chi)(t) = \begin{cases} \phi_0(t - t_0), t_0 - \tau \leq t \leq t_0 \\ \\ \phi_0(0) + \int_{t_0}^{t} F(s, \chi_s) ds, t_0 \leq t \leq t_0 + \Delta. \end{cases}$$

Then $T\chi$ is continuous and, since $\|F(s, \chi_s)\| < B$, $|(T\chi)(t) - \phi_0(0)| \leq B\Delta \leq b$ for $t_0 \leq t \leq t_0 + \Delta$. Thus $T\chi \in S$, that is, $T : S \to S$. Choose $x_{(0)} \in S$ and construct the successive approximations $x_{(1)} = Tx_{(0)}, x_{(2)} = Tx_{(1)}, \ldots$. Note that for each l, $x_{(l)}(t) = \phi_0(t - t_0)$ on $[t_0 - \tau, t_0]$. We will now prove that the sequence $x_l(t)$ converges. For each $l = 0, 1, 2, \ldots$ when $t_0 \le t \le t_0 + \Delta$

$$\begin{aligned}\|x_{(l+2)}(t) - x_{(l+1)}(t)\| &= \int_{t_0}^{t} [F(s, x_{(l+1)s}) - F(s, x_{(l)s})] ds\| \\ &\le \int_{t_0}^{t} K\|x_{(l+1)s} - x_{(l)s}\|_\tau ds.\end{aligned}$$

Note that $\|x_{(1)}(t) - x_{(0)}(t)\| \le 2b$ on $[t_0 - \tau, t_0 + \Delta]$. Thus $\|x_{(1)t} - x_{(0)t}\|_\tau \le 2b$ on $[t_0, t_0 + \Delta]$ and

$$\begin{aligned}\|x_{(2)}(t) - x_{(1)}(t)\| &\le \int_{t_0}^{t} K\|x_{(1)s} - x_{(0)s}\| ds \\ &\le 2bK(t - t_0)\end{aligned}$$

on $[t_0, t_0 + \Delta]$, which further implies that $\|x_{(2)t} - x_{(1)t}\|_\tau \le 2bK(t - t_0)$ on $[t_0, t_0 + \Delta]$. This leads to

$$\begin{aligned}\|x_{(3)}(t) - x_{(2)}(t)\| &\le \int_{t_0}^{t} K\|x_{(2)s} - x_{(1)s}\| ds \\ &\le 2b\frac{K^2(t - t_0)^2}{2}.\end{aligned}$$

Using induction it can be shown that

$$\|x_{(l+1)}(t) - x_{(t)}\| \le 2b\frac{K^l(t - t_0)^l}{l!}$$

on $[t_0, t_0 + \Delta]$. This together with $x_{(l+1)}(t) = x_{(l)}(t)$ on $[t_0 - \tau, t_0]$ gives

$$\|x_{(l+1)}(t) - x_{(t)}\| \le 2b\frac{K^l\Delta^l}{l!}$$

on $[t_0 - \tau, t_0 + \Delta]$. Now the series

$$x_{(0)}(t) + \sum_{p=0}^{\infty} [x_{(p+1)}(t) - x_{(p)}(t)]$$

converges uniformly on $[t_0 - \tau, t_0 + \Delta]$ by the Weierstrass $M-$Test, but

$$x_{(l)}(t) = x_{(0)}(t) + \sum_{p=0}^{l-1} [x_{(p+1)}(t) - x_{(p)}(t)],$$

and so the sequence $\{x_{(l)}(t)\}$ converges uniformly on $[t_0-\tau, t_0+\Delta]$. Let $x(t) = \lim_{l\to\infty} x_{(l)}(t)$ for $t_0 - \tau \le t \le t_0\Delta$. Clearly, $x(t)$ is continuous on $[t_0-\tau, t_0+\Delta]$ and $x_{t_0} = \phi_0$. Further

$$\|x(t) - x_{(l)}(t)\| \le 2b \sum_{p=l}^{\infty} \frac{(K\Delta)^p}{p!}$$

for $t_0 - \tau \le t \le t_0\Delta$ and $\|x_t - x_{(l)t}\|_\tau \le 2b\sum_{p=l}^{\infty} \frac{(K\Delta)^p}{p!}$ for $t_0 \le t \le t_0\Delta$. Thus, for $t_0 \le t \le t_0\Delta$,

$$\begin{aligned}\|x(t) - \phi_0(0)\| &\le \|x(t) - x_{(l)}(t)\| + \|x_{(l)}(t) - \phi_0(0)\| \\ &\le 2b \sum_{p=l}^{\infty} \frac{(K\Delta)^p}{p!} + b \\ &\le b,\end{aligned}$$

and $x_t \in \mathcal{C}_D$. Finally for $t \in [t_0, t_0+\Delta]$

$$\begin{aligned}\|x(t) - \phi_0(0) - \int_{t_0}^{t} F(s, x_s)ds\| &\le \|x(t) - x_{(l)}(t)\| + \int_{t_0}^{t} \|F(s, x_{(l-1)s}) - F(s, x_s)\|ds \\ &\le 2b \sum_{p=l}^{\infty} \frac{(K\Delta)^p}{p!} + K\Delta 2b \sum_{p=l-1}^{\infty} \frac{(K\Delta)^p}{p!}.\end{aligned}$$

Taking the limit as $l \to \infty$ of this inequality then gives

$$\|x(t) - \phi_0(0) - \int_{t_0}^{t} F(s, x_s)ds\| = 0,$$

that is, $x(t)$ satisfies (2.21). Uniqueness follows from Theorem 2.2.2.1.

Theorem 2.2.2.11(Continuous Dependence on Initial Conditions) Let $F : [t_0, \alpha] \times \mathcal{C}_D \to \mathbb{R}^n$ be continuous and globally Lipschitz constant K. Let $\phi_0 \in \mathcal{C}_D$ and $\tilde{\phi_0} \in \mathcal{C}_D$ be given and let x and $\tilde{x}$ be unique solutions of (2.18) with $x_{t_0} = \phi_0$ and $\tilde{x}_{t_0}$, respectively. If x and $\tilde{x}$ are both valid on $[t_0 - \tau, \beta)$, then

$$\|x(t) - \tilde{x(t)}\| \le \|\phi_0 - \tilde{\phi_0}\|_\tau e^{K(t-t_0)}$$

for $t_0 \le t < \beta$.

Proof. Since x and $\tilde{x}$ are solutions of the given initial value problems, x satisfies (2.21) and $\tilde{x}$ satisfies

$$\begin{cases} \tilde{x}_{t_0} = \tilde{\phi}_0 \\ \tilde{x}(t) = \tilde{\phi}_0(0) + \int_{t_0}^{t} F(s, \tilde{x}_s)ds, t_0 \le t < \beta. \end{cases}$$

Thus for $t_0 \leq t < \beta$

$$\begin{aligned}
\|x(t) - \tilde{x}(t)\| &= \|\phi_0(0) - \tilde{\phi}_0(0) + \int_{t_0}^{t} [F(s,x_s) - F(s,\tilde{x}_s)]ds\| \\
&\leq \|\phi_0(0) - \tilde{\phi}_0(0)\| + \int_{t_0}^{t} \|[F(s,x_s) - F(s,\tilde{x}_s)]\|ds \\
&\leq \|\phi_0 - \tilde{\phi}_0\|_\tau + \int_{t_0}^{t} K\|x_s - \tilde{x}_s)\|_\tau ds, \text{ for } t_0 \leq t \leq \beta.
\end{aligned}$$

Since $\|x(t) - \tilde{x}(t)\| \leq \|\phi_0 - \tilde{\phi}_0\|_\tau$ on $[t_0 - \tau, t_0]$, it follows that

$$\|x_t - \tilde{x}_t\| \leq \|\phi_0 - \tilde{\phi}_0\|_\tau + \int_{t_0}^{t} K\|x_s - \tilde{x}_s)\|_\tau ds \text{ for } t_0 \leq t \leq \beta.$$

Applying the generalized Gronwall's Lemma with $C = \|\phi_0 - \tilde{\phi}_0\|_\tau$ and $k(s) = K$ yields

$$\begin{aligned}
\|x(t) - \tilde{x}(t)\| &\leq \|x_t - \tilde{x}_t\|_\tau \\
&\leq \|\phi_0 - \tilde{\phi}_0\|_\tau e^{K(t-t_0)}, \text{ for } t_0 \leq t \leq \beta.
\end{aligned}$$

Theorem 2.2.2.12(Continuous Dependence on F) Let $F, \tilde{F} : [t_0, \alpha) \times C_D \to \mathbb{R}^n$ be continuous, and let F be globally Lipschitz with Lipschitz constant K. Given $\phi_0, \tilde{\phi}_0 \in C_D$, let $x(t)$ and $\tilde{x}(t)$ be the unique solutions of (2.20) and

$$\begin{cases}
\tilde{x}'(t) = \tilde{F}(t, \tilde{x}_t), t \geq t_0 \\
\tilde{x}(t) = \tilde{\phi}_0(t - t_0), t_0 - \tau \leq t \leq t_0,
\end{cases} \tag{2.23}$$

respectively. If x and $\tilde{x}$ are both valid on $[t_0 - \tau, \beta)$ and $\|F(t,\psi) - \tilde{F}(t,\psi)\| \leq \mu$ for all $t \in [t_0, \alpha)$, $\psi \in C_D$ then

$$\|x(t) - \tilde{x}(t)\| \leq \|\phi_0 - \tilde{\phi}_0\|_\tau e^{K(t-t_0)} + \frac{\mu}{K}[e^{K(t-t_0)} - 1], \text{ for } t_0 \leq t < \beta.$$

Proof. $x(t)$ and $\tilde{x}(t)$ must satisfy the integral equations (2.21) and

$$\begin{cases}
\tilde{x}_{t_0} = \tilde{\phi}_0 \\
\tilde{x}(t) = \tilde{\phi}_0(0) + \int_{t_0}^{t} \tilde{F}(s, \tilde{x}_s)ds, t_0 \leq t < \beta.
\end{cases}$$

Thus on $[t_0 - \tau, t_0]$

$$\begin{aligned}
\|x(t) - \tilde{x}(t)\| &= \|\phi_0(t - t_0) - \tilde{\phi}_0(t - t_0)\| \\
&\leq \|\phi_0 - \tilde{\phi}_0\|_\tau
\end{aligned}$$

on $[t_0,\beta)$

$$\begin{aligned}\|x(t)-\tilde{x}(t)\| &\leq \|\phi_0(0)-\tilde{\phi}_0(0)\|+\int_{t_0}^{t}\|F(s,x_s)-\tilde{F}(s,\tilde{x}_s)\|ds\\ &\leq \|\phi_0-\tilde{\phi}_0\|_\tau+\int_{t_0}^{t}\|F(s,x_s)-F(s,\tilde{x}_s)\|ds\\ &\quad+\int_{t_0}^{t}\|F(s,\tilde{x}_s)-\tilde{F}(s,\tilde{x}_s)\|ds\\ &\leq \|\phi_0-\tilde{\phi}_0\|_\tau+\int_{t_0}^{t}K\|x_s-\tilde{x}_s\|_\tau ds+\int_{t_0}^{t}\mu ds.\end{aligned}$$

Since the right hand side of this last inequality is an increasing function of t, it follows that

$$\|x-\tilde{x}\|_\tau \leq \|\phi_0-\tilde{\phi}_0\|_\tau+\mu(t-t_0)+\int_{t_0}^{t}K\|x_s-\tilde{x}_s\|ds, \text{ for } t_0\leq t<\beta.$$

Applying the generalized Gronwall's Lemma with $c(t)=\|\phi_0-\tilde{\phi}_0\|_\tau+\mu(t-t_0)$ and $k(t)=K$ yields the result.

For $y\in C^{n-1}([-\tau,0],\mathbb{R})$ we may define the function y_t' on $[-\tau,0]$ as follows

$$\begin{aligned}y_t'(\theta) &= y'(t+\theta)\\ &= \lim_{h\to0^+}\frac{y(t+\theta+h)-y(t+\theta)}{h}\\ &= \lim_{h\to0^+}\frac{y_t(\theta+h)-y_t(\theta)}{h} \quad -\tau\leq\theta\leq0.\end{aligned}$$

We may define the functions $y_t'',y_t''',\ldots,y_t^{(n-1)}$ on $[-\tau,0]$ in a similar manner. Then for $J\subset\mathbb{R}$ and $G:J\times[C([-\tau,0],\mathbb{R}]^n$ we can consider the scalar nth order delay differential equation

$$y^{(n)}(t)=G(t,y_t,y_t',y_t'',\ldots,y_t^{(n-1)}) \tag{2.24}$$

with initial conditions

$$\begin{cases} y_{t_0}=\phi_0\\ y_{t_0}'=\phi_1\\ \vdots\\ y_{t_0}^{(n-1)}=\phi_{n-1}\end{cases} \tag{2.25}$$

where $t_0 \in J$ and $\phi_j \in C([-\tau, 0], \mathbb{R})$. Solutions of (2.24) will be $(n-1)$ times differentiable functions. The initial value problem (2.24)-(2.25) can be reduced to a delay differential equation on $J \times \mathcal{C}$ in the usual way, that is, by defining $x \in \mathbb{R}^n$

$$x = [y, y', y'', ..., y^{(n-1)}]^T.$$

The theorems in the previous section may then be applied to this initial value problem.

2.3 Fixed Point Theorems

This section contains an elementary set of definitions and theorems relevant to our study. Every qualitative property we discuss is formulated in a complete metric space. Let it be noted that in this thesis there is usually a Banach space $(\mathbb{B}, ||.||)$ in the background. A subset S of $\mathbb{B}$ is selected and $(S, ||.||)$ is the complete metric space in which we work, where the metric on S is defined by the norm inherited from the Banach space. Thus the notation almost always suggests a norm $||.||$ instead of a metric ρ. Even if the zero function, say θ, is not in S, then for $\phi \in S$, $||\phi||$ is interpreted as $\rho(\phi, \theta) = ||\phi - \theta||$.

Definition 2.3.1 A pair (S, ρ) is a metric space if S is a set and $\rho : S \times S \to [0, \infty)$ such that when y, z, and u are in S then

(a) $\rho(y, z) \geq 0$, $\rho(y, y) = 0$ and $\rho(y, z) = 0$ implies $y = z$,

(b) $\rho(y, z) = \rho(z, y)$, and

(c) $\rho(y, z) \leq \rho(y, u) + \rho(u, z)$.

The metric space is complete if every Cauchy sequence in (S, ρ) has a limit in that space. A sequence $\{x_n\} \subset S$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $\rho(x_n, x_m) < \varepsilon$.

If the functions in a differential or difference equation of interest all satisfy a local Lipschitz condition, then contraction mappings will usually be our first choice in our investigations. If the functions are not Lipschitz then we will turn to fixed point theorems of Krasnoselskii's type. These require compactness instead of completeness and are formulated in complete normed spaces.

Definition 2.3.2 A set L in a metric space (S,ρ) is compact if each sequence $\{x_n\} \subset L$ has a subsequence with limit in L.

Definition 2.3.3 Let U be an interval on R and let $\{f_n\}$ be a sequence of functions with $f_n : U \to \mathbb{R}^d$. Denote by $|.|$ any norm on $\mathbb{R}^d$.

(a) $\{f_n\}$ is uniformly bounded on U if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in U$.

(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in U$, and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$ for all n.

Definition 2.3.4 A subset A of X is said to be closed if for any convergent sequence $\{x_n\}_{n=1}^{\infty} \subset A$, the limit point is also in A.

Definition 2.3.5 A function $f : X \to Y$ from a metric space to a metric space is said to be Lipschitz continuous if there exists $L \in \mathbb{R}$ such that $d(f(u), f(v)) \leq Ld(u,v)$ for every $u, v \in X$. We call L a Lipschitz constant, and write $Lip(f)$ for the smallest Lipschitz constant that works.

Definition 2.3.6 A contraction is a Lipschitz continuous function from a metric space to itself that has Lipschitz constant less than one.

Definition 2.3.7 A fixed point of a function $T : X \to X$ is a point $x \in X$ such that $Tx = x$.

Here is the standard result on how one verifies equicontinuity.

Theorem 2.3.1[Burton (2006)] Let $S \subset C([a,b])$, $-\infty < a \leq b < \infty$. If each $f \in S$ is differentiable in (a,b) and there is K such that $|f'(x)| \leq K$ holds for all $f \in S$, and all $x \in (a,b)$, then S is equicontinuous.

The proof of the next result can be found in any text on real variables or in Burton (1985).

Theorem 2.3.2 (Ascoli-Arzela) If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a,b]$, then there is a subsequence which converges uniformly on $[a,b]$ to a continuous function.

If our t-intervals are infinite, then the following extension of Theorem 2.3.1 which is stated below will be used to prove compactness. This theorem is stated in Burton and Furumochi (2001b).

Theorem 2.3.3[Burton (2006)] Let $\mathbb{R}^+ = [0,\infty)$ and let $q : \mathbb{R}^+ \to R^+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k(t)\}$ is an equicontinuous sequence of $\mathbb{R}^d$-valued functions on $\mathbb{R}^+$ with $|\phi_k(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Definition 2.3.7 A vector space $(V,+,.)$ is a normed space if for each $x,y \in V$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(a) $\|x\| = 0$ if and only if $x = 0$,

(b) $\|\alpha x\| = |\alpha|\|x\|$ for each $\alpha \in \mathbb{R}$, and

(c) $\|x+y\| \leq \|x\| + \|y\|$.

A normed space is a vector space and it is a metric space with $\rho(x,y) = \|x-y\|$. But a vector space with a metric is not always a normed space.

Definition 2.3.8 A banach space is a complete normed space.

We will now state some useful fixed point theorems. We begin by stating the contraction mapping principle which generally goes under the name Banach-Caccioppoli Theorem, or Banach's (1932) Contraction Mapping Principle. A proof can be found in many places such as Smart (1980) or Burton (1985). It gains more respect every day. The real power of the result lies in its application with cleverly chosen metrics.

Theorem 2.3.4 (The Contraction Mapping Principle) Let (S,ρ) be a complete metric space and let $H : S \to S$. If there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in S$ we have

$$\rho(H\phi_1, H\phi_2) \leq \alpha\rho(\phi_1, \phi_2),$$

then there is one and only one point $\phi \in S$ with $H\phi = \phi$.

Definition 2.3.9 Let (M,d) be a metric space and $B : M \to M$. B is said to be a large contraction if for each pair $\phi, \psi \in M$ with $\phi \neq \psi$ then $d(B\phi, B\psi) < d(\phi, \psi)$ and if for each $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\phi, \psi \in M, d(\phi, \psi) \geq \varepsilon] \Rightarrow d(B\phi, B\psi) < \delta d(\phi, \psi).$$

Theorem 2.3.5[Burton (2006)] Let (M,d) be a complete metric space and B a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.

Krasnoselskii (1958) studied a paper of Schauder (1932) and obtained the following working hypothesis: The inversion of a perturbed differential operator yields the sum of a contraction and compact map. Accordingly, he formulated the following fixed point theorem which is a combination of the contraction mapping principle and Schauder's fixed point theorems.

Theorem 2.3.6[Burton (2006)] Let M be a closed convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into S such that

(a) $Ax + By \in M$ for all $x, y \in M$,

(b) A is continuous and AM is contained in a compact set,

(c) B is a contraction with constant $\alpha < 1$.

Then there is a $y \in M$ with $Ay + By = y$.

Burton (1996) studied the theorem of Krasnoselskii and observed that Theorem 2.3.6 can be more interesting in applications with certain changes and formulated the following results.

Theorem 2.3.7[Burton (2006)] Let M be a bounded convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into M such that

(a) $Ax + By \in M$ for all $x, y \in M$,

(b) A is continuous and AM is contained in a compact subset of M,

(c) B is a large contraction.

Then there exists $y \in M$ with $Ay + By = y$.

2.3.1. Formulation of Fixed Point Problems in Differential Equations

This section is an elementary introduction to the formulations of fixed point problems in differential equations. In this thesis we will be primarily interested in functional differential and difference equations, but we begin with an ordinary differential equation

$$x'(t) = g(t, x(t)), \tag{2.26}$$

where $g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Perhaps the most basic problem concerning (2.26) is to find a solution through a given point $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$ defined on some interval $[t_0, t_0 + \gamma]$ and satisfying (2.26) on that interval.

For this problem, our first guess would be that the set S should consist of differentiable functions $\phi : [t_0, t_0 + \gamma] \to \mathbb{R}^n$ with $\phi(t_0) = x_0$. Next, the simplest way to find a mapping is to formally integrate (2.26) and obtain

$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds,$$

so that the mapping P on S is defined by

$$(P\phi)(t) = x_0 + \int_{t_0}^{t} g(s, \phi(s))ds.$$

A fixed point will certainly satisfy the equation. Since our mapping is given by an integral, our second approximation to S is the set of continuous functions; differentiability will be automatic. There is now a vast array of fixed point theorems which will yield a fixed point of that mapping and satisfy our initial value problem. We will use the Contraction Mapping Principle for this problem. For our illustration here, it is easiest to complete the solution by asking that g satisfy a global Lipschitz condition of the form

$$|g(t,x)-g(t,y)| \leq K|x-y|$$

for $t \geq t_0$, $K > 0$, and for all $x, y \in \mathbb{R}^n$, where $|.|$ is any norm on $\mathbb{R}^n$. This will allow us to give a contraction mapping argument. For any fixed interval $[t_0, t_0+\gamma]$, our set S with the supremum metric is a complete metric space and $P : S \to S$. Checking our contraction requirement, we have

$$\begin{aligned} |(P\phi_1)(t)-(P\phi_2)(t)| &\leq \int_{t_0}^{t} K|\phi_1(s)-\phi_2(s)|ds \\ &\leq K\gamma\|\phi_1(t)-\phi_2(t)\|, \end{aligned}$$

so that if $\alpha := K\gamma < 1$ then P is a contraction with unique fixed point ϕ, a solution of our differential equation and it satisfies the initial condition.

So much more can be done. But everything begins with a suitable mapping; that is the central problem. It can be relatively easy to state and prove theorems, once we have a proper mapping, but the real problem is in constructing the mapping. Seldom will we see a problem in which it is so easy to find a suitable mapping as the one we just finished.

2.4 Difference Calculus

Many of the calculations involved in solving and analyzing difference equations can be simplified by use of the difference calculus, a collection of mathematical tools quite similar to the differential equations.

2.4.1. The Difference Operator

Just as the differential operator plays the central role in the differential calculus, the difference operator is the basic component of calculations involving finite differences.

Definition 2.4.1.1

Let $y(n)$ be a function of a real or complex variable n. The difference operator Δ is defined by

$$\Delta y(n) = y(n+1) - y(n).$$

For the most part, we will take the domain of y to be a set of consecutive integers such as the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$.

Occasionally we will apply the difference operator to a function of two or more variables. In this case, a subscript will be used to indicate which variable is to be shifted by one unit. Higher order differences are defined by composing the difference operator with itself. The second order difference is

$$\begin{aligned}
\Delta^2 y(n) &= \Delta(\Delta y(n)) \\
&= \Delta(y(n+1) - y(n)) \\
&= (y(n+2) - y(n+1)) - (y(n+1) - y(n)) \\
&= y(n+2) - 2y(n+1) + y(n).
\end{aligned}$$

An elementary operator that is often used in conjunction with the difference operator is the shift operator.

Definition 2.4.1.2 The shift operator E is defined by

$$Ey(n) = y(n+1).$$

The fundamental properties of Δ are given in the following theorem.

Theorem 2.4.1.3

(a) $\Delta^m(\Delta^k y(n)) = \Delta^{m+k} y(n)$ for all positive integers m and k.

(b) $\Delta(y(n)+z(n)) = \Delta y(n) + \Delta z(n)$.

(c) $\Delta(Cy(n)) = C\Delta y(n)$ if C is a constant.

(d) $\Delta(y(n)z(n)) = y(n)\Delta z(n) + Ez(n)\Delta y(n)$.

(e) $\Delta\left(\frac{y(n)}{z(n)}\right) = \frac{z(n)\Delta y(n) - y(n)\Delta z(n)}{z(n)Ez(n)}$.

2.4.2. Summation

To make effective use of the difference operator, we introduce in this section its right inverse operator, which is sometimes called the "indefinite sum."

Definition 2.4.2.1 An "indefinite sum" (or "antidifference") of $y(n)$, denoted $\sum y(n)$, is any function so that

$$\Delta\left(\sum y(n)\right) = y(n)$$

for all n in the domain of y.

Theorem 2.4.2.2 If $z(n)$ is an indefinite sum of $y(n)$, then every indefinite sum of $y(n)$ is given by

$$\sum y(n) = z(n) + C(n),$$

where $C(n)$ has the same domain as y and $\Delta C(n) = 0$.

In what follows it will be convenient to use the convention

$$\sum_{k=a}^{b} y(k) = 0$$

whenever $a > b$. Observe that for m fixed and $n \geq m$,

$$\Delta_n\left(\sum_{k=m}^{n-1} y(k)\right) = y(n),$$

and for p fixed and $p \geq n$,

$$\Delta_n\left(\sum_{k=n}^{p} y(k)\right) = -y(n).$$

The following theorem contains a useful formula for computing definite sums, which is analogous to the fundamental theorem of calculus.

Theorem 2.4.2.3 If $z(n)$ is an indefinite sum of $y(n)$, then

$$\sum_{k=m}^{n-1} y(k) = [z(k)]_m^n = z(n) - z(m).$$

2.4.3. Linear Difference Equations

Let $p(n)$ and $r(n)$ be given functions with $p(n) \neq 0$ for all n. The first order linear difference equation is

$$y(n+1) - p(n)y(n) = r(n). \tag{2.27}$$

Equation (2.27) is said to be of the first order because it involves the values of y at n and only $n+1$ only, as in the first order difference operator $\Delta y(n) = y(n+1) - y(n)$. If $p(n) = 1$ for all n, then Eq. (2.27) is simply

$$\Delta y(n) = r(n),$$

so the solution is

$$y(n) = \sum r(n) + C(n),$$

where $\Delta C(n) = 0$. For simplicity, we assume that the domain of interest is a discrete set $t = a,\, a+1, a+2, \cdots$. Consider first the equation

$$u(n+1) = p(n)u(n), \tag{2.28}$$

which is easily solved by iteration:

$$u(a+1) \;=\; p(a)u(a).$$

Thus,

$$\begin{aligned} u(a+2) &= p(a+1)u(a+1) \\ &= p(a+1)p(a)u(a), \\ &\vdots \\ u(a+n) &= u(a)\prod_{k=0}^{n-1} p(a+k). \end{aligned}$$

We can write the solution in the more convenient form

$$u(n) \quad = \quad u(a)\prod_{s=a}^{n-1} p(s),\ (n = a, a+1, \ldots),$$

where it is understood that $\prod_a^{a-1} p(s) = 1$ and for $n \geq a+1$, the product is taken over $a, a+1, \cdots, n-1$.

2.5 Basic Measure Theory

In this section we provide some basic definitions and theorems in measure theory.

2.5.1. σ-algebra and Measure

Definition 2.5.1.1 Let X be a set. A collection $\mathfrak{M}$ of subsets of X is a $\sigma-$ algebra if $\mathfrak{M}$ has the following properties:

(a) $X \in \mathfrak{M}$,

(b) $A \in \mathfrak{M} \Rightarrow X \setminus A \in \mathfrak{M}$,

(c) $A_1, A_2, A_3, ... \in \mathfrak{M} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

The pair $(X, \mathfrak{M})$ is called a measurable space and elements of $\mathfrak{M}$ are called measurable sets.

Definition 2.5.1.2 Let $(X, \mathfrak{M})$ be a measurable space. A measure (also called positive measure) is a function

$$\mu : \mathfrak{M} \to [0, \infty]$$

such that

(a) $\mu(\phi) = 0$,

(b) μ is countably additive, that is, if $A_1, A_2, A_3, ... \in \mathfrak{M}$ are pairwise disjoint, then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i).$$

The triple $(X,\mathfrak{M},\mu)$ is called a measure space.

If $\mu(X) < \infty$, then μ is called a finite measure.

The following theorem provides some elementary properties of measures.

Theorem 2.5.1.3 Let $(X,\mathfrak{M},\mu)$ be a measure space. Then

(a) If the sets $A_1, A_2, ..., A_n \in \mathfrak{M}$ are pairwise disjoint then $\mu(A_1 \cup ... \cup A_n) = \mu(A_1) + \mu(A_2) + ... + \mu(A_n)$.

(b) If $A, B \in \mathfrak{M}, A \subset B$ and $\mu(B) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(c) If $A, B \in \mathfrak{M}, A \subset B$, then $\mu(A) \leq \mu(B)$.

(d) If $A_1, A_2, A_3, ... \in \mathfrak{M}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(e) If $A_1, A_2, A_3, ... \in \mathfrak{M}, \mu(A_i) = 0$, for $i = 1, 2, 3, ...$ then $\mu(\cup_{i=1}^{\infty} A_i) = 0$.

(f) If $A_1, A_2, A_3, ... \in \mathfrak{M}$, $A_1 \subset A_2 \subset A_3 \subset ...$ then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$$

(g) If $A_1, A_2, A_3, ... \in \mathfrak{M}$, $A_1 \supset A_2 \supset A_3 \supset ...$ and $\mu(A_1) < \infty$, then

$$\mu\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$$

2.5.2. Lebesgue Measure on the Real Line

The length of a bounded interval I (open, closed, half-open) with endpoints a and b $(a < b)$ is defined by $\ell(I) := b - a$. If I is $(a,\infty), (-\infty,b)$, or $(-\infty,\infty)$, then $\ell(I) = \infty$. Is it possible to extend this concept of length (or measure) to arbitrary subsets of $\mathbb{R}$? When one attempts to do this, one is led rather naturally to what has become known as Lebesgue measure.

Given a set E of real numbers, $\mu(E)$ will denote its Lebesgue measure if it is defined.

The following are the properties of the Lebesgue measure.

(a) Extends length: For every interval I, $\mu(I) = \ell(I)$.

(b) Monotone: If $A \subset B \subset \mathbb{R}$, then $0 \leq \mu(A) \leq \mu(B) \leq \infty$.

(c) Translation invariant: For each subset A of $\mathbb{R}$ and for each point $x_0 \in \mathbb{R}$ we define $A + x_0 := \{x + x_0 : x \in A\}$. Then $\mu(A + x_0) = \mu(A)$.

(d) Countably additive: If A and B are disjoint subsets of $\mathbb{R}$, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}$ is a sequence of disjoint sets, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Let $M = M(\mathbb{R})$ denote the largest family of subsets of $\mathbb{R}$ for which conditions (a) - (d) hold with $\mu : M \to [0, \infty]$. Members of $M = M(\mathbb{R})$ are called Lebesgue measurable subsets of $\mathbb{R}$.

Definition 2.5.2.1 For each subset E of $\mathbb{R}$ we define its Lebesgue outer measure $\mu^*(E)$ by

$$\mu^*(E) := \inf\Big\{ \sum_{k=1}^{\infty} \ell(I_k) : \{I_k\} \text{ a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \Big\}.$$

It is obvious that $0 \leq \mu^*(E) \leq \infty$ for every set $E \subset \mathbb{R}$.

Theorem 2.5.2.1 The Lebesgue outer measure $\mu^*(E)$ is zero if E is countable; extends length; is monotone; translation invariant; and is countably subadditive: for every sequence $E_i \subset \mathbb{R}$,

$$\mu^*\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

That is, every subset of $\mathbb{R}$ has a Lebesgue outer measure which satisfies properties (a)-(c), but satisfies only part of property (d).

Definition 2.5.2.2 A set $E \subset \mathbb{R}$ is called Lebesgue measurable if for every subset A of $\mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Definition 2.5.2.3 If E is a Lebesgue measurable set, the Lebesgue measure of E is defined to be its outer measure $\mu^*(E)$ and is written $\mu(E)$.

Theorem 2.5.2.2 The collection M of Lebesgue measurable sets has the following properties:

(a) Both ϕ and $\mathbb{R}$ are measurable; $\mu(\phi) = 0$ and $\mu(\mathbb{R}) = \infty$.

(b) If E is measurable, then so is E^c.

(c) If $\mu^*(E) = 0$, then E is measurable.

(d) If E_1 and E_2 are measurable, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable.

(e) If E is measurable, then $E + x_0$ is measurable.

(f) Every interval is measurable and $\mu(I) = \mu^*(I) = \ell(I)$.

(g) If $\{E_i : 1 \leq i \leq n\}$ is a finite collection of disjoint measurable sets, then for all $A \subset \mathbb{R}$,

$$\mu^*\Big(\bigcup_{i=1}^{n} A \cap E_i\Big) = \mu^*\Big(A \cap \Big(\bigcup_{i=1}^{n} E_i\Big)\Big) = \sum_{i=1}^{n} \mu^*(A \cap E_i).$$

In particular, when $A = \mathbb{R}$ we have

$$\mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \mu(E_i).$$

(h) If $\{E_i\}$ is a sequence of measurable sets, then

$$\bigcup_{i=1}^{\infty} E_i \text{ and } \bigcap_{i=1}^{\infty} E_i$$

are also measurable sets.

(i) If $\{E_i\}$ is an arbitrary sequence of disjoint measurable sets, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

(j) Every open set and every closed set is measurable.

2.5.3. Measurable Functions

Definition 2.5.3.1 Let f be a function on $[a,b]$. We say that f is a measurable function if, for every $s \in \mathbb{R}$ the set

$$\{x | f(x) > s\}$$

is a measurable set.

That is, f is a measurable function if, for every real s, the inverse image under f of $(0,\infty)$ is a measurable set. It follows immediately that every continuous function g on $[a,b]$ is measurable. If g is continuous, then, the inverse image under g of (s,∞) is open. But open sets are measurable. Hence $\{x|g(x)>s\}$ is a measurable set, and so g is a measurable function.

Here are other criteria for measurability equivalent to definition 2.5.3.1

Theorem 2.5.3.2 The function f on $[a,b]$ is measurable if and only if any one (and hence all) of the following statements hold.

(a) For every $s \in \mathbb{R}$ the set $\{x|f(x) \geq s\}$ is a measurable set.

(b) For every $s \in \mathbb{R}$ the set $\{x|f(x) < s\}$ is a measurable set.

(c) For every $s \in \mathbb{R}$ the set $\{x|f(x) \leq s\}$ is a measurable set.

Theorem 2.5.3.3 If f is a measurable function on $[a,b]$, and if $c \in \mathbb{R}$, then the functions $f+c$ and cf are measurable.

Theorem 2.5.3.4 If f and g are measurable functions on $[a,b]$, then so are $f+g$, $f-g$, and fg. Furthermore, if $g(x) \neq 0$ $(a \leq x \leq b)$, then f/g is also measurable.

2.5.4. Definition and existence of the Lebesgue integral for bounded functions

Definition 2.5.4.1 Let f be a bounded function on $[a,b]$, and let E be a subset of $[a,b]$. Then we define

$$M[f;E] = l.u.b._{x\in E} f(x),$$

$$m[f;E] = g.l.b._{x\in E} f(x).$$

Definition 2.5.4.2 By a measurable partition P of $[a,b]$ we mean a finite collection $\{E_1, E_2, ..., E_n\}$ of measurable subsets of $[a,b]$ such that

$$\cup_{k=1}^{n} E_k = [a,b]$$

and such that

$$\mu(E_j \cap E_k) = 0 \quad (j,k = 1,...,n; j \neq k).$$

The sets $E_1, E_2, ..., E_n$ are called the components of P.

If P and Q are measurable partitions, then Q is called a refinement of P if every component of Q is wholly contained in some component of P.

Definition 2.5.4.3 Let f be a bounded function on $[a,b]$ and let $P = \{E_1, ..., E_n\}$ be any measurable partition of $[a,b]$. We define the upper sum $U[f;P]$ as

$$U[f;P] = \sum_{k=1}^{n} M[f;E_k]\mu(E_k).$$

Similarly, we define the lower sum

$$L[f;P] = \sum_{k=1}^{n} m[f;E_k]\mu(E_k).$$

Definition 2.5.4.4 Let f be a bounded function on $[a,b]$. We define

$$\ell\overline{\int_a^b} f(x)dx,$$

called the Lebesgue upper integral of f over $[a,b]$, as

$$\ell\overline{\int_a^b} f(x)dx = g.l.b._P U[f;P]$$

where the $g.l.b.$ is taken over all measurable partitions P of $[a,b]$. Similarly, we define

$$\ell\underline{\int_a^b} f(x)dx,$$

called the Lebesgue lower integral of f over $[a,b]$, as

$$\ell\underline{\int_a^b} f(x)dx = l.u.b._P L[f;P].$$

For simplicity we will denote the upper and lower integrals of f respecticely by

$$\ell\overline{\int_a^b} f \text{ and } \ell\underline{\int_a^b} f.$$

Definition 2.5.4.5 If f is a bounded function on $[a,b]$, we say that f is Lebesgue integrable on $[a,b]$ if

$$\ell\overline{\int_a^b} f = \ell\underline{\int_a^b} f.$$

If f is Lebesgue integrable on $[a,b]$, we write $f \in \ell[a,b]$.

Theorem 2.5.4.6(Lebesgue Dominated Convergence Theorem) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\ell[a,b]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ almost everywhere } (a \le x \le b).$$

Suppose there exists $g \in \ell[a,b]$ such that

$$|f_n(x)| \le g(x) \text{ almost everywhere } (a \le x \le b; n \in I).$$

Then $f \in \ell[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

2.6 The time scale calculus

Definition 2.6.1 A time scale is an arbitrary nonempty closed subset of the real numbers.

Thus

$$\mathbb{R},\ \mathbb{Z},\ \mathbb{N},\ \mathbb{N}_0,$$

that is, the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales.

Definition 2.6.2 Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we put $\inf \phi = \sup \mathbb{T}$(that is, $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \phi = \inf \mathbb{T}$ (that is, $\rho(t) = t$ if $\mathbb{T}$ has a minimum t) , where ϕ denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, t is called right-dense, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. We also need below the set $\mathbb{T}^{\kappa}$ which is derived from the time scale $\mathbb{T}$ as follow: If $\mathbb{T}$ has a left-scattered maximum m, then

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup\mathbb{T}), \sup\mathbb{T}), \text{ if } \sup\mathbb{T} < \infty \\ \mathbb{T}, \text{ if } \sup\mathbb{T} = \infty. \end{cases}$$

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T},$$

that is $f^{\sigma} = f o\, \sigma$.

Example 2.6.1 Let us briefly consider the two examples, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t,\infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) \equiv 0 \text{ for all } t \in \mathbb{T}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, \ldots\} = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) \equiv 1 \text{ for all } t \in \mathbb{T}.$$

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^{\kappa}$.

Definition 2.6.3 Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t (that is, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Example 2.6.2

(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is constant, then $f^\Delta(t) = 0$. This is clear because for any $\varepsilon > 0$,

$$\begin{aligned} |[f(\sigma(t)) - f(s)] - 0.[\sigma(t) - s]| &= |\alpha - \alpha| \\ &\leq \varepsilon|\sigma(t) - s| \text{ holds for all } s \in \mathbb{T}. \end{aligned}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^\Delta(t) = 1$. This follows since for any $\varepsilon > 0$,

$$\begin{aligned} |[f(\sigma(t)) - f(s)] - 1.[\sigma(t) - s]| &= |\sigma(t) - s - (\sigma(t) - s)| \\ &= 0 \\ &\leq \varepsilon|\sigma(t) - s| \text{ holds for all } s \in \mathbb{T}. \end{aligned}$$

Some easy and useful relationships concerning the delta derivative are given next.

Theorem 2.6.1 Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

Proof. Part (i). Assume that f is differentiable at t. Let $\varepsilon\in(0,1)$. Define

$$\varepsilon^{*}=\varepsilon[1+|f^{\Delta}(t)|+2\mu]^{-1}.$$

Then $\varepsilon^{*}\in(0,1)$. By definition 2.6.3 there exists a neighbourhood U of t such that

$$|f(\sigma(t))-f(s)-[\sigma(t)-s]f^{\Delta}(t)|\leq\varepsilon^{*}|\sigma(t)-s| \text{ for all } s\in U.$$

Therefore we have for all $s\in U\cap(t-\varepsilon^{*},t+\varepsilon)$

$$\begin{aligned}
|f(t)-f(s)| &= |\{f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\}\\
&\quad -\{f(\sigma(t))-f(t)-\mu(t)f^{\Delta}(t)\}+(t-s)f^{\Delta}(t)|\\
&\leq \varepsilon^{*}|\sigma(t)-s|+\varepsilon^{*}\mu(t)+|t-s||f^{\Delta}(t)|\\
&\leq \varepsilon^{*}[\mu(t)+|t-s|+\mu(t)+|f^{\Delta}(t)|]\\
&= \varepsilon.
\end{aligned}$$

It follows that f is continuous at t.

Part (ii). Assume f is continuous at t and t is right-scattered. By continuity

$$\begin{aligned}
\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s} &= \frac{f(\sigma(t))-f(t)}{\sigma(t)-t}\\
&= \frac{f(\sigma(t))-f(t)}{\mu(t)}.
\end{aligned}$$

Hence, given $\varepsilon>0$, there is a neighbourhood U of t such that

$$\left|\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}-\frac{f(\sigma(t))-f(t)}{\mu(t)}\right|\leq\varepsilon$$

for all $s \in U$. It follows that

$$\left|[f(\sigma(t)) - f(s)] - \frac{f(\sigma(t)) - f(t)}{\mu(t)}[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Hence we get the desired result

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Part (iii). Assume f is differentiable at t and t is right-dense. Let $\varepsilon > 0$ be given. Since f is differentiable at t, there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]|\varepsilon|\sigma(t) - s|$$

for all $s \in U$. Since $\sigma(t) = t$ we have that

$$|[f(t) - f(s)] - f^{\Delta}(t)(t - s)| \leq \varepsilon|t - s|$$

for all $s \in U$. It follows that

$$\left|\frac{f(t) - f(s)}{t - s} - f^{\Delta}(t)\right| \leq \varepsilon$$

for all $s \in U$, $s \neq t$. Therefore we get the desired result

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Part (iv). If $\sigma(t) = t$, then $\mu(t) = 0$ and we have that

$$f(\sigma(t)) = f(t) = f(t) + \mu(t)f^{\Delta}(t).$$

On the other hand if $\sigma(t) > t$, then by (ii)

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t).\frac{f(\sigma(t)) - f(t)}{\mu(t)} \\ &= f(t) + \mu(t)f^{\Delta}(t), \end{aligned}$$

and the proof of part (iv) is complete.

Next we would like to find the derivatives of sums, products, and quotients of differentiable functions. This is possible according to the following theorem.

Theorem 2.6.2 Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then:

(i) The sum $f+g:\mathbb{T}\to\mathbb{R}$ is differentiable at t with

$$(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f:\mathbb{T}\to\mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t)=\alpha f^{\Delta}(t).$$

(iii) The product $fg:\mathbb{T}\to\mathbb{R}$ is differentiable at t with

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)\\ &= f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t)f(\sigma(t))\neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t)=-\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t))\neq 0$, then $\frac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Proof. Assume that f and g are delta differentiable at $t\in\mathbb{T}^{\kappa}$.

Part (i). Let $\varepsilon>0$. Then there exist neighbourhoods U_1 and U_2 of t with

$$|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|\leq\frac{\varepsilon}{2}|\sigma(t)-s| \text{ for all } s\in U_1$$

and

$$|g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)|\leq\frac{\varepsilon}{2}|\sigma(t)-s| \text{ for all } s\in U_2.$$

Let $U=U_1\cap U_2$. Then we have for all $s\in U$

$$\begin{aligned}&|(f+g)(\sigma(t))-(f+g)(s)-[f^{\Delta}(t)+g^{\Delta}(t)](\sigma(t)-s)|\\ &=|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)+g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)|\\ &\leq|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|+|g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)|\\ &\leq\frac{\varepsilon}{2}|\sigma(t)-s|+\frac{\varepsilon}{2}|\sigma(t)-s|\\ &=\varepsilon|\sigma(t)-s|.\end{aligned}$$

Therefore $f+g$ is differentiable at t and $(f+g)^{\Delta} = f^{\Delta} + g^{\Delta}$ holds at t.

Part (iii). Let $\varepsilon \in (0,1)$. Define $\varepsilon^* = \varepsilon[1+|f(t)|+|g(\sigma(t))|+|g^{\Delta}(t)|]^{-1}$. Then $\varepsilon^* \in (0,1)$ and hence there exist neighbourhoods U_1, U_2, and U_3 of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s)| \leq \varepsilon^* |\sigma(t)-s| \text{ for all } s \in U_1$$

and

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t)-s)| \leq \varepsilon^* |\sigma(t)-s| \text{ for all } s \in U_2.$$

and

$$|f(t) - f(s)| \leq \varepsilon^* \text{ for all } s \in U_3.$$

Put $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. Then

$$\begin{aligned}
&|(fg)(\sigma(t)) - (fg)(s) - [f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)](\sigma(t)-s)| \\
&= |[f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s)]g(\sigma(t)) \\
&\quad + [g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t)-s)]f(t) \\
&\quad + [g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t)-s)][f(s)-f(t)] \\
&\quad + (\sigma(t)-s)g^{\Delta}(t)[f(s)-f(t)]| \\
&\leq \varepsilon^*|\sigma(t)-s||g(\sigma(t))| + \varepsilon^*|\sigma(t)-s||f(t)| \\
&\quad + \varepsilon^*\varepsilon^*|\sigma(t)-s| + \varepsilon^*|\sigma(t)-s||g^{\Delta}(t)| \\
&= \varepsilon^*|\sigma(t)-s|[|g(\sigma(t))| + |f(t)| + \varepsilon^* + |g^{\Delta}(t)|] \\
&\leq \varepsilon^*|\sigma(t)-s|[1 + |f(t)| + |g(\sigma(t))| + |g^{\Delta}(t)|] \\
&= \varepsilon|\sigma(t)-s|.
\end{aligned}$$

Thus $(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta}$ holds at t.

Definition 2.6.4 A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \setminus D$ is countable and contains no right scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Theorem 2.6.3 Let f and g be real-valued functions defined on $\mathbb{T}$, both pre-differentiable with $D \subset \mathbb{T}$. Then

(i) $|f^{\Delta}(t)| \leq g^{\Delta}(t)$ for all $t \in D$ implies

$$|f(s) - f(r)| \leq g(s) - g(r) \text{ for all } r, s \in \mathbb{T},\ r \leq s. \tag{2.29}$$

(ii) If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \sup_{t \in U^{\kappa} \cap D} |f^{\Delta}(t)||s - r|. \tag{2.30}$$

In order to describe classes of functions that are "integrable", we introduce the following concepts.

Definition 2.6.5 A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exists (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.6.6 A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by in this thesis by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, R).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is *rd*-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, R).$$

Some results concerning *rd*-continuous and regulated functions are contained in the following theorem.

Theorem 2.6.4

(i) If f is continuous, then f is *rd*-continuous.

(ii) If f is *rd*-continuous then f is regulated.

(iii) The jump operator σ is *rd*-continuous.

(iv) If f is regulated or *rd*-continuous, then so if f^{σ}.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then fog has that property too.

Theorem 2.6.5(Existence of Pre-Antiderivatives) Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in D.$$

Definition 2.6.7 Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 2.6.4 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^{\kappa}.$$

Theorem 2.6.6(Existence of Antiderivatives) Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \text{ for } t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 2.6.7 If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 2.6.8 If $f^{\Delta} \geq 0$, then f is nondecreasing.

Theorem 2.6.9 If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then

(i)

$$\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$$

(ii)

$$\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$$

(iii)

$$\int_a^b (f)(t)\Delta t = -\int_b^a f(t)\Delta t;$$

(iv)

$$\int_a^b (f)(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$$

(v)

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$$

(vi)

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$$

(vii)

$$\int_a^a f(t)\Delta t = 0;$$

(viii) if $|f(t)| \leq g(t)$ on $[a,b)$, then

$$\Big|\int_a^b f(t)\Delta t\Big| \leq \int_a^b g(t)\Delta t;$$

(ix) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$.

We next define the improper integral $\int_a^{\infty} f(t)\Delta t$ as one would expect.

Definition 2.6.8 If $a \in \mathbb{T}, \sup \mathbb{T} = \infty$, and f is *rd*- continuous on $[0,\infty)$, then we define the improper integral by

$$\int_a^{\infty} f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Theorem 2.6.10(Chain Rule) Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with

$$(fog)^\Delta(t) = f'(g(c))g^\Delta(t).$$

We next present a chain rule which calculates $(fog)^\Delta$, where

$$g : \mathbb{T} \to \mathbb{R} \text{ and } f : \mathbb{R} \to \mathbb{R}.$$

Theorem 2.6.11(Chain Rule) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^\kappa$, then

$$(w \circ \nu)^\Delta = (w^{\tilde{\Delta}} \circ \nu)\nu^\Delta.$$

Theorem 2.6.12(Substitution) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\nu^\Delta(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\tilde{\Delta} s.$$

Definition 2.6.9 A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 2.6.10 Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\,\Delta z\Big),$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.6.1 Let $p, q \in \mathcal{R}$. Then

(i) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ where, $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Next we consider the first order nonhomogeneous linear equation

$$y^{\Delta} = p(t)y + f(t) \tag{2.31}$$

and the corresponding homogeneous equation

$$y^{\Delta} = p(t)y \tag{2.32}$$

on a time scale $\mathbb{T}$.

Theorem 2.6.13 Suppose (2.32) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem

$$y^{\Delta} = p(t)y, \;\; y(t_0) = y_0 \tag{2.33}$$

is given by

$$y(t) = e_p(t,t_0)y_0.$$

2.7 Qualitative Properties

2.7.1. Basic Definitions for Neutral Functional Differential Equations

Consider a neutral functional differential equation of the form

$$x'(t) = f(t,x(t),x(t-\tau(t)),x'(t-\tau(t))),\; t \geq t_0 \tag{2.34}$$

where $\tau : [t_0,\infty) \to [t_0,\infty), f : [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, f(t,0,0,0) = 0$, and all the functions are continuous in their respective arguments. Note that $x \equiv 0$ is a solution of (2.34). For a given $t_0 \geq 0$, let $\beta_0 = \inf_{t \geq t_0} t - \tau(t)$. Then we define $E_{t_0} = [\beta_0, t_0]$ which is the initial interval.

To specify a solution for equation (2.34) we need $t_0 \geq 0$ and a continuous function $\psi : E_{t_0} \to \mathbb{R}^n$. We say that ψ is the initial function on the interval E_{t_0} with ψ' continuous.

Definition 2.7.1.1 $x(t,\psi)$ is a solution of (2.34) if $x(t,\psi)$ is defined on an interval $[\beta_0, t_0+\gamma), 0 < \gamma \leq \infty$, $x(t_0,\psi) = \psi$, and satisfies (2.34) for $t_0 < t < \gamma$.

Definition 2.7.1.2 The zero solution of (2.34) is said to be stable if for each $t_0 \geq 0$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $||\psi|| < \delta$ on E_{t_0} implies that any solution of (2.34) satisfies $|x(t,t_0,\psi)| < \psi$ for all $t \geq t_0$.

Definition 2.7.1.3 The zero solution of (2.34) is said to be asymptotically stable if it is stable and if for each $t_0 \geq 0$ there is an $\eta > 0$ such that $||\psi|| < \eta$ on E_{t_0} implies that any solution of (2.34) satisfies $x(t,t_0,\psi) \to 0$ as $t \to \infty$.

Definition 2.7.1.4 The solution $x(t)$ of (2.34) is said to be periodic if $x(t+T) = x(t)$ for $T > 0$ and for all $t \in (-\infty,\infty)$. T is called the period of x.

Definition 2.7.1.5 The solution $x(t)$ of (2.34) is said to be positive if $x(t) > 0$ for all t.

2.7.2. Basic Definitions for Neutral Functional Difference Equations

Consider a neutral functional difference equation of the form

$$x(n+1) = f(n,x(n),x(n-\tau(n)),\Delta x(n-\tau(n))), \tag{2.35}$$

where $\tau : [n_0,\infty) \cap \mathbb{Z} \to \mathbb{R}$, $f : [n_0,\infty) \cap \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f(n,0,0,0) = 0$ and f is continuous in its respective arguments. Note that $x \equiv 0$ is a solution of (2.35).

For a given $n_0 \geq 0$, let $\alpha_0 = \inf_{n \geq n_0} n - \tau(n)$. Then we define $D_{n_0} = [\alpha_0, n_0] \cap \mathbb{Z}$ which is the initial interval. To specify a solution for equation (2.35) we need $n_0 \geq 0$ and an initial bounded function $\psi : D_{n_0} \to \mathbb{R}^n$.

Definition 2.7.2.1 $x(t,\psi)$ is a solution of (2.35) if $x(t,\psi)$ is defined on an interval $[\alpha_0, n_0] \cap \mathbb{Z}$, $x(n_0,\psi) = \psi$, and satisfies (2.35) for $n \geq n_0$.

Definition 2.7.2.2 The zero solution of (2.35) is said to be stable if for each $n_0 \geq 0$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $||\psi|| < \delta$ on D_{n_0} implies that any solution of (2.35) satisfies $|x(n,n_0,\psi)| < \varepsilon$ for all $n \geq n_0$.

Definition 2.7.2.3 The zero solution of (2.35) is said to be asymptotically stable if it is stable and if for each $n_0 \geq 0$ there is an $\eta > 0$ such that $||\psi|| < \eta$ on D_{n_0} implies that any solution of (2.35) satisfies $x(n,n_0,\psi) \to 0$ as $n \to \infty$..

Definition 2.7.2.4 The solution $x(n)$ of (2.35) is said to be periodic if $x(n+N) = x(n)$ for $N \in \mathbb{Z}^+$ and for all $n \in (-\infty,\infty) \cap \mathbb{Z}$. N is called the period of x.

Definition 2.7.2.5 The solution $x(n)$ of (2.35) is said to be positive if $x(n) > 0$ for all $n \in \mathbb{Z}$.

CHAPTER THREE

PERIODIC SOLUTIONS FOR FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

3.1 Introduction

In this Chapter we obtain sufficient conditions for the existence of periodic solutions for totally nonlinear neutral differential equations of the first order. In particular, we consider the totally nonlinear neutral differential equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-g(t)) + q\big(t, x(t), x(t-g(t))\big), \tag{3.1}$$

where $a(t)$ is a real valued function, $c(t)$ is continuously differentiable, $g(t)$ is twice continuously differentiable, $h : \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument and $q : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is also continuous with respect to its arguments.

Equations of form similar to (3.1) where $h(x(t)) = x(t)$ have gained the attention of many researchers in recent times. This include the work of Burton and Furumochi (2001a), Raffoul (2003), and Djoudi and Khemis (2006). In all the above mentioned papers, the method of variation of parameters was applied directly to invert the equations, however, for (3.1) the method of variation of parameters cannot be applied directly. We therefore resort to the idea of adding and subtracting a linear term. Burton (2002) noted that the added terms destroys a contraction already present in part of the equation but replaces it with the so called large contraction mapping which is suitable for fixed point theory.

Remark 3.1.1 The content of this Chapter has been published as:

E. Yankson, " Periodic solutions for totally nonlinear neutral differential equations with functional delay," Opuscula Mathematica, No. 3, 2012.

The rest of the Chapter is organized as follows. First, some preliminary material is provided in section two. Our main results in this Chapter are presented in the third section.

3.2 Preliminaries

Let $T > 0$ and define the set $P_T = \{\phi \in C(\mathbb{R},\mathbb{R}) : \phi(t+T) = \phi(t)\}$ and the norm $\|x(t)\| = \max_{t\in[0,T]} |x(t)|$, where C is the space of continuous real valued functions. Then $(P_T, ||.||)$ is a Banach space. Also, for any $L > 0$, define

$$\mathbb{M}_L = \{\varphi \in P_T \ : \ ||\varphi|| \leq L, \varphi' \text{ is bounded}\}.$$

In this Chapter we make the following assumptions.

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad g(t+T) = g(t), \quad g(t) \geq g^* > 0 \tag{3.2}$$

with $c(t)$ continuously differentiable, $g(t)$ twice continuously differentiable and g^* is constant. Also,

$$\int_0^T a(s)ds > 0. \tag{3.3}$$

We also assume that $q(t,x,y)$ is continuous and periodic in t and Lipschitz continuous in x and y. That is

$$q(t+T,x,y) = q(t,x,y) \tag{3.4}$$

and some positive constants K and E,

$$|q(t,x,y) - q(t,z,w)| \leq K\|x-z\| + E\|y-w\|. \tag{3.5}$$

Also, we assume that for all t, $0 \leq t \leq T$,

$$g'(t) \neq 1. \tag{3.6}$$

Since $g(t)$ is periodic, condition (3.6) implies that $g'(t) < 1$.

Lemma 3.2.1. Suppose (3.2)-(3.3) and (3.6) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of equation (3.1) if and only if

$$x(t) = \frac{c(t)}{1-g'(t)}x(t-g(t)) + \left(1-e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \left[a(u)H(x(u)) - r(u)x(u-g(u)) + q(u,x(u),x(u-g(u)))\right] e^{-\int_u^t a(s)ds} du, \tag{3.7}$$

where

$$r(u) = \frac{\Big(c'(u)-c(u)a(u)\Big)\Big(1-g'(u)\Big) + g''(u)c(u)}{(1-g'(u))^2}, \tag{3.8}$$

and

$$H(x(u)) = x(u) - h(x(u)). \tag{3.9}$$

Proof. Let $x(t) \in P_T$ be a solution of (3.1). We first rewrite (3.1) in the form

$$x'(t) + a(t)x(t) = a(t)H(x(t)) + c(t)x'(t-g(t)) + q\big(t,x(t),x(t-g(t))\big). \tag{3.10}$$

Multiply both sides of (3.10) by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\int_{t-T}^{t} \left[x(u)e^{\int_0^u a(s)ds}\right]' du = \int_{t-T}^{t} \left[a(u)H(x(u)) + c(u)x'(u-g(u)) + q(u,x(u),x(u-g(u)))\right] e^{\int_0^u a(s)ds} du.$$

Thus we obtain,

$$x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} = \int_{t-T}^{t} \left[a(u)H(x(u)) + c(u)x'(u-g(u)) + q(u,x(u),x(u-g(u)))\right] e^{\int_0^u a(s)ds} du.$$

By dividing both sides of the above equation by $\exp(\int_0^t a(s)ds)$ and using the fact that $x(t) = x(t-T)$, we obtain

$$x(t) = \left(1-e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \left[a(u)H(x(u)) + c(u))x'(u-g(u)) + q(u,x(u),x(u-g(u)))\right] e^{-\int_u^t a(s)ds} du. \tag{3.11}$$

Rewrite

$$\int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du$$
$$= \int_{t-T}^{t} \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))} e^{-\int_u^t a(s)ds}du.$$

Let

$$U = \frac{c(u)}{1-g'(u)} e^{-\int_u^t a(s)ds}, \quad \text{and} \quad dV = x'(u-g(u))(1-g'(u))du.$$

It follows that

$$\begin{aligned}
dU &= \frac{(1-g'(u))[c'(u)e^{-\int_u^t a(s)ds} + c(u)a(u)e^{-\int_u^t a(s)ds}] - c(u)e^{-\int_u^t a(s)ds}(-g''(u))}{[1-g'(u)]^2} \\
&= \frac{\Big((1-g'(u))[c'(u)+c(u)a(u)] - c(u)(-g''(u))\Big)e^{-\int_u^t a(s)ds}}{[1-g'(u)]^2}. \\
&= r(u)e^{-\int_u^t a(s)ds}.
\end{aligned}$$

Also, with $z = u - g(u)$ we obtain

$$\begin{aligned}
V &= \int x'(u-g(u))(1-g'(u))du \\
&= \int x'(z)dz \\
&= x(u-g(u)).
\end{aligned}$$

We therefore obtain

$$\int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du$$
$$= \frac{c(t)}{1-g'(t)} x(t-g(t))\big(1-e^{-\int_{t-T}^t a(s)ds}\big) - \int_{t-T}^{t} r(u)e^{-\int_u^t a(s)ds}x(u-g(u))du, \tag{3.12}$$

where $r(u)$ is given by (3.8). Then substituting (3.12) into (3.11) gives the desired results. Since each step in the above work is reversible, the proof is complete.

3.3 Existence of periodic solution

In this section we state and prove our main results in this Chapter. In proving the results in this chapter, we employ theorem 2.3.7 in which the notion of a large contraction is required as one of the sufficient conditions. In view of this we first define the operator P by

$$\begin{aligned}(P\varphi)(t) &= \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1-e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \\ &\times \int_{t-T}^{t} \left[a(u)H(\varphi(u)) - r(u)\varphi(u-g(u))\right. \\ &\left.+ q(u,\varphi(u),\varphi(u-g(u)))\right] e^{-\int_u^t a(s)ds}\,du.\end{aligned} \tag{3.13}$$

where r and H are given in (3.8) and (3.9) respectively. It therefore follows from Lemma 3.2.1 that fixed points of P are solutions of (3.1) and vice versa.

In order to employ Theorem 2.3.7 we need to express the operator P as a sum of two operators, one of which is completely continuous and the other is a large contraction. Let $(P\varphi)(t) = A\varphi(t) + B\varphi(t)$ where $A, B : P_T \to P_T$ are defined by

$$(B\varphi)(t) = \left(1-e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \left[a(u)H(\varphi(u))\right] e^{-\int_u^t a(r)dr}du, \tag{3.14}$$

and

$$\begin{aligned}(A\varphi)(t) \;&=\; \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1-e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \\ &\times \int_{t-T}^{t} \left[-r(u)\varphi(u-g(u)) + q(u,\varphi(u),\varphi(u-g(u)))\right] e^{-\int_u^t a(s)ds}du.\end{aligned} \tag{3.15}$$

In the rest of the Chapter we require the following conditions.

$$KL + EL + |q(t,0,0)| \le \beta L a(t), \tag{3.16}$$

$$|r(t)| \le \delta a(t), \tag{3.17}$$

$$\max_{t\in[0,T]} \left|\frac{c(t)}{(1-g'(t))}\right| = \alpha, \tag{3.18}$$

and

$$J(\beta+\alpha+\delta) \le 1, \tag{3.19}$$

where α, β, δ, L and J are constants with $J \ge 3$.

Next we state our main result and present its proof in four lemmas.

Theorem 3.3.1. Let L be a fixed positive number and let $(P_T, ||.||)$ be the Banach space of continuous T-periodic real functions. Suppose (3.2)-(3.4), and (3.16)-(3.19) hold. Then equation (3.1) possesses a periodic solution in the subset $\mathbb{M}_L$.

The proof is based on the following four lemmas.

Lemma 3.3.2. Suppose that conditions (3.2)-(3.4) and (3.16)-(3.19) hold. Then for the L defined in Theorem 3.3.1, $A : \mathbb{M}_L \to \mathbb{M}_L$ is continuous in the supremum norm and maps $\mathbb{M}_L$ into a compact subset of $\mathbb{M}_L$.

Proof. We first show that $(A\varphi)(t+T) = (A\varphi)(t)$. Substituting $t+T$ into $(A\varphi)(t)$ gives

$$\begin{aligned}
(A\varphi)(t+T) &= \frac{c(t+T)}{1-g'(t+T)}\varphi(t+T-g(t+T)) + \left(1-e^{-\int_t^{t+T} a(s)ds}\right)^{-1} \\
&\times \int_t^{t+T} \left[-r(u)\varphi(u-g(u))\right. \\
&\left. + q(u,\varphi(u),\varphi(u-g(u)))\right] e^{-\int_u^{t+T} a(s)ds}\, du.
\end{aligned}$$

With $v = s-T$ and $k = u-T$ we obtain

$$\begin{aligned}
(A\varphi)(t+T) &= \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1-e^{-\int_{t-T}^{t} a(v+T)dv}\right)^{-1} \\
&\times \int_{t-T}^{t} \left[-r(k+T)\varphi(k+T-g(k+T))\right. \\
&\left. + q(k+T,\varphi(k+T),\varphi(k+T-g(k+T)))\right] e^{-\int_{k+T}^{t+T} a(s)ds}\, dk \\
&= \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1-e^{-\int_{t-T}^{t} a(v)dv}\right)^{-1} \\
&\times \int_{t-T}^{t} \left[-r(k)\varphi(k-g(k))\right. \\
&\left. + q(k,\varphi(k),\varphi(k-g(k)))\right] e^{-\int_k^{t} a(v)dv}\, dk \\
&= (A\varphi)(t).
\end{aligned}$$

We will next show that A maps $\mathbb{M}_L$ into itself. Note that

$$|q(t,x,y)| \le |q(t,x,y)-q(t,0,0)| + |q(t,0,0)| \le K|x| + E|y| + |q(t,0,0)|.$$

Thus, for any $\varphi \in \mathbb{M}_L$, we have

$$\begin{aligned}
|(A\varphi)(t)| &\leq \Big|\frac{c(t)\varphi(t-g(t))}{1-g'(t)}\Big| + \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \\
&\quad \times \int_{t-T}^{t} |r(u)\varphi(u-g(u))| e^{-\int_u^t a(s)ds} du \\
&\quad + \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} |q(u,\varphi(u),\varphi(u-g(u)))| e^{-\int_u^t a(s)ds} du \\
&\leq \alpha L + \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} \delta a(u) L e^{-\int_u^t a(s)ds} du \\
&\quad + \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} (KL+EL+|q(t,0,0)|) e^{-\int_u^t a(s)ds} du
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha L + \delta L \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)ds} du \\
&\quad + \beta L \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)ds} du \\
&\leq (\alpha+\delta+\beta) L \leq \frac{L}{J} < L.
\end{aligned}$$

Thus showing that A maps $\mathbb{M}_L$ into itself.

Now we show that A is continuous. Let $\varphi, \psi \in \mathbb{M}_L$, and let

$$a = \max_{t\in[0,T]} \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1}, \quad b = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds},$$

$$\sigma = \max_{t\in[0,T]} r(t), \quad \lambda = \max_{t\in[0,T]} |q(t,0,0)|, \tag{3.20}$$

$$\nu = \max_{t\in[0,T]} \Big|\frac{c'(t)}{(1-g'(t))}\Big|, \quad \mu = \max_{t\in[0,T]} \Big|\frac{g''(t)c(t)}{(1-g'(t))^2}\Big|.$$

Given $\varepsilon > 0$, take $\delta = \varepsilon/F$ such that $\|\varphi-\psi\| < \delta$. Then we get

$$\begin{aligned}
\big\|\big(A\varphi(t)\big) - \big(A\psi(t)\big)\big\| &\leq \alpha\|\varphi-\psi\| + ab\int_{t-T}^{t} \big[L\|\varphi-\psi\| + E\|\varphi-\psi\| + \sigma\|\varphi-\psi\|\big] du \\
&\leq F\|\varphi-\psi\| < \varepsilon
\end{aligned}$$

where $F = \alpha + Tab[\sigma + L + E]$. This proves A is continuous. To show A is compact, we let $\varphi_n \in \mathbb{M}_L$ where n is a positive integer. Then as before we have that

$$\|A(\varphi_n(t))\| \leq L. \tag{3.21}$$

Moreover, a direct calculation shows that

$$\begin{aligned}(A\varphi_n)'(t) &= q(t,\varphi_n(t),\varphi_n(t-g(t))) - r(t)\varphi_n(t-g(t)) - a(t)\big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \\ &\quad \times \int_{t-T}^{t} [q(u,\varphi_n(u),\varphi_n(u-g(u))) - r(t)\varphi_n(u-g(u))]e^{-\int_u^t a(s)ds}du \\ &\quad + \frac{c'(t)\varphi_n(t)+c(t)\varphi_n'(t)}{1-g'(t)} + \frac{g''(t)c(t)\varphi_n(t)}{(1-g'(t))^2}.\end{aligned}$$

By invoking conditions (3.5), (3.16)-(3.18), (3.20) and (3.21) we obtain

$$|(A\varphi_n)'(t)| \le KL + EL + \lambda + \delta||a||L + ||a||L + \nu L + \alpha L' + \mu L \le D,$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that the subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous T- periodic function. Thus, A is compact.

Lemma 3.3.3. Suppose (3.2)-(3.5), and (3.16) hold. Suppose also that for the L defined in Theorem 3.3.1,

$$\big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} \big[|a(u)||H(\varphi(u))|\big]e^{-\int_u^t a(r)dr}du \le \frac{(J-1)L}{J}. \tag{3.22}$$

For B,A defined by (3.14) and (3.15), if $\varphi,\psi \in \mathbb{M}_L$ are arbitrary, then

$$A\varphi + B\psi : \mathbb{M}_L \to \mathbb{M}_L.$$

Proof. Let $\varphi,\psi \in \mathbb{M}_L$ be arbitrary. Using the definition of B and the result of Lemma 3.3.2, we obtain

$$\begin{aligned}&|(A\varphi)(t)+(B\psi)(t)| \\ &\le |\frac{c(t)}{1-g'(t)}\varphi(t-g(t))| + \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \\ \times \quad &\int_{t-T}^{t} \big[|r(u)\varphi(u-g(u))| + |q(u,\varphi(u),\varphi(u-g(u)))|\big]e^{-\int_u^t a(s)ds}du \\ &+ \big(1-e^{-\int_{t-T}^{t} a(s)ds}\big)^{-1} \times \int_{t-T}^{t} \big|a(u)H(\varphi(u))\big|e^{-\int_u^t a(r)dr}du \\ &\le \frac{L}{J} + \frac{(J-1)L}{J} = L.\end{aligned}$$

Thus $A\varphi + B\psi \in \mathbb{M}_L$. This completes the proof.

In the next lemma we prove that H is a large contraction on $\mathbb{M}_L$. To this end we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous and differentiable on $U_L = [-L, L]$.

(H2) h is strictly increasing on U_L.

(H3) $\sup_{s\in U_L} h'(s) \le 1$.

(H4) $(s-r)\Big\{ \sup_{t\in U_L} h'(t) \Big\} \ge h(s) - h(r) \ge (s-r)\Big\{ \inf_{t\in U_L} h'(t) \Big\} \ge 0$ for $s, r \in U_L$ with $s \ge r$.

Lemma 3.3.4. Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H4)$. Then for the L defined in Theorem 3.3.1, the mapping H is a large contraction on the set $\mathbb{M}_L$.

Proof. Let $\phi, \varphi \in \mathbb{M}_L$ with $\phi \ne \varphi$. Then $\phi(t) \ne \varphi(t)$ for some $t \in \mathbb{R}$. Define the set

$$D(\phi, \varphi) = \Big\{ t \in \mathbb{R} : \phi(t) \ne \varphi(t) \Big\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{R}$ whenever $\varphi \in \mathbb{M}_L$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \tag{3.23}$$

holds for all $t \in D(\phi, \varphi)$. By (H3) we have

$$1 \ge \sup_{t\in U_L} h'(t) \ge \inf_{s\in U_L} h'(s) \ge 0. \tag{3.24}$$

Define the set $U_t \subset U_L$ by $U_t = [\varphi(t), \phi(t)] \cap U_L$ if $\phi(t) > \varphi(t)$, and $U_t = [\phi(t), \varphi(t)] \cap U_L$ if $\phi(t) < \varphi(t)$, for $t \in D(\phi, \varphi)$. Hence, for a fixed $t_0 \in D(\phi, \varphi)$ we get by (H4) and (3.23) that

$$\sup\{h'(u) : u \in U_{t_0}\} \ge \frac{h(\phi(t_0)) - h(\varphi(t_0))}{\phi(t_0) - \varphi(t_0)} \ge \inf\{h'(u) : u \in U_{t_0}\}.$$

Since $U_t \subset U_L$ for every $t \in D(\phi, \varphi)$, we find

$$\sup_{u\in U_L} h'(u) \ge \sup\{h'(u) : u \in U_{t_0}\} \ge \inf\{h'(u) : u \in U_{t_0}\} \ge \inf_{u\in U_L} h'(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L} h'(u) \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \geq \inf_{u \in U_L} h'(u) \geq 0 \tag{3.25}$$

for all $t \in D(\phi, \varphi)$. So, (3.25) yields

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \Big| 1 - \Big(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \Big) \Big| \\ &\leq |\phi(t) - \varphi(t)| \Big(1 - \inf_{u \in U_L} h'(u) \Big) \end{aligned} \tag{3.26}$$

for all $t \in D(\phi, \varphi)$. Thus, (3.25) and (3.26) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_L$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [-L,L]} |\phi(t) - \varphi(t)| \geq \varepsilon.$$

If $|\phi(t) - \varphi(t)| \leq \varepsilon/2$ for some $t \in D(\phi, \varphi)$, then from (3.26)

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{3.27}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then from (3.25) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &\leq |\phi(t) - \varphi(t)| \Big\{ 1 - \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \Big\} \\ &\leq (1 - \lambda) \|\phi(t) - \varphi(t)\|, \end{aligned} \tag{3.28}$$

where

$$\lambda := \frac{1}{2L} \min \Big\{ h(u + \frac{\varepsilon}{2}) - h(u), u \in [-L, L] \Big\} > 0.$$

Consequently, it follows from (3.27) and (3.28) that

$$|(H\phi(t) - (H\varphi)(t)| \leq \delta \|\phi - \varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1$. The proof is complete.

The next result gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 3.3.5. If H is a large contraction on $\mathbb{M}_L$, then so is the mapping B.

Proof. If H is a large contraction on $\mathbb{M}_L$, then for $x, y \in \mathbb{M}_L$, with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. Thus, it follows from the equality

$$a(u)e^{-\int_u^t a(r)dr} = \frac{d}{du}[e^{-\int_u^t a(r)dr}],$$

that

$$\begin{aligned}
|Bx(t) - By(t)| &\leq \left(1 - e^{-\int_{t-T}^t a(s)ds}\right)^{-1} \times \int_{t-T}^t a(u)\left|H(x(u)) - H(y(u))\right| \\
&\quad \times e^{-\int_u^t a(r)dr} du \\
&\leq \frac{\|x-y\|}{\left(1 - e^{-\int_{t-T}^t a(s)ds}\right)^{-1}} \int_{t-T}^t a(u)e^{-\int_u^t a(r)dr} du \\
&= \|x-y\|.
\end{aligned}$$

Thus,

$$\|Bx - By\| \leq \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \leq \delta\|x - y\|$$

holds if we know the existence of a $0 < \delta < 1$, such that for all $\varepsilon > 0$

$$\left[x, y \in \mathbb{M}_L,\ \|x - y\| \geq \varepsilon\right] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete.

By Lemma 3.2.1, φ is a solution of (3.1) if

$$\varphi = A\varphi + B\varphi,$$

where B and A are given by (3.14) and (3.15) respectively. By Lemma 3.3.2, $A : \mathbb{M}_L \to \mathbb{M}_L$ is completely continuous. By Lemma 3.3.3, $A\varphi + B\psi \in \mathbb{M}_L$ whenever $\varphi, \psi \in \mathbb{M}_L$. Moreover, $B : \mathbb{M}_L \to \mathbb{M}_L$ is a large contraction by Lemma 3.3.5. Thus all the hypotheses of Theorem 2.3.7 are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}_L$ such that $\varphi = A\varphi + B\varphi$. Hence (3.1) has a $T-$ periodic solution.

CHAPTER FOUR

ASYMPTOTIC STABILITY FOR NEUTRAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

4.1 Introduction

In this Chapter we prove that the zero solution of totally nonlinear neutral differential equations are asymptotically stable. As pointed out in the introduction of Chapter three, a number of authors have considered similar forms of the equation we consider in this Chapter where $h(x(t)) = x(t)$. In the previous Chapter, we obtained sufficient conditions for the existence of periodic solutions of the equation which we consider in this Chapter.

We consider the totally nonlinear neutral differential equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-g(t)) + b(t)q\big(x(t-g(t))\big), t \geq 0 \tag{4.1}$$

with an initial function $x(t) = \psi(t)$, $t \in [n_0, 0]$, with $\psi \in C([n_0, 0], \mathbb{R})$, $[n_0, 0] = \{z \leq 0 \mid z = t - g(t),\ t \geq 0\}$. We assume in this paper that $a, h, b, q \in C(\mathbb{R}_+, \mathbb{R})$ with $a(t) \geq 0$, $c \in C^1(\mathbb{R}_+, \mathbb{R})$ and $g \in C^2(\mathbb{R}_+, \mathbb{R})$ such that

$$g'(t) \neq 1, t \in \ \mathbb{R}_+. \tag{4.2}$$

We further assume that $h(0) = 0$, $q(0) = 0$, and there is positive constant ρ such that

$$|q(x) - q(y)| \leq \rho|x - y|, \tag{4.3}$$

for all $t \geq 0$.

The rest of the Chapter is organized as follows. In section two we introduce some preliminary material relevant for our work in this Chapter. The last section contains a statement of our main results of this Chapter together with its proof.

4.2 Preliminaries

Define S to be the Banach space of bounded continuous functions $\varphi : [n_0, \infty) \to \mathbb{R}$ with the supremum norm $||.||$. Suppose L is a positive real number. Then let

$$\begin{aligned}\mathbb{M} &= \Big\{\varphi \in S \mid \varphi(t) = \psi(t) \text{ if } t \in [n_0, 0], |\varphi(t)| \leq L \text{ for } t \in [n_0, \infty), \\ &\quad \text{and } \varphi(t) \to 0 \text{ as } t \to \infty\Big\}.\end{aligned}$$

The set $\mathbb{M}$ is therefore convex, bounded and complete when endowed with the supremum norm $||.||$.

In this chapter we make the following assumptions. Let $\mu(t) = \frac{c(t)}{(1-g'(t))}$ and assume that there are constants $k_1, k_3 > 0$ such that for $0 \leq t_1 < t_2$

$$\Big|\int_{t_1}^{t_2} a(u)du\Big| \leq k_1|t_2 - t_1|, \tag{4.4}$$

and

$$|\mu(t_2) - \mu(t_1)| \leq k_3|t_2 - t_1|. \tag{4.5}$$

Suppose that for $t \geq 0$,

$$\sup_{t \geq 0} \Big|\frac{c(t)}{(1-g'(t))}\Big| = \alpha, \tag{4.6}$$

$$|b(t)| < a(t), \tag{4.7}$$

$$|r(t)| < \delta a(t), \tag{4.8}$$

and

$$J(\rho + \alpha + \delta) \leq 1, \tag{4.9}$$

where α, ρ, δ and J are positive constants with $J > 3$.

We assume further that

$$t - g(t) \to \infty \text{ and } \int_0^t a(u)du \to \infty \text{ as } t \to \infty. \tag{4.10}$$

Lemma 4.2.1. Suppose (4.2) hold. Then $x(t)$ is a solution of equation (4.1) if and only if

$$x(t) = \Big[\psi(0) - \frac{c(0)\psi(-g(0))}{1-g'(0)}\Big]e^{-\int_0^t a(s)ds} + \frac{c(t)}{1-g'(t)}x(t-g(t)) + \int_0^t \big[a(u)H(x(u)) - r(u)x(u-g(u)) + b(u)q(x(u-g(u)))\big]e^{-\int_u^t a(s)ds}\,du\,, \tag{4.11}$$

where

$$r(u) = \frac{\Big(c'(u) - c(u)a(u)\Big)\Big(1-g'(u)\Big) + g''(u)c(u)}{(1-g'(u))^2}, \tag{4.12}$$

and

$$H(x(u)) = x(u) - h(x(u)). \tag{4.13}$$

Proof. We first rewrite (4.1) in the form

$$x'(t) + a(t)x(t) = a(t)H(x(t)) + c(t)x'(t-g(t)) + b(t)q\big(x(t-g(t))\big). \tag{4.14}$$

Multiply both sides of (4.14) by $e^{\int_0^t a(s)ds}$ and then integrate from 0 to t to obtain

$$x(t) = \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t \big[a(u)H(x(u)) + c(u)x'(u-g(u)) + b(u)q(x(u-g(u)))\big]e^{\int_0^u a(s)ds}du. \tag{4.15}$$

Rewrite

$$\int_0^t c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du = \int_0^t \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))}e^{-\int_u^t a(s)ds}du.$$

We integrate the above integral by parts. Let

$$U = \frac{c(u)}{1-g'(u)}e^{-\int_u^t a(s)ds}, \quad \text{and} \quad dV = x'(u-g(u))(1-g'(u))du.$$

Thus

$$\begin{aligned}
dU &= \frac{(1-g'(u))[c'(u)e^{-\int_u^t a(s)ds}+c(u)a(u)e^{-\int_u^t a(s)ds}]-c(u)e^{-\int_u^t a(s)ds}(-g''(u))}{[1-g'(u)]^2} \\
&= \frac{\Big((1-g'(u))[c'(u)+c(u)a(u)]-c(u)(-g''(u))\Big)e^{-\int_u^t a(s)ds}}{[1-g'(u)]^2}. \\
&= r(u)e^{-\int_u^t a(s)ds}.
\end{aligned}$$

Also, with $z = u - g(u)$ we obtain

$$\begin{aligned}
V &= \int x'(u-g(u))(1-g'(u))du \\
&= \int x'(z)dz \\
&= x(u-g(u)).
\end{aligned}$$

We therefore have that

$$\begin{aligned}
&\int_0^t c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du \\
&= \frac{c(t)}{1-g'(t)}x(t-g(t)) - \frac{c(0)\psi(-g(0))}{1-g'(0)}e^{-\int_0^t a(s)ds} \\
&\quad - \int_0^t r(u)e^{-\int_u^t a(s)ds}x(u-g(u))du,
\end{aligned} \tag{4.16}$$

where $r(u)$ is given by (4.12). Then substituting (4.16) into (4.15) gives the desired results. Since each step in the above work is reversible, the proof is complete.

In order to employ Theorem 2.3.7 we need to define an operator P that is a sum of two operators, one of which is completely continuous and the other a large contraction. Thus let $(P\varphi)(t) = A\varphi(t) + B\varphi(t)$ where $A, B : \mathbb{M} \to \mathbb{M}$ are defined by

$$\begin{aligned}
(B\varphi)(t) &= \Big[\psi(0) - \frac{c(0)\psi(-g(0))}{1-g'(0)}\Big]e^{-\int_0^t a(s)ds} \\
&\quad + \int_0^t a(u)H(\varphi(u))e^{-\int_u^t a(s)ds}du,
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
(A\varphi)(t) &= \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) - \int_0^t r(u)\varphi(u-g(u))e^{-\int_u^t a(s)ds}du \\
&\quad + \int_0^t b(u)q(\varphi(u-g(u)))e^{-\int_u^t a(s)ds}du
\end{aligned} \tag{4.18}$$

respectively.

4.3 Asymptotic Stability

In this section we state and prove our main results of this Chapter.

Lemma 4.3.1. Suppose that conditions (4.3), (4.6)-(4.10) hold. Then for the map A defined in (4.18), $|A\varphi(t)| \le L/J < L$. Moreover, $A\varphi(t) \to 0$ as $t \to \infty$.

Proof. Let $\varphi \in \mathbb{M}$. Using the expression of the map A and the conditions (4.6)-(4.9) we have that

$$
\begin{aligned}
|A\varphi(t)| &\le \Big|\frac{c(t)}{1-g'(t)}\varphi(t-g(t))\Big| + \int_0^t |r(u)\varphi(u-g(u))| e^{-\int_u^t a(s)ds} du \\
&\quad + \int_0^t |b(u)q(\varphi(u-g(u)))| e^{-\int_u^t a(s)ds} du \\
&\le \alpha L + L\delta \int_0^t a(u) e^{-\int_u^t a(s)ds} du \\
&\quad + L\rho \int_0^t a(u) e^{-\int_u^t a(s)ds} du \\
&= \alpha L + L\delta \int_0^t \frac{d}{du}(e^{-\int_u^t a(s)ds}) du \\
&\quad + L\rho \int_0^t \frac{d}{du}(e^{-\int_u^t a(s)ds}) du \\
&\le L(\alpha+\delta+\rho) \\
&\le \frac{L}{J} < L.
\end{aligned}
$$

Thus showing that $A\varphi(t)$ is bounded by L.

We will next prove that $A\varphi(t) \to 0$ as $t \to \infty$. The first term on the right hand side of the map A tends to zero by the condition that $t - g(t) \to \infty$ as $t \to \infty$ and the fact that $\varphi \in \mathbb{M}$. Finally we show that the remaining integral terms goes to zero as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - g(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T > 0$ such that $t \ge T$ implies $|\varphi(t-g(t))| < \varepsilon$. Thus for $t \ge T$ we have

$$
\begin{aligned}
&\Big|\int_0^t \Big[b(u)q(\varphi(u-g(u))) - r(u)\varphi(u-g(u))\Big] e^{-\int_u^t a(s)ds} du\Big| \\
&\le \int_0^T \Big|b(u)q(\varphi(u-g(u))) - r(u)\varphi(u-g(u))\Big| e^{-\int_u^t a(s)ds} du \\
&+ \int_T^t \Big|b(u)q(\varphi(u-g(u))) - r(u)\varphi(u-g(u))\Big| e^{-\int_u^t a(s)ds} du \\
&\qquad \le L(\rho+\delta) e^{-\int_T^t a(s)ds} + \varepsilon(\rho+\delta).
\end{aligned}
$$

In view of condition (4.10), the term $L(\rho+\delta)e^{-\int_T^t a(s)ds}$ is arbitrarily small. This completes the proof.

Lemma 4.3.2. Suppose (4.6)-(4.10) hold. Suppose also that

$$\int_0^t \left[|a(u)||H(\varphi(u))|\right] e^{-\int_u^t a(r)dr} du \leq \frac{2L}{3}. \tag{4.19}$$

For B, A defined by (4.17) and (4.18), if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then

$$||A\varphi + B\psi|| \leq L.$$

Moreover, $B\varphi \to 0$ as $t \to \infty$.

Proof. Let $L > 0$ be given. Choose $\gamma > 0$ such that

$$\Big(1 - \Big|\frac{c(0)}{1-g'(0)}\Big|\Big)\gamma e^{-\int_0^t a(s)ds} + \frac{L}{J} + \frac{2L}{3} \leq L. \tag{4.20}$$

Let $\psi : [n_0, 0] \to \mathbb{R}$ be a given initial function such that $||\psi|| \leq \gamma$. Also let $\varphi, \psi \in \mathbb{M}$ be arbitrary. Using the definition of B and the result of Lemma 4.2.1, we obtain

$$\begin{aligned}
&|(A\varphi)(t) + (B\phi)(t)| \\
&\leq \Big|\frac{c(t)}{1-g'(t)}\varphi(t-g(t))\Big| + \int_0^t |r(u)\varphi(u-g(u))| e^{-\int_u^t a(s)ds} du \\
&\quad + \int_0^t |b(u)q(\varphi(u-g(u)))| e^{-\int_u^t a(s)ds} du \\
&\quad + \Big|\psi(0) - \frac{c(0)\psi(-g(0))}{1-g'(0)}\Big| e^{-\int_0^t a(s)ds} \\
&\quad + \int_0^t a(u)|H(\phi(u))| e^{-\int_u^t a(s)ds} du \\
&\leq \Big(1 - \Big|\frac{c(0)}{1-g'(0)}\Big|\Big)\gamma e^{-\int_0^t a(s)ds} + \frac{L}{J} + \frac{2L}{3} \\
&\leq L.
\end{aligned}$$

Thus $||A\varphi + B\psi|| \leq L$.

Since $0 \in \mathbb{M}$, we have also proved that $|B\psi(t)| \leq L$.

We next show that $B\varphi \to 0$ as $t \to \infty$. The first term on the right hand side of the map B tends to zero because of condition (4.10). Now, let $\varphi \in \mathbb{M}$. Then suppose that in view of the fact that $\varphi(t) \to 0$ as $t \to \infty$, we can find a $T \geq 0$ such that for $t \geq T$ there exist an $\varepsilon > 0$ such that $|\varphi(t) - h(\varphi(t))| \leq \varepsilon$. Then

$$\begin{aligned}
&\Big| \int_0^t a(u)H(\varphi(u))e^{-\int_u^t a(s)ds}du \Big| \\
&\leq \int_0^T |a(u)H(\varphi(u))|e^{-\int_u^t a(s)ds}du \\
&\quad + \int_T^t |a(u)H(\varphi(u))|e^{-\int_u^t a(s)ds}du \\
&\leq \frac{2L}{3}e^{-\int_T^t a(s)ds} + \varepsilon
\end{aligned}$$

In view of condition (4.10), the term $\frac{2L}{3}e^{-\int_T^t a(s)ds}$ is arbitrarily small. This completes the proof.

Lemma 4.3.3. Suppose that (4.3), (4.6)-(4.9) hold. Then, the mapping A is continuous on $\mathbb{M}$.

Proof. Let $\varepsilon > 0$ be given. Choose $\eta = \varepsilon J$ such that $||\varphi - \phi|| \leq \eta$.

$$\begin{aligned}
|(A\varphi)(t) - (A\phi)(t)| &\leq \alpha|\varphi(t-g(t)) - \phi(t-g(t))| \\
&\quad + \Big| \int_0^t r(u)[\varphi(u-g(u)) - \phi(u-g(u))] \Big| e^{-\int_u^t a(s)ds}du \\
&\quad + \Big| \int_0^t b(u)[q(\varphi(u-g(u))) - q(\phi(u-g(u)))] \Big| \\
&\times \; e^{-\int_u^t a(s)ds}du \\
&\leq \alpha||\varphi - \phi|| \\
&\quad + \delta \int_0^t a(u)\Big| e^{-\int_u^t a(s)ds}du||\varphi - \phi|| \\
&\quad + \rho \int_0^t a(u)e^{-\int_u^t a(s)ds}du|||\varphi - \phi|| \\
&\leq (\alpha + \delta + \rho)||\varphi - \phi|| \\
&\leq \frac{1}{J}||\varphi - \phi|| \\
&\leq \varepsilon.
\end{aligned}$$

Therefore A is continuous. This completes the proof.

Lemma 4.3.4. Suppose that (4.3), (4.4)-(4.8) hold. Then, the operator A maps $\mathbb{M}$ into a compact subset of $\mathbb{M}$.

Proof. It follows from Lemma 4.3.1 that $||A(\varphi_n)|| \leq L$, and so the family $A\varphi$ of functions are uniformly bounded. We next show that $A\varphi$ is equicontinuous. Let $\varepsilon > 0$ be given. Choose $\delta_1 = \varepsilon/K$ where $K = Lk_3 + 3L\delta k_1 + 3L\rho k_1$ such that $|t_2 - t_1| < \delta_1$. Let $\varphi \in \mathbb{M}$ and let $0 \leq t_1 < t_2$. Then,

$$
\begin{aligned}
& |A\varphi_n(t_2) - A\varphi_n(t_1)| \\
\leq\ & \Big|\frac{c(t_2)}{1-g'(t_2)}\varphi_n(t_2 - g(t_2)) - \frac{c(t_1)}{1-g'(t_1)}\varphi_n(t_1 - g(t_1))\Big| \\
& + \Big|\int_0^{t_2} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_2} a(s)ds}du \\
& - \int_0^{t_1} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_1} a(s)ds}du\Big| \\
& + \Big|\int_0^{t_2} |b(u)q(\varphi_n(u-g(u)))|e^{-\int_u^{t_2} a(s)ds}du \\
& - \int_0^{t_1} |b(u)q(\varphi_n(u-g(u)))|e^{-\int_u^{t_1} a(s)ds}du\Big|
\end{aligned}
\tag{4.21}
$$

By condition (4.5), we have that

$$
\begin{aligned}
& |\mu(t_2)\varphi_n(t_2 - g(t_2)) - \mu(t_1)\varphi_n(t_1 - g(t_1))\Big| \\
\leq\ & L|\mu(t_2) - \mu(t_1)| \\
\leq\ & Lk_3|t_2 - t_1|.
\end{aligned}
\tag{4.22}
$$

Moreover,

$$
\begin{aligned}
& \Big|\int_0^{t_2} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_2} a(s)ds}du - \int_0^{t_1} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_1} a(s)ds}du\Big| \\
=\ & \Big|\int_0^{t_1} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_2} a(s)ds}du + \int_{t_1}^{t_2} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_2} a(s)ds}du \\
& - \int_0^{t_1} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_1} a(s)ds}du\Big| \\
=\ & \Big|\int_0^{t_1} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_1} a(s)ds}\Big(e^{-\int_{t_1}^{t_2} a(s)ds} - 1\Big)du \\
& + \int_{t_1}^{t_2} r(u)\varphi_n(u-g(u))e^{-\int_u^{t_2} a(s)ds}du\Big| \\
\leq\ & L\Big|e^{-\int_{t_1}^{t_2} a(s)ds} - 1\Big|\int_0^{t_1} \delta a(u)e^{-\int_u^{t_1} a(s)ds}du + L\int_{t_1}^{t_2} |r(u)|e^{-\int_u^{t_2} a(s)ds}du \\
\leq\ & L\delta\int_{t_1}^{t_2} a(u)du + L\int_{t_1}^{t_2} e^{-\int_u^{t_2} a(s)ds}d\Big(\int_{t_1}^{u} |r(v)|dv\Big) \\
\leq\ & L\delta\int_{t_1}^{t_2} a(u)du + L\Big[e^{-\int_u^{t_2} a(s)ds}\Big(\int_{t_1}^{u} |r(v)|dv\Big)\Big]_{t_1}^{t_2}
\end{aligned}
$$

$$\begin{aligned}
&\quad + L\int_{t_1}^{t_2} a(u)e^{-\int_u^{t_2} a(s)ds}\int_{t_1}^{u}|r(v)|dvdu \\
&\leq \; L\delta\int_{t_1}^{t_2} a(u)du + L\int_{t_1}^{t_2}|r(u)|du\Big(1+\int_{t_1}^{t_2} a(u)e^{-\int_u^{t_2} a(s)ds}du\Big) \\
&\leq \; L\delta\int_{t_1}^{t_2} a(u)du + 2L\int_{t_1}^{t_2}|r(u)|du \\
&\leq \; L\delta\int_{t_1}^{t_2} a(u)du + 2\delta L\int_{t_1}^{t_2} a(u)du \\
&\leq \; 3L\delta k_1|t_2-t_1|. \qquad (4.23)
\end{aligned}$$

Also, by (4.4) and (4.7) we obtain

$$\begin{aligned}
&\quad \Big|\int_0^{t_2}|b(u)q(\varphi_n(u-g(u)))|e^{-\int_u^{t_2} a(s)ds}du \\
&\quad - \int_0^{t_1}|b(u)q(\varphi_n(u-g(u)))|e^{-\int_u^{t_1} a(s)ds}du\Big| \\
&= \; \Big|\int_0^{t_1} b(u)q(\varphi_n(u-g(u)))e^{-\int_u^{t_1} a(s)ds}\Big(e^{-\int_{t_1}^{t_2} a(s)ds}-1\Big)du \\
&\quad + \int_{t_1}^{t_2} b(u)q(\varphi_n(u-g(u)))e^{-\int_u^{t_2} a(s)ds}du\Big| \\
&\leq \; L\Big|e^{-\int_{t_1}^{t_2} a(s)ds}-1\Big|\int_0^{t_1}\rho a(u)e^{-\int_u^{t_1} a(s)ds}du + L\rho\int_{t_1}^{t_2}|b(u)|e^{-\int_u^{t_2} a(s)ds}du \\
&\leq \; L\rho\int_{t_1}^{t_2} a(u)du + L\rho\int_{t_1}^{t_2} e^{-\int_u^{t_2} a(s)ds}d\Big(\int_{t_1}^{u}|b(v)|dv\Big) \\
&\leq \; L\rho\int_{t_1}^{t_2} a(u)du + L\rho\Big(\Big[e^{-\int_u^{t_2} a(s)ds}\int_{t_1}^{u}|b(v)|dv\Big]_{t_1}^{t_2} \\
&\quad + \int_{t_1}^{t_2} a(u)e^{-\int_u^{t_2} a(s)ds}\int_{t_1}^{u}|b(v)|dvdu\Big) \\
&\leq \; L\rho\int_{t_1}^{t_2} a(u)du + L\rho\int_{t_1}^{t_2}|b(u)|du\Big(1+\int_{t_1}^{t_2} a(u)e^{-\int_u^{t_2} a(s)ds}du\Big) \\
&\leq \; L\rho\int_{t_1}^{t_2} a(u)du + 2L\rho\int_{t_1}^{t_2}|b(u)|du \\
&\leq \; L\rho\int_{t_1}^{t_2} a(u)du + 2L\rho\int_{t_1}^{t_2} a(u)du \\
&\leq \; 3L\rho k_1|t_2-t_1|. \qquad (4.24)
\end{aligned}$$

Substituting (4.22)-(4.24) into (4.21) gives

$$\begin{aligned}
&\quad |A\varphi_n(t_2)-A\varphi_n(t_1)| \\
&\leq \; Lk_3|t_2-t_1| + 3L\delta k_1|t_2-t_1| + 3L\rho k_1|t_2-t_1| \\
&\leq \; K|t_2-t_1| < \varepsilon. \qquad (4.25)
\end{aligned}$$

Therefore, $A\mathbb{M}$ is equicontinuous. Then by the Ascoli-Arzela theorem we that $A\mathbb{M}$

resides in a compact set. $\square$

In the next lemma we prove that H is a large contraction on $\mathbb{M}$. To this end we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous and differentiable on $U_L = [-L, L]$.

(H2) h is strictly increasing on U_L.

(H3) $\sup_{s \in U_L} h'(s) \le 1$.

(H4) $(s-r)\Big\{\sup_{t \in U_L} h'(t)\Big\} \ge h(s) - h(r) \ge (s-r)\Big\{\inf_{t \in U_L} h'(t)\Big\} \ge 0$ for $s, r \in U_L$ with $s \ge r$.

Lemma 4.3.5. Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H4)$. Then the mapping H is a large contraction on the set $\mathbb{M}$.

Proof. Let $\phi, \varphi \in \mathbb{M}$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{R}$. Define the set

$$D(\phi, \varphi) = \Big\{\, t \in \mathbb{R} : \phi(t) \neq \varphi(t) \Big\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{R}$ whenever $\varphi \in \mathbb{M}$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \tag{4.26}$$

holds for all $t \in D(\phi, \varphi)$. By (H3) we have

$$1 \ge \sup_{t \in U_L} h'(t) \ge \inf_{s \in U_L} h'(s) \ge 0. \tag{4.27}$$

Define the set $U_t \subset U_L$ by $U_t = [\varphi(t), \phi(t)] \cap U_L$ if $\phi(t) > \varphi(t)$, and $U_t = [\phi(t), \varphi(t)] \cap U_L$ if $\phi(t) < \varphi(t)$, for $t \in D(\phi, \varphi)$. Hence, for a fixed $t_0 \in D(\phi, \varphi)$ we get by (H4) and (4.26) that

$$\sup\{h'(u) : u \in U_{t_0}\} \ge \frac{h(\phi(t_0)) - h(\varphi(t_0))}{\phi(t_0) - \varphi(t_0)} \ge \inf\{h'(u) : u \in U_{t_0}\}.$$

Since $U_t \subset U_L$ for every $t \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_L} h'(u) \ge \sup\{h'(u) : u \in U_{t_0}\} \ge \inf\{h'(u) : u \in U_{t_0}\} \ge \inf_{u \in U_L} h'(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L} h'(u) \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \geq \inf_{u \in U_L} h'(u) \geq 0 \tag{4.28}$$

for all $t \in D(\phi, \varphi)$. So, (4.28) yields

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \Big| 1 - \Big(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \Big) \Big| \\ &\leq |\phi(t) - \varphi(t)| \Big(1 - \inf_{u \in U_L} h'(u) \Big) \end{aligned} \tag{4.29}$$

for all $t \in D(\phi, \varphi)$. Thus, (4.28) and (4.29) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [-L,L]} |\phi(t) - \varphi(t)| \geq \varepsilon.$$

If $|\phi(t) - \varphi(t)| \leq \varepsilon/2$ for some $t \in D(\phi, \varphi)$, then from (4.29)

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{4.30}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then from (4.28) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &\leq |\phi(t) - \varphi(t)| \Big\{ 1 - \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \Big\} \\ &\leq (1 - \lambda) \|\phi(t) - \varphi(t)\|, \end{aligned} \tag{4.31}$$

where

$$\lambda := \frac{1}{2L} \min \Big\{ h(u + \frac{\varepsilon}{2}) - h(u), u \in [-L, L] \Big\} > 0.$$

Consequently, it follows from (4.30) and (4.31) that

$$|(H\phi(t) - (H\varphi)(t)| \leq \delta \|\phi - \varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1$. The proof is complete.

The next result gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 4.3.6. If H is a large contraction on $\mathbb{M}$ then so is the mapping B.

Proof. If H is a large contraction on $\mathbb{M}$, then for $x, y \in \mathbb{M}$, with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. Thus, it follows from the equality

$$a(u)e^{-\int_u^t a(r)dr} = \frac{d}{du}[e^{-\int_u^t a(r)dr}],$$

that

$$\begin{aligned}
|Bx(t) - By(t)| &\leq \left(1 - e^{-\int_{t-T}^t a(s)ds}\right)^{-1} \times \int_{t-T}^t a(u)\big|H(x(u)) - H(y(u))\big|e^{-\int_u^t a(r)dr}du \\
&\leq \frac{\|x-y\|}{\left(1 - e^{-\int_{t-T}^t a(s)ds}\right)^{-1}} \int_{t-T}^t a(u)e^{-\int_u^t a(r)dr}du \\
&= \|x-y\|.
\end{aligned}$$

Thus,

$$\|Bx - By\| \leq \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \leq \delta\|x - y\|$$

holds if we know the existence of a $0 < \delta < 1$, such that for all $\varepsilon > 0$

$$\left[x, y \in \mathbb{M},\ \|x-y\| \geq \varepsilon\right] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete.

Theorem 4.3.7. Let L be a fixed positive number. Suppose that conditions (4.2), (4.4)-(4.10) hold. If ψ is a given initial function which is sufficiently small then there is a solution $x(t, 0, \psi)$ of (4.1) with $|x(t, 0, \psi)| \leq L$ and $x(t, 0, \psi) \to 0$ as $t \to \infty$.

Proof. From the hypothesis of Lemmas 4.3.1 we have that A is bounded by L, and $A\varphi(t) \to 0$ as $t \to \infty$. So, A maps $\mathbb{M}$ into $\mathbb{M}$. It also follows from Lemmas 4.3.2 that for arbitrary $\varphi, \phi \in \mathbb{M}$, $A\varphi + B\phi \in \mathbb{M}$, since both $A\varphi$ and $B\phi$ are bounded by L and $B\phi \to 0$ as $t \to \infty$. Also, in view of Lemmas 4.3.3 and 4.3.4, we have that A is continuous and $A\mathbb{M}$ resides in a compact set. Finally, B is a large contraction by Lemma 4.3.6. Thus, all the conditions of Theorem 2.3.7 are satisfied. Therefore, there exists a solution of (4.1) with $|x(t,0,\psi)| \leq L$ and $x(t,0,\psi) \to 0$ as $t \to \infty$. The proof is complete.

CHAPTER FIVE

Existence and positivity of solutions for nonlinear periodic differential equations

5.1 Introduction

Let $T > 0$ be fixed. We consider the non-linear neutral periodic equation

$$\begin{aligned} x'(t) &= -a(t)x^3(t) + c(t)x'(g(t))g'(t) + q(t, x^3(g(t))), \\ x(t) &= x(t+T). \end{aligned} \tag{5.1}$$

In recent years, there have been several papers written on the stability and periodicity of solutions for equations of forms similar to equation (5.1); see Burton (2002), Deham and Djoudi (2008) and Deham and Djoudi (2010). In the above mentioned papers, the nonlinear term q and the function a are assumed to be continuous in all arguments. The objective of this Chapter is to prove the existence and positivity of solutions of the periodic differential equation (5.1) by imposing much weaker conditions on the nonlinear term q and the argument function a. Specifically, we assume that q satisfies Carath$\acute{e}$odory conditions and $a \in L^1(\mathbb{R}, \mathbb{R})$.

The map $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carath$\acute{e}$odory conditions with respect to $L^1[0,T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,T]$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,T], R)$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|f(t,z)| \leq \alpha_r(t)$.

The rest of the Chapter is organized as follows. First, we present some preliminary material that we will employ to show the existence and positivity of solutions in this Chapter. In section three, we present our existence of periodic solutions results together with its proof. Finally, we obtain conditions for positivity of solutions in section 4.

Remark 5.1.1

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5.2 Preliminaries

Define the set $P_T = \{\phi \in C(\mathbb{R},\mathbb{R}) : \phi(t+T) = \phi(t)\}$ and the norm $||\phi|| = \sup_{t\in[0,T]} |\phi(t)|$, where C is the space of continuous real valued functions. Then $(P_T, ||.||)$ is a Banach space. In this Chapter we make the following assumptions.

(D1) $a \in L^1(\mathbb{R},\mathbb{R})$ is bounded and satisfies $a(t+T) = a(t)$ for all t and

$$1 - e^{-\int_{t-T}^{t} a(r)dr} \equiv \frac{1}{\rho} \neq 0.$$

(D2) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t.

(D3) $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies $g(t+T) = g(t)$ for all t.

(D4) q satisfies Carath$\acute{e}$odory conditions with respect to $L^1[0,T]$, and $q(t+T,x) = q(t,x)$.

Lemma 5.2.1. Suppose that conditions (D1), (D2), (D3), and (D4) hold. Then $x \in P_T$ is a solution of equation (5.1) if and only if, $x \in P_T$ satisfies

$$\begin{aligned} x(t) &= c(t)x(g(t)) + \rho \int_{t-T}^{t} a(u)[x(u) - x^3(u)]e^{-\int_u^t a(r)dr} du \\ &\quad + \rho \int_{t-T}^{t} [q(u, x^3(g(u))) - r(u)x(g(u))]e^{-\int_u^t a(r)dr} du \qquad (5.2) \end{aligned}$$

where $r(u) = a(u)c(u) + c'(u)$.

Proof. Let $x \in P_T$ be a solution of (5.1). We first rewrite (5.1) in the form

$$x'(t) + a(t)x(t) = a(t)x(t) - a(t)x^3(t) + c(t)x'(g(t))g'(t) + q(t, x^3(g(t))).$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate the resulting equation from $t - T$ to t. Thus we obtain,

$$\begin{aligned} x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} &= \int_{t-T}^{t} \Big[a(u)\big(x(u) - x^3(u)\big) \\ &\quad + c(u)x'(g(u))g'(u) \\ &\quad + q(u, x^3(g(u)))\Big] e^{\int_0^u a(s)ds} du. \end{aligned} \tag{5.3}$$

Dividing both sides of (5.3) by $e^{\int_0^t a(s)ds}$ and using the fact that $x \in P_T$ we obtain

$$\begin{aligned} x(t)\frac{1}{\rho} &= \int_{t-T}^{t} \Big[a(u)\big(x(u) - x^3(u)\big) + c(u)x'(g(u))g'(u) \\ &\quad + q(u, x^3(g(u)))\Big] e^{-\int_u^t a(s)ds} du. \end{aligned} \tag{5.4}$$

Integrating the second term on the right hand side of (5.4) by parts gives

$$\begin{aligned} \int_{t-T}^{t} c(u)x'(g(u))g'(u)e^{-\int_u^t a(s)ds} du &= c(t)x(g(t)) \\ &\quad - e^{-\int_{t-T}^t a(s)ds} c(t-T)x(g(t-T)) \\ &\quad - \int_{t-T}^{t} \frac{d}{du}\big[c(u)e^{-\int_u^t a(s)ds}\big] x(g(u))du. \end{aligned}$$

Since $c(t) = c(t-T), g(t) = g(t-T)$, and $x \in P_T$, then

$$\begin{aligned} \int_{t-T}^{t} c(u)x'(g(u))g'(u)e^{-\int_u^t a(s)ds} du &= \frac{1}{\rho} c(t)x(g(t)) \\ &\quad - \int_{t-T}^{t} \frac{d}{du}\big[c(u)e^{-\int_u^t a(s)ds}\big] x(g(u))du. \end{aligned} \tag{5.5}$$

Substituting the right hand side of (5.5) into (5.4) and simplifying gives the desired result.

The converse implication is easily obtained and the proof is complete.

5.3 Existence of periodic solution

In this section we state and prove our existence results. In view of this we first define the operator H by

$$\begin{aligned}(H\varphi)(t) &= c(t)\varphi(g(t)) + \rho\int_{t-T}^{t} a(u)[\varphi(u)-\varphi^3(u)]e^{-\int_u^t a(r)dr}du \\ &\quad + \rho\int_{t-T}^{t}[q(u,\varphi^3(g(u))) - r(u)\varphi(g(u))]e^{-\int_u^t a(r)dr}du, \qquad (5.6)\end{aligned}$$

where r is given in Lemma 5.2.1. It therefore follows from Lemma 5.2.1 that fixed points of H are solutions of (5.1) and vice versa.

In order to employ Theorem 2.3.7 we need to express the operator H as a sum of two operators, one of which is completely continuous and the other is a large contraction. Let $(H\varphi)(t) = A\varphi(t) + B\varphi(t)$ where $A, B : P_T \to P_T$ are defined by

$$(B\varphi)(t) = \rho\int_{t-T}^{t} a(u)[\varphi(u)-\varphi^3(u)]e^{-\int_u^t a(r)dr}du, \qquad (5.7)$$

and

$$(A\varphi)(t) = c(t)\varphi(g(t)) + \rho\int_{t-T}^{t}[q(u,\varphi^3(g(u))) - r(u)\varphi(g(u))]e^{-\int_u^t a(r)dr}du \qquad (5.8)$$

respectively.

Lemma 5.3.1. Suppose that conditions (D1), (D2), (D3), and (D4) hold. Then $A : P_T \to P_T$ is completely continuous.

Proof. It follows from (5.8) and conditions (D1), (D2), that $r(\sigma+T) = r(\sigma)$ and $e^{-\int_{\sigma+T}^{t+T} a(r)dr} = e^{-\int_\sigma^t a(u)du}$. Consequently, we have that

$$\begin{aligned}(A\varphi)(t+T) &= c(t+T)\varphi(g(t+T)) \\ &\quad + \rho\int_{t}^{t+T}[q(u,\varphi^3(g(u))) - r(u)\varphi(g(u))]e^{-\int_u^{t+T} a(r)dr}du \\ &= c(t)\varphi(g(t)) \\ &\quad + \rho\int_{t-T}^{t}[q(k+T,\varphi^3(g(k+T))) - r(k+T)\varphi(g(k+T))] \\ &\quad \times e^{-\int_{k+T}^{t+T} a(r)dr}du\end{aligned}$$

$$
\begin{aligned}
&= c(t)\varphi(g(t)) \\
&\quad + \rho \int_{t-T}^{t} [q(k, \varphi^3(g(k))) - r(k)\varphi(g(k))] e^{-\int_k^t a(s)ds} du \\
&= (A\varphi)(t).
\end{aligned}
$$

That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T.

To see that A is continuous let $\{\varphi_i\} \subset P_T$ be such that $\varphi_i \to \varphi$. By the Dominated Convergence Theorem,

$$
\begin{aligned}
& \lim_{i \to \infty} |A\varphi_i(t) - A\varphi(t)| \\
\leq\ & \lim_{i \to \infty} \Big(|c(t)||\varphi_i(g(t)) - \varphi(g(t))| \\
& + \rho \int_{t-T}^{t} \Big(\big| q(u, \varphi_i^3(g(u))) - q(u, \varphi^3(g(u))) \big| \\
& + |r(u)| \big| \varphi_i(g(u)) - \varphi(g(u)) \big| \Big) e^{-\int_u^t a(r)dr} du \Big) \to 0.
\end{aligned}
$$

Hence $A : P_T \to P_T$.

We next show that A is completely continuous. Let $Q \subset P_T$ be a closed bounded subset and let μ be such that $||\varphi|| \leq \mu$ for all $\varphi \in Q$. Then

$$
\begin{aligned}
|A\varphi(t)| &\leq \nu\mu + \rho \int_{t-T}^{t} \Big(|q(u, \varphi^3(g(u)))| + |r(u)||\varphi(g(u))| \Big) e^{-\int_u^t a(r)dr} du \\
&\leq \nu\mu + \rho N \Big(\int_{t-T}^{t} \alpha_\mu(u) du + \mu \int_{t-T}^{t} |r(u)| du \Big) \equiv K,
\end{aligned}
$$

where $\nu = \max_{t \in [0,T]} c(t)$ and $N = \max_{u \in [t-T,t]} e^{-\int_u^t a(r)dr}$. And so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in Q$. Without loss of generality, we can pick $\tau < t$ such that $t - \tau < T$. Then

$$
\begin{aligned}
& |A\varphi(t) - A\varphi(\tau)| \\
&= \Big| c(t)\varphi(t) + \rho \int_{t-T}^{t} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^t a(r)dr} ds \\
& - c(\tau)\varphi(\tau) - \rho \int_{\tau-T}^{\tau} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^\tau a(r)dr} ds \Big| \\
&= \Big| c(t)\varphi(t) + \rho \int_{t-T}^{\tau} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^t a(r)dr} ds \\
& + \rho \int_{\tau}^{t} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^t a(r)dr} ds \\
& - c(\tau)\varphi(\tau) - \rho \int_{\tau-T}^{t-T} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^\tau a(r)dr} ds
\end{aligned}
$$

$$
\begin{aligned}
&- \rho \int_{t-T}^{\tau} \Big(q(s, \varphi^3(g(s))) - r(s)\varphi(g(s)) \Big) e^{-\int_s^{\tau} a(r)dr} ds \Big| \\
&\leq \Big| c(t)\varphi(t) - c(\tau)\varphi(\tau) \Big| + \rho \int_{\tau}^{t} \Big(|q(s, \varphi^3(g(s)))| + |r(s)||\varphi(g(s))| \Big) e^{-\int_s^{t} a(r)dr} ds \\
&+ \rho \int_{t-T}^{\tau} \Big(|q(s, \varphi^3(g(s)))| + |r(s)||\varphi(g(s))| \Big) \Big| e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr} ds \Big| \\
&+ \rho \int_{\tau-T}^{t-T} \Big(|q(s, \varphi^3(g(s)))| + |r(s)||\varphi(g(s))| \Big) e^{-\int_s^{\tau} a(r)dr} ds \\
&\leq \Big| c(t)\varphi(t) - c(\tau)\varphi(\tau) \Big| + 2\rho N \Big(\int_{\tau}^{t} \alpha_{\mu}(s) + \mu |r(s)| ds \Big) \\
&+ \rho \int_{t-T}^{\tau} \Big(\alpha_{\mu}(s) + \mu |r(s)| \Big) \Big| e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr} \Big| ds.
\end{aligned}
$$

Now $|c(t)\varphi(t) - c(\tau)\varphi(\tau)\Big| \to 0$ and $\int_{\tau}^{t} \alpha_{\mu}(s) + \mu|r(s)|ds \to 0$ as $(t-\tau) \to 0$. Also, since

$$
\begin{aligned}
&\int_{t-T}^{\tau} \Big(\alpha_{\mu}(s) + \mu |r(s)| \Big) \Big| e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr} \Big| ds \\
&\leq \int_{0}^{T} \Big(\alpha_{\mu}(s) + \mu |r(s)| \Big) \Big| e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr} \Big| ds,
\end{aligned}
$$

and $|e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr}\Big| \to 0$ as $(t-\tau) \to 0$, then by the Dominated Convergence Theorem,

$$
\int_{t-T}^{\tau} \Big(\alpha_{\mu}(s) + \mu |r(s)| \Big) \Big| e^{-\int_s^{t} a(r)dr} ds - e^{-\int_s^{\tau} a(r)dr} \Big| ds \to 0
$$

as $(t-\tau) \to 0$. Thus $|A\varphi(t) - A\varphi(\tau)| \to 0$ as as $(t-\tau) \to 0$ independently of $\varphi \in Q$. It therefore follows that the family of $A\varphi$ is equicontinuous on Q.

By the Arzel$\grave{a}$-Ascoli Theorem, A is completely continuous and the proof is complete.

Lemma 5.3.2. Let $||.||$ be the supremum norm, and

$$
\mathbb{M} = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C, ||\varphi|| \leq \sqrt{3}/3\}.
$$

If $(F\varphi)(t) = \varphi(t) - \varphi^3(t)$. Then F is a large contraction of the set $\mathbb{M}$.

Proof. For each $t \in \mathbb{R}$ we have, for φ, ψ real functions,

$$\begin{aligned}|(F\varphi)(t)-(F\psi)(t)| &= |\varphi(t)-\varphi^3(t)-\psi(t)+\psi^3(t)| \\ &= |\varphi(t)-\psi(t)||1-(|\varphi^2(t)+\varphi(t)\psi(t)+\psi^2(t))|.\end{aligned}$$

Then for

$$|\varphi(t)-\psi(t)|^2 = \varphi^2(t)-2\varphi(t)\psi(t)+\psi^2(t)| \le 2(\varphi^2(t)+\psi^2(t))$$

and for $\varphi^2(t)+\psi^2(t) < 1$, we have

$$\begin{aligned}|(F\varphi)(t)-(F\psi)(t)| &= |\varphi(t)-\psi(t)|\Big[1-(\varphi^2(t)+\psi^2(t))+|\varphi(t)\psi(t)|\Big] \\ &\le |\varphi(t)-\psi(t)|\Big[1-(\varphi^2(t)+\psi^2(t))+\frac{\varphi^2(t)+\psi^2(t)}{2}\Big] \\ &\le |\varphi(t)-\psi(t)|\Big[1-\frac{\varphi^2(t)+\psi^2(t)}{2}\Big].\end{aligned}$$

Thus, we have shown that pointwise F is a large contraction. It is easy to see that this implies a large contraction in the supremum norm.

For a given $\varepsilon \in (0,1)$, let $\varphi, \psi \in \mathbb{M}$ with $||\varphi-\psi|| \ge \varepsilon$.

(a) Suppose that for some t we have $\varepsilon/2 \le |\varphi(t)-\psi(t)|$ so that

$$(\varepsilon/2)^2 \le |\varphi(t)-\psi(t)|^2 \le 2(\varphi^2(t)+\psi^2(t))$$

or

$$\varphi^2(t)+\psi^2(t) \ge \varepsilon^2/8.$$

For all such t we have

$$\begin{aligned}|(F\varphi)(t)-(F\psi)(t)| &\le |\varphi(t)-\psi(t)|\big[1-\frac{\varepsilon^2}{16}\big] \\ &\le ||\varphi-\psi|\big[1-\frac{\varepsilon^2}{16}\big].\end{aligned}$$

(b) Suppose that for some t, we have $|\varphi(t)-\psi(t)| \le \varepsilon/2$. Then

$$\begin{aligned}|(F\varphi)(t)-(F\psi)(t)| &\le |\varphi(t)-\psi(t)| \\ &\le (1/2)||\varphi-\psi||.\end{aligned}$$

Thus, for all t we have

$$|(F\varphi)(t)-(F\psi)(t)| \leq \min[1/2, 1-\frac{\varepsilon^2}{16}]||\varphi-\psi||.$$

The proof is complete.

For the rest of the Chapter we define

$$\mathbb{M} = \{\varphi \in P_T \mid ||\varphi|| \leq L\},$$

where $L = \sqrt{3}/3$.

We also need the following condition on the nonlinear term q.

(D5) There exists periodic functions $\alpha, \beta, \in L^1[0,T]$, with period T, such that

$$|q(t,x)| \leq \alpha(t)|x| + \beta(t),$$

for all $x \in \mathbb{R}$.

Lemma 5.3.3. Suppose that (D5) hold. Also suppose there exist constants $\lambda > 0, R > 0$, $J \geq 3$ and $\gamma > 0$ such that

$$|\alpha(t)|L^3 + |\beta(t)| \leq \lambda L a(t), \tag{5.9}$$

$$|r(t)| \leq R a(t), \tag{5.10}$$

$$\gamma = \max_{t\in[0,T]} |c(t)|, \tag{5.11}$$

and

$$J(\gamma+\lambda+R) \leq 1. \tag{5.12}$$

For A defined by (5.8), if $\varphi \in \mathbb{M}$, then $|(A\varphi)(t)| \leq L/J \leq L$ for all t.

Proof. Let $\varphi \in \mathbb{M}$. Then $||\varphi|| \leq L$. Thus for A defined by (5.8) we have that

$$\begin{aligned} |(A\varphi)(t)| &\leq |c(t)\varphi(g(t))| \\ &\quad + \rho \int_{t-T}^{t} |q(u,\varphi^3(g(u)))| e^{-\int_u^t a(r)dr} du \\ &\quad + \rho \int_{t-T}^{t} |r(u)\varphi(g(u))| e^{-\int_u^t a(r)dr} du \end{aligned}$$

It follows from conditions (D5), (5.9), (5.10), (5.11) and (5.12) that

$$\begin{aligned} |(A\varphi)(t)| &\leq \gamma L \\ &\quad + \rho \int_{t-T}^{t} [|\alpha(u)|L^3 + |\beta(u)|] e^{-\int_u^t a(r)dr} du \\ &\quad + \rho R \int_{t-T}^{t} a(u) L e^{-\int_u^t a(r)dr} du \\ &\leq \gamma L \\ &\quad + \rho \lambda L \int_{t-T}^{t} a(u) e^{-\int_u^t a(r)dr} du \\ &\quad + \rho R L \int_{t-T}^{t} a(u) e^{-\int_u^t a(r)dr} du \\ &\leq (\gamma + \lambda + R)L \leq \frac{L}{J} < L. \end{aligned}$$

Therefore A maps $\mathbb{M}$ into itself. This completes the proof.

Lemma 5.3.4. Suppose (D1), (D2), (D3), (D4) and (D5) hold. Suppose also that the hypotheses in Lemma 5.3.3 hold. For B, A defined by (5.7) and (5.8), if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then

$$A\varphi + B\psi : \mathbb{M} \to \mathbb{M}.$$

Moreover, B is a large contraction on M with a unique fixed point in $\mathbb{M}$.

Proof. Let $\varphi, \psi \in \mathbb{M}$ be arbitrary. Note that $|\psi(t)| \leq \sqrt{3}/3$ implies

$$|\psi(t) - \psi^3(t)| \leq (2\sqrt{3})/9.$$

Using the definition of B and the result of Lemma 5.3.3, we obtain

$$
\begin{aligned}
&|(A\varphi)(t)+(B\psi)(t)| \\
&\le |c(t)\varphi(g(t))|+\rho\int_{t-T}^{t}|q(u,\varphi^3(g(u)))|e^{-\int_u^t a(r)dr}du \\
&\quad +\rho\int_{t-T}^{t}|r(u)\varphi(g(u))|e^{-\int_u^t a(r)dr}du \\
&\quad +\left|\rho\int_{t-T}^{t}a(u)|\psi(u)-\psi^3(u)|e^{-\int_u^t a(r)dr}du\right| \\
&\le \frac{\sqrt{3}}{3J}+\frac{2\sqrt{3}}{9}\le L.
\end{aligned}
$$

Thus $A\varphi+B\psi\in\mathbb{M}$.

We will next show that B is a large contraction with a unique fixed point in $\mathbb{M}$. Lemma 5.3.2 shows that $\psi-\psi^3$ is a large contraction in the supremum norm. Thus for any ε, we found a $\delta<1$ from the proof of that proposition such that

$$
\begin{aligned}
|(B\varphi)(t)-(B\psi)(t)| &\le \rho\int_{t-T}^{t}a(u)\delta||\varphi-\psi||e^{-\int_u^t a(r)dr}du \\
&\le \delta||\varphi-\psi||.
\end{aligned}
$$

Furthermore, since $0\in\mathbb{M}$ the above inequality shows that, $B:\mathbb{M}\to\mathbb{M}$ when $\psi=0$. This completes the proof.

Theorem 5.3.5. Let $(P_T,||.||)$ be the Banach space of continuous T-periodic real functions and $\mathbb{M}=\{\varphi\in P_T\mid ||\varphi||\le L\}$, where $L=\sqrt{3}/3$. Suppose (D1), (D2), (D3), (D4), (D5) and (5.9)-(5.12) hold. Then equation (5.1) possesses a periodic solution φ in the subset $\mathbb{M}$.

Proof. By Lemma 5.2.1, φ is a solution of (5.1) if

$$\varphi=A\varphi+B\varphi,$$

where B and A are given by (5.7) and (5.8) respectively. By Lemma 5.3.1, $A:\mathbb{M}\to\mathbb{M}$ is completely continuous. By Lemma 5.3.4, $A\varphi+B\psi\in\mathbb{M}$ whenever $\varphi,\psi\in\mathbb{M}$. Moreover, $B:\mathbb{M}\to\mathbb{M}$ is a large contraction. Thus all the hypotheses of Theorem 2.3.7 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi\in\mathbb{M}$ such that $\varphi=A\varphi+B\varphi$. Hence (5.1) has a $T-$ periodic solution. This completes the proof.

5.4 Existence of positive solutions

In this section we obtain sufficient conditions under which there exists positive solutions of (5.1). We begin by defining some quantities. Let

$$z \equiv \min_{s\in[t-T,t]} e^{-\int_s^t a(r)dr}, \quad Z \equiv \max_{s\in[t-T,t]} e^{-\int_s^t a(r)dr}.$$

Given constants $0 < L < K$, define the set $\mathbb{M}_p = \{\psi \in P_T \,:\, L \le \psi(t) \le K, t \in [0,T]\}$.

In this section we make the following assumptions.

(D6) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t and there exists a $c^* > 0$ such that $c^* < c(t)$ for all $t \in [0,T]$.

(D7) There exits α such that $||c|| \le \alpha < 1$.

(D8) There exists constants $0 < L < K$ such that

$$\frac{(1-c^*)L}{\rho zT} \le a(u)[\sigma - \sigma^3] + q(u,\sigma^3) - r(u)\sigma \le \frac{(1-\alpha)K}{\rho ZT}$$

for all $\sigma \in \mathbb{M}$ and $u \in [t-T,t]$.

Theorem 5.4.1. Suppose that conditions (D1), (D3), (D4), (D6), (D7) and (D8) hold. Then there exists a positive solution of (5.1).

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned} A\varphi(t) + B\psi(t) &= c(t)\varphi(g(t)) + \rho\int_{t-T}^{t}\Big[a(u)[\psi(u) - \psi^3(u)] \\ &\quad + q(u,\varphi^3(g(u))) - r(u)\varphi(g(s))\Big]e^{-\int_u^t a(r)dr}du \\ &\ge c^*L + \rho zT\frac{(1-c^*)L}{\rho zT} = L. \end{aligned}$$

Likewise,

$$A\varphi(t) + B\psi(t) \le \alpha K + \rho ZT\frac{(1-\alpha)K}{\rho ZT} = K.$$

Thus condition (i) of Theorem 2.3.6 is satisfied. From Lemma 5.3.1 the operator A is completely continuous and from Lemma 5.3.4 the operator B is a large contraction. Therefore, by Theorem 2.3.7, the operator H has a fixed point in $\mathbb{M}_p$. This fixed point is a positive solution of (5.1).

CHAPTER SIX

POSITIVE PERIODIC SOLUTIONS FOR NEUTRAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

6.1 Introduction

Motivated by the work of Zeng (1997), Li and Shen (1997), Wang (1999) and Lui and Ge (2003) on the existence of periodic solutions of second order differential equations we consider the second order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=c\frac{d}{dt}x(t-\tau(t))+f(t,h(x(t)),g(x(t-\tau(t)))), \tag{6.1}$$

where p and q are positive continuous real-valued functions. The function $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous in its respective arguments and $c\in\mathbb{R}$. None of the above papers considered neutral second order differential equations. In fact, the equations that were considered by Zeng (1997), Li and Shen (1997), and Wang (1999) were without delays. However, Lui and Ge (2003) extended the results on the existence of periodic solutions to second order equations with delay. Results on the existence of positive periodic solutions for second order neutral delay differential equations of the form of (6.1) are not available. Thus, our main objective in this chapter is to fill this gap, by obtaining sufficient conditions for the existence of positive periodic solutions for neutral differential equation of the second order.

The rest of the Chapter is organized as follows. We introduce our notation and state some preliminary results in section two. In section three, we state our results and provide its proof. In the last section, we extend the results obtained in section three to totally nonlinear neutral delay differential equations of the second order.

Remark 6.1. The content of this Chapter has been published as:

E. Yankson, ”Positive Periodic Solutions for Neutral Differential Equations of the Second-Order,” Electronic journal of differential equations, No. 14, 2012.

6.2 Preliminaries

For $T > 0$, let P_T be the set of continuous scalar functions x that are periodic in t, with period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

In this Chapter we make the following assumptions.

$$p(t+T) = p(t), \quad q(t+T) = q(t), \quad \tau(t+T) = \tau(t), \tag{6.2}$$

with τ being scalar function, continuous, and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T p(s)ds > 0, \ \int_0^T q(s)ds > 0. \tag{6.3}$$

We also assume that $f(t,h,g)$ is periodic in t with period T; that is,

$$f(t+T,h,g) = f(t,h,g). \tag{6.4}$$

Next we state some lemmas that will be relevant for our work in this Chapter.

Lemma 6.2.1.[Liu and Ge (2004)] Suppose that (6.2) and (6.3) hold and

$$\frac{R_1[\exp(\int_0^T p(u)du) - 1]}{Q_1 T} \geq 1, \tag{6.5}$$

where

$$R_1 = \max_{t\in[0,T]} \Big| \int_t^{t+T} \frac{\exp(\int_t^s p(u)du)}{\exp(\int_0^T p(u)du) - 1} q(s)ds \Big|,$$
$$Q_1 = \Big(1 + \exp\Big(\int_0^T p(u)du\Big)\Big)^2 R_1^2.$$

Then there are continuous and T-periodic functions a and b such that $b(t) > 0$, $\int_0^T a(u)du > 0$, and

$$a(t) + b(t) = p(t), \quad \frac{d}{dt}b(t) + a(t)b(t) = q(t), \quad \text{for } t \in \mathbb{R}.$$

Lemma 6.2.2.[Wang, Lian and Ge (2007)] Suppose the conditions of Lemma 6.1.1 hold and $\phi \in P_T$. Then the equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u b(v)dv + \int_u^s a(v)dv]du + \int_s^{t+T} \exp[\int_t^u b(v)dv + \int_u^{s+T} a(v)dv]du}{[\exp\big(\int_0^T a(u)du\big) - 1][\exp\big(\int_0^T b(u)du\big) - 1]}.$$

Corollary 6.2.3.[Wang, Lian and Ge (2007)] Green's function G satisfies the following properties

$$\begin{aligned} G(t,t+T) = G(t,t), \quad & G(t+T,s+T) = G(t,s), \\ \frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - & \frac{\exp\big(\int_t^s b(v)dv\big)}{\exp\big(\int_0^T b(v)dv\big) - 1}, \\ \frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + & \frac{\exp\big(\int_t^s a(v)dv\big)}{\exp\big(\int_0^T a(v)dv\big) - 1}. \end{aligned}$$

We next state and prove the following lemma which will play an essential role in obtaining our results.

Lemma 6.2.4. Suppose (6.2)-(6.5) hold. If $x \in P_T$, then x is a solution of (6.1) if and only if

$$\begin{aligned} x(t) = & \int_t^{t+T} cE(t,s)x(s-\tau(s))ds \\ & + \int_t^{t+T} G(t,s)[-a(s)cx(s-\tau(s)) + f(s,h(x(s)),g(x(s-\tau(s))))]ds, \end{aligned} \tag{6.6}$$

where

$$E(t,s) = \frac{\exp(\int_t^s b(v)dv)}{\exp(\int_0^T b(v)dv) - 1}. \tag{6.7}$$

Proof. Let $x \in P_T$ be a solution of (6.1). From Lemma 6.1.2, we have

$$x(t) = \int_t^{t+T} G(t,s)[c\frac{\partial}{\partial s}x(s-\tau(s)) + f(s,h(x(s)),g(x(s-\tau(s))))]ds. \tag{6.8}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_t^{t+T} cG(t,s)\frac{\partial}{\partial s}x(s-\tau(s))ds \\
&= -\int_t^{t+T} c[\frac{\partial}{\partial s}G(t,s)]x(s-\tau(s))ds \\
&= \int_t^{t+T} cx(s-\tau(s))[E(t,s)-a(s)G(t,s)]ds,
\end{aligned} \tag{6.9}$$

where E is given by (6.7). Then substituting (6.9) in (6.8) completes the proof.

Lemma 6.2.5.[Wang, Lian and Ge (2007)] Let $A = \int_0^T p(u)du$,
$B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$. If

$$A^2 \geq 4B, \tag{6.10}$$

then

$$\begin{aligned}
\min\Big\{\int_0^T a(u)du, \int_0^T b(u)du\Big\} \geq \frac{1}{2}(A-\sqrt{A^2-4B}) := l, \\
\max\Big\{\int_0^T a(u)du, \int_0^T b(u)du\Big\} \leq \frac{1}{2}(A+\sqrt{A^2-4B}) := m.
\end{aligned}$$

Corollary 6.2.6.[Wang, Lian and Ge (2007)] Functions G and E satisfy

$$\frac{T}{(e^m-1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2}, \quad |E(t,s)| \leq \frac{e^m}{e^l-1}.$$

To simplify notation, we introduce the constants

$$\beta = \frac{e^m}{e^l-1}, \quad \alpha = \frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2}, \quad \gamma = \frac{T}{(e^m-1)^2}. \tag{6.11}$$

6.3 Existence of positive periodic solutions

We present our existence results in this section by considering two cases; $c \geq 0$, $c \leq 0$. For some non-negative constant K and a positive constant L we define the set

$$\mathbb{D} = \{\varphi \in P_T : K \leq \varphi \leq L\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that there exist a positive constant σ such that

$$\sigma < E(t,s), \quad \text{for all } (t,s) \in [0,T]\times[0,T], \tag{6.12}$$

$$c \geq 0 \tag{6.13}$$

and for all $s \in \mathbb{R}$, $\mu \in \mathbb{D}$

$$\frac{K(1-\sigma cT)}{\gamma T} \leq f(s,h(\mu),g(\mu)) - ca(s)\mu \leq \frac{L(1-\beta cT)}{\alpha T}. \tag{6.14}$$

To apply Theorem 2.3.6, we construct two mappings in which one is a contraction and the other is completely continuous. Thus, we define the map $\mathcal{A} : \mathbb{D} \to P_T$ by

$$\begin{aligned} &(\mathcal{A}\varphi)(t) \\ &= \int_t^{t+T} G(t,s)[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) - ca(s)\varphi(s-\tau(s))]ds. \end{aligned} \tag{6.15}$$

Similarly, we define the map $\mathcal{B} : \mathbb{D} \to P_T$ by

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds. \tag{6.16}$$

Lemma 6.3.1. If $\mathcal{B}$ is given by (6.16) with

$$c\beta T < 1, \tag{6.17}$$

then $\mathcal{B} : \mathbb{D} \to P_T$ is a contraction.

Proof. Let $\varphi \in \mathbb{D}$, then

$$(\mathcal{B}\varphi)(t+T) \;=\; \int_{t+T}^{t+2T} cE(t+T,s)\varphi(s-\tau(s))ds.$$

With $k = s - T$, we obtain

$$\begin{aligned} (\mathcal{B}\varphi)(t+T) &= \int_t^{t+T} cE(t+T,k+T)\varphi(k+T-\tau(k+T))dk \\ &= \int_t^{t+T} cE(t,k)\varphi(k-\tau(k))dk \\ &= (\mathcal{B}\varphi)(t). \end{aligned}$$

Let $\varphi, \psi \in \mathbb{D}$ then

$$\begin{aligned} \|\mathcal{B}\varphi - \mathcal{B}\psi\| &= \sup_{t\in[0,T]} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \\ &= \sup_{t\in[0,T]} \Big| \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds - \int_t^{t+T} cE(t,s)\psi(s-\tau(s))ds \Big| \\ &\leq c\beta \int_t^{t+T} \|\varphi-\psi\| ds \\ &\leq c\beta T \|\varphi - \psi\|. \end{aligned}$$

Hence $\mathcal{B} : P_T \to P_T$ is a contraction.

Lemma 6.3.2. Suppose that conditions (6.2)-(6.4), and (6.12)-(6.14),(6.17) hold. Then $\mathcal{A} : P_T \to P_T$ is completely continuous on $\mathbb{D}$.

Proof. Let $\mathcal{A}$ be defined by (6.15) and $\varphi \in \mathbb{D}$. Then

$$\begin{aligned}(\mathcal{A}\varphi)(t+T) &= \int_{t+T}^{t+2T} G(t+T,s)[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) \\ &\quad -ca(s)\varphi(s-\tau(s))]ds.\end{aligned}$$

With $k = s-T$, we obtain

$$\begin{aligned}(\mathcal{A}\varphi)(t+T) &= \int_{t}^{t+T} G(t+T,k+T) \\ &\quad \times[f(k+T,h(\varphi(k+T)),g(\varphi(k+T-\tau(k+T)))) \\ &\quad -ca(k+T)\varphi(k+T-\tau(k+T))]dk \\ &= \int_{t}^{t+T} G(t,k) \\ &\quad \times[f(k,h(\varphi(k)),g(\varphi(k-\tau(k)))) \\ &\quad -ca(k)\varphi(k-\tau(k))]dk \\ &= (\mathcal{A}\varphi)(t).\end{aligned}$$

For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$ we have that

$$\begin{aligned}|(\mathcal{A}\varphi)(t)| &\leq |\int_{t}^{t+T} G(t,s)[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) \\ &\quad -ca(s)\varphi(s-\tau(s))]ds| \\ &\leq \int_{t}^{t+T} \alpha\frac{L(1-\beta cT)}{\alpha T}ds \\ &\leq T\alpha\frac{L(1-\beta cT)}{\alpha T} = L(1-\beta cT).\end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have

$$\|\mathcal{A}\varphi\| \leq L(1-\beta cT).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. We next show that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi \in \mathbb{D}$. By using (6.2), (6.3) and (6.4) we obtain by taking the derivative in (6.15) that

$$
\begin{aligned}
\frac{d}{dt}(\mathcal{A}\varphi)(t) &= G(t,t+T)[f(t+T,h(\varphi(t+T)),g(\varphi(t+T-\tau(t+T))))\\
&\quad - ca(t+T)\varphi(t+T-\tau(t+T))]\\
&\quad - G(t,t)[f(t,h(\varphi(t)),g(\varphi(t-\tau(t))))\\
&\quad - ca(t)\varphi(t-\tau(t))]\\
&\quad + \int_t^{t+T}\frac{\partial}{\partial t}G(t,s)\\
&\quad \times[f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\\
&\quad -ca(s)\varphi(s-\tau(s))]ds\\
&= G(t,t)[f(t,h(\varphi(t)),g(\varphi(t-\tau(t))))\\
&\quad - ca(t)\varphi(t-\tau(t))]\\
&\quad - G(t,t)[f(t,h(\varphi(t)),g(\varphi(t-\tau(t))))\\
&\quad - ca(t)\varphi(t-\tau(t))]\\
&\quad + \int_t^{t+T}[-b(t)G(t,s)+\frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right)-1}]\\
&\quad \times[f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\\
&\quad -ca(s)\varphi(s-\tau(s))]ds\\
&= \int_t^{t+T}[-b(t)G(t,s)+\frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right)-1}]\\
&\quad \times[f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\\
&\quad -ca(s)\varphi(s-\tau(s))]ds.
\end{aligned}
$$

Consequently, by invoking (6.11), and (6.14), we obtain

$$|\frac{d}{dt}(\mathcal{A}\varphi)(t)| \le T(\|b\|\alpha+\beta)\frac{L(1-\beta cT)}{\alpha T} \le M,$$

for some positive constant M. Hence $(\mathcal{A}\varphi)$ is equicontinuous. Then by the Ascoli-Arzela theorem we obtain that $\mathcal{A}$ is a compact map. Due to the continuity of all the terms in (6.15), we have that $\mathcal{A}$ is continuous. This completes the proof.

Theorem 6.3.3. Let α, β and γ be given by (6.11). Suppose that conditions (6.2)-(6.5), (6.10),(6.13),(6.14) and (6.17) hold, then Equation (6.1) has a positive periodic solution z satisfying $K \le z \le L$.

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (6.15) and (6.16) we obtain

$$
\begin{aligned}
&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\
&= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s,h(\psi(s)),g(\psi(s-\tau(s)))) \\
&\quad - ca(s)\psi(s-\tau(s))]ds \\
&\leq c\beta LT + \alpha \int_t^{t+T} [f(s,h(\psi(s)),g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds \\
&\leq c\beta LT + \alpha T \frac{L(1-\beta cT)}{\alpha T} = L.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\
&= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s,h(\psi(s)),g(\psi(s-\tau(s)))) \\
&\quad - ca(s)\psi(s-\tau(s))]ds \\
&\geq c\sigma KT + \gamma \int_t^{t+T} [f(s,h(\psi(s)),g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds \\
&\geq c\sigma KT + \gamma T \frac{K(1-\sigma cT)}{\gamma T} = K.
\end{aligned}
$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Thus all the hypotheses of Theorem 2.3.6 are satisfied and therefore equation (6.1) has a periodic solution in $\mathbb{D}$. This completes the proof.

We next consider the case when $c \leq 0$. To this end we substitute conditions (6.13) and (6.14) with the following conditions respectively.

$$c \leq 0 \tag{6.18}$$

and for all $s \in \mathbb{R}, \mu \in \mathbb{D}$

$$\frac{K - c\beta LT}{\gamma T} \leq f(s,h(\mu),g(\mu)) - ca(s)\mu \leq \frac{L - c\sigma KT}{\alpha T}. \tag{6.19}$$

Theorem 6.3.4. Let α, β and γ be given by (6.11). Suppose that conditions (6.2)-(6.5), (6.10),(6.17), (6.18), and (6.19) hold, then (6.1) has a positive periodic solution z satisfying $K \leq z \leq L$.

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (6.15) and (6.16) we obtain

$$
\begin{aligned}
&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\
&= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s,h(\psi(s)),g(\psi(s-\tau(s)))) \\
&\quad - ca(s)\psi(s-\tau(s))]ds \\
&\leq c\sigma LT + \alpha \int_t^{t+T} [f(s,h(\psi(s)),g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds \\
&\leq c\sigma LT + \alpha T\Big(\frac{L - c\sigma KT}{\alpha T}\Big) = L.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\
&= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s,h(\psi(s)),g(\psi(s-\tau(s)))) \\
&\quad - ca(s)\psi(s-\tau(s))]ds \\
&\geq c\beta KT + \gamma \int_t^{t+T} [f(s,h(\psi(s)),g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds \\
&\geq c\beta KT + \gamma T\Big(\frac{K - c\beta LT}{\gamma T}\Big) = K.
\end{aligned}
$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Thus all the hypotheses of Theorem 2.3.6 are satisfied and therefore equation (6.1) has a periodic solution in $\mathbb{D}$. This completes the proof.

6.4 Totally nonlinear neutral second order differential equations

In this section we turn our attention to totally nonlinear second order differential equations. In particular, we establish sufficient conditions for the existence of positive periodic solution of the equation

$$
\begin{aligned}
\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)h(x(t)) &= \frac{d}{dt}c(t,x(t-\tau(t))) \\
&\quad + f(t,h(x(t)),g(x(t-\tau(t)))),
\end{aligned}
\tag{6.20}
$$

where p and q are positive continuous real-valued functions. The functions $f: \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, c:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $h:\mathbb{R}\to\mathbb{R}$, are continuous in their respective arguments.

Equation (6.20) is a more general form of (6.1)in the sense that, the neutral term $\frac{d}{dt}c(t,x(t-\tau(t)))$ in (6.20) produces non-linearity in the derivative term $\frac{d}{dt}x(t-\tau(t))$, whereas the neutral term in (6.1) enters linearly. Also, $h(x(t)) = x(t)$ in (6.1), thus making (6.20) totally nonlinear. In view of the above differences, the results obtained in the previous section does not carry over to equation (6.20).

In addition to the assumptions in section 6.1 we also assume that $c(t,x)$ is periodic in t with period T. That is,

$$c(t+T,x) = c(t,x). \tag{6.21}$$

We further assume that there exist positive constants σ, c^* and μ such that

$$\sigma < E(t,s), \text{ for all } (t,s) \in [0,T] \times [0,T], \tag{6.22}$$

$$c^* < c(t,t-\tau(t)), \tag{6.23}$$

$$\|c(t,x)\| \leq \mu, \tag{6.24}$$

$$\beta\mu T < L, \;\; c^*\sigma T < K, \tag{6.25}$$

and for all $s \in \mathbb{R}$, $\phi, \varphi \in \mathbb{D}$

$$\frac{K - c^*\sigma T}{\gamma T} \leq q(s)[\phi(s) - h(\phi(s))] + f(s,h(\varphi),g(\varphi)) - a(s)c(s,\varphi) \leq \frac{L - \beta\mu T}{\alpha T}. \tag{6.26}$$

We next state and prove the following lemma which will play an essential role in obtaining our results.

Lemma 6.4.1. Suppose (6.2)-(6.5) hold. If $x \in P_T$, then x is a solution of (6.20) if and only if

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)q(s)[x(s) - h(x(s))]ds \\ &+ \int_t^{t+T} \Big[c(s,x(s-\tau(s)))[E(t,s) - a(s)G(t,s)] \\ &+ G(t,s)f(s,h(x(s)),g(x(s-\tau(s))))\Big]ds, \end{aligned} \tag{6.27}$$

where $E(t,s)$ is given by (6.7).

Proof. Let $x \in P_T$ be a solution of (6.20). Rewrite (6.20) as

$$\begin{aligned}\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t) &= q(t)[x(t)-h(x(t))]+\frac{d}{dt}c(t,x(t-\tau(t))) \\ &\quad + f(t,h(x(t)),g(x(t-\tau(t)))).\end{aligned}$$

From Lemma 6.1.2, we have

$$\begin{aligned}x(t) &= \int_t^{t+T} G(t,s)q(s)[x(s)-h(x(s))]ds \\ &\quad + \int_t^{t+T} G(t,s)\Big[\frac{\partial}{\partial s}c(s,x(s-\tau(s)))+f(s,h(x(s)),g(x(s-\tau(s))))\Big]ds.\end{aligned} \tag{6.28}$$

Integrating by parts, we have

$$\begin{aligned}&\int_t^{t+T} G(t,s)\frac{\partial}{\partial s}c(s,x(s-\tau(s)))ds \\ &= -\int_t^{t+T}\Big[\frac{\partial}{\partial s}G(t,s)\Big]c(s,x(s-\tau(s)))ds \\ &= \int_t^{t+T} c(s,x(s-\tau(s)))[E(t,s)-a(s)G(t,s)]ds,\end{aligned} \tag{6.29}$$

where E is given by (6.7). Then substituting (6.29) in (6.28) completes the proof.

To apply theorem 2.3.7, we construct two mappings in which one is a large contraction and the other is completely continuous. Thus, we set the map $\mathcal{A}:\mathbb{D}\to\mathbb{D}$

$$\begin{aligned}(\mathcal{A}\varphi)(t) &= \int_t^{t+T}\Big[c(s,\varphi(s-\tau(s)))[E(t,s)-a(s)G(t,s)] \\ &\quad + G(t,s)f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\Big]ds.\end{aligned} \tag{6.30}$$

Similarly, we define the map $\mathcal{B}:\mathbb{D}\to\mathbb{D}$

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s)q(s)[\varphi(s)-h(\varphi(s))]ds. \tag{6.31}$$

In the next lemma we prove that $\mathcal{F}(\varphi(s)) = \varphi(s)-h(\varphi(s))$ is a large contraction on $\mathbb{D}$. To this end we make the following assumptions on the function $h:\mathbb{R}\to\mathbb{R}$.

(H1) h is continuous and differentiable on $U=[K,L]$.

(H2) h is strictly increasing on U.

(H3) $\sup_{s\in U} h'(s) \leq 1$.

(H4) $(s-r)\Big\{\sup_{t\in U} h'(t)\Big\} \geq h(s)-h(r) \geq (s-r)\Big\{\inf_{t\in U} h'(t)\Big\} \geq 0$ for $s,r\in U$ with $s\geq r$.

Lemma 6.4.2. Let $h:\mathbb{R}\to\mathbb{R}$ be a function satisfying $(H1)-(H4)$. Then $\mathcal{F} = \varphi(s)-h(\varphi(s))$ is a large contraction on the set $\mathbb{D}$.

Proof. Let $\phi,\varphi\in\mathbb{D}$ with $\phi\neq\varphi$. Then $\phi(t)\neq\varphi(t)$ for some $t\in\mathbb{R}$. Define the set

$$S(\phi,\varphi) = \Big\{\ t\in\mathbb{R} : \phi(t)\neq\varphi(t)\Big\}.$$

Note that $\varphi(t)\in U$ for all $t\in\mathbb{R}$ whenever $\varphi\in\mathbb{D}$. Since h is strictly increasing

$$\frac{h(\varphi(t))-h(\phi(t))}{\varphi(t)-\phi(t)} = \frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)} > 0 \tag{6.32}$$

holds for all $t\in S(\phi,\varphi)$. By (H3) we have

$$1 \geq \sup_{t\in U} h'(t) \geq \inf_{s\in U} h'(s) \geq 0. \tag{6.33}$$

Define the set $U_t\subset U$ by $U_t = [\varphi(t),\phi(t)]\cap U$ if $\phi(t)>\varphi(t)$, and $U_t = [\phi(t),\varphi(t)]\cap U$ if $\phi(t)<\varphi(t)$, for $t\in S(\phi,\varphi)$. Hence, for a fixed $t_0\in S(\phi,\varphi)$ we get by (H4) and (6.32) that

$$\sup\{h'(u) : u\in U_{t_0}\} \geq \frac{h(\phi(t_0))-h(\varphi(t_0))}{\phi(t_0)-\varphi(t_0)} \geq \inf\{h'(u) : u\in U_{t_0}\}.$$

Since $U_t\subset U$ for every $t\in S(\phi,\varphi)$, we find

$$\sup_{u\in U} h'(u) \geq \sup\{h'(u) : u\in U_{t_0}\} \geq \inf\{h'(u) : u\in U_{t_0}\} \geq \inf_{u\in U} h'(u),$$

and therefore,

$$1 \geq \sup_{u\in U} h'(u) \geq \frac{h(\varphi(t))-h(\phi(t))}{\varphi(t)-\phi(t)} \geq \inf_{u\in U} h'(u) \geq 0 \tag{6.34}$$

for all $t\in S(\phi,\varphi)$. So, (6.34) yields

$$\begin{aligned} |(\mathcal{F}\phi)(t)-(\mathcal{F}\varphi)(t)| &= |\phi(t)-h(\phi(t))-\varphi(t)+h(\varphi(t))| \\ &= |\phi(t)-\varphi(t)|\Big|1-\Big(\frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)}\Big)\Big| \\ &\leq |\phi(t)-\varphi(t)|\Big(1-\inf_{u\in U} h'(u)\Big) \end{aligned} \tag{6.35}$$

for all $t \in S(\phi, \varphi)$. Thus, (6.34) and (6.35) imply that $\mathcal{F}$ is a large contraction in the supremum norm. To see this choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{D}$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [K,L]} |\phi(t) - \varphi(t)| \geq \varepsilon.$$

If $|\phi(t) - \varphi(t)| \leq \varepsilon/2$ for some $t \in S(\phi, \varphi)$, then from (6.35)

$$|(\mathcal{F}\phi)(t) - (\mathcal{F}\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2}\|\phi - \varphi\|. \tag{6.36}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[K, L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in S(\phi, \varphi)$, then from (6.34) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |(\mathcal{F}\phi)(t) - (\mathcal{F}\varphi)(t)| &\leq |\phi(t) - \varphi(t)| \Big\{1 - \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)}\Big\} \\ &\leq (1 - \lambda)\|\phi(t) - \varphi(t)\|, \end{aligned} \tag{6.37}$$

where

$$\lambda := \frac{1}{2L} \min\Big\{h(u + \frac{\varepsilon}{2}) - h(u), u \in [K, L]\Big\} > 0.$$

Consequently, it follows from (6.36) and (6.37) that

$$|(\mathcal{F}\phi(t) - (\mathcal{F}\varphi)(t)| \leq \delta\|\phi - \varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1 - \lambda\right\} < 1$. The proof is complete.

The next result gives a relationship between the mappings $\mathcal{F}$ and $\mathcal{B}$ in the sense of large contraction.

Lemma 6.4.3. Suppose that

$$\frac{K}{\gamma T} \leq q(s)[\varphi(s) - h(\varphi(s))] \leq \frac{L - \beta \mu T}{\alpha T}, \tag{6.38}$$

then $\mathcal{B}$ maps $\mathbb{D}$ into itself. Moreover, if $\mathcal{F}$ is a large contraction on $\mathbb{D}$, and

$$\alpha ||q|| T \leq 1, \tag{6.39}$$

then so is the mapping $\mathcal{B}$.

Proof. We first show that $(\mathbb{B}\varphi)(t+T) = (\mathbb{B}\varphi)(t)$. Evaluating $(\mathbb{B}\varphi)(t)$ at $t+T$ gives

$$(\mathcal{B}\varphi)(t+T) = \int_{t+T}^{t+2T} G(t+T,s)q(s)[\varphi(s) - h(\varphi(s))]ds.$$

With $k = s - T$ we obtain

$$\begin{aligned} (\mathcal{B}\varphi)(t+T) &= \int_{t}^{t+T} G(t+T,k+T)q(k+T)[\varphi(k+T) - h(\varphi(k+T))]dk \\ &= \int_{t}^{t+T} G(t,k)q(k)[\varphi(k) - h(\varphi(k))]dk \\ &= (\mathbb{B}\varphi)(t). \end{aligned}$$

Let $\varphi \in \mathbb{D}$ then

$$\begin{aligned} (\mathcal{B}\varphi)(t) &\leq \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) \\ &= L - \beta\mu T < L. \end{aligned}$$

On the other hand,

$$(\mathcal{B}\varphi)(t) \;\geq\; \gamma T(\frac{K}{\gamma T}) = K.$$

Thus showing that $\mathcal{B}$ maps $\mathbb{D}$ into itself.

If $\mathcal{F}$ is a large contraction on $\mathbb{D}$, then for $x,y \in \mathbb{D}$, with $x \neq y$, we have $\|\mathcal{F}x - \mathcal{F}y\| \leq \|x-y\|$. Thus,

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| \;\leq\; \alpha||q||T||x-y|| \leq ||x-y||.$$

Thus,

$$\|Bx - By\| \;\leq\; \|x-y\|.$$

One may also show in a similar way that

$$\|\mathcal{B}x - \mathcal{B}y\| \;\leq\; \delta\|x-y\|$$

holds if we know the existence of a $0 < \delta < 1$, such that for all $\varepsilon > 0$

$$\big[x,y \in \mathbb{D},\ \|x-y\| \geq \varepsilon\big] \Rightarrow \|\mathcal{F}x - \mathcal{F}y\| \leq \delta\|x-y\|.$$

The proof is complete.

Lemma 6.4.4. Suppose that conditions (6.2)-(6.4), (6.21) and (6.22)-(6.26). hold. Then $\mathcal{A} : \mathbb{D} \to \mathbb{D}$ is completely continuous on $\mathbb{D}$.

Proof. Let $\mathcal{A}$ be defined by (6.30). We begin by showing that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. Evaluating $(\mathcal{A}\varphi)(t)$ at $t+T$ gives

$$\begin{aligned}
(\mathcal{A}\varphi)(t+T) &= \int_{t+T}^{t+2T} \Big[c(s,\varphi(s-\tau(s)))[E(t+T,s)-a(s)G(t+T,s)] \\
&\quad + G(t+T,s)f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\Big]ds.
\end{aligned}$$

With $k = s-T$ we obtain,

$$\begin{aligned}
(\mathcal{A}\varphi)(t+T) &= \int_{t}^{t+T} \Big[c(k+T,\varphi(k+T-\tau(k+T)))[E(t+T,k+T) \\
&\quad - a(k+T)G(t+T,k+T)] \\
&\quad + G(t+T,k+T) \\
&\quad \times f(k+T,h(\varphi(k+T)),g(\varphi(k+T-\tau(k+T))))\Big]dk \\
&= \int_{t}^{t+T} \Big[c(k,\varphi(k-\tau(k)))[E(t,k)-a(k)G(t,k)] \\
&\quad + G(t,k)f(k,h(\varphi(k)),g(\varphi(k-\tau(k))))\Big]dk \\
&= (\mathcal{A}\varphi)(t).
\end{aligned}$$

For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$ we have that

$$\begin{aligned}
(\mathcal{A}\varphi)(t) &\leq \mu\beta T + \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) \\
&\leq L.
\end{aligned}$$

Also,

$$\begin{aligned}
(\mathcal{A}\varphi)(t) &\geq c^*\sigma T + \gamma T\Big(\frac{K-c^*\sigma T}{\gamma T}\Big) \\
&\geq K.
\end{aligned}$$

Thus $\mathcal{A}$ maps $\mathbb{D}$ into itself.

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\leq \Big|\int_t^{t+T} c(s,\varphi(s-\tau(s)))E(t,s)ds\Big| \\
&\quad + \Big|\int_t^{t+T} G(t,s)\Big[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) \\
&\quad - a(s)c(s,\varphi(s-\tau(s)))\Big]ds\Big| \\
&\leq \mu\beta T + \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) \\
&\leq L.
\end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have that

$$\|\mathcal{A}\varphi\| \leq L.$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. We next show that $\mathcal{A}(\mathbb{D})$ is equicontinuous by first computing $\frac{d}{dt}(\mathcal{A}\varphi_n(t))$. We obtain by taking the derivative in (6.30) that

$$\begin{aligned}
\frac{d}{dt}(\mathcal{A}\varphi)_n(t) &= \frac{\exp\big(\int_t^{t+T} b(v)dv-1\big)}{\exp\big(\int_0^T b(v)dv\big)-1}c(t,\varphi_n(t-\tau(t))) \\
&\quad + \int_t^{t+T} c(s,\varphi_n(s-\tau(s)))\Big[-b(t)E(t,s)-a(s)\Big(-b(t)G(t,s) \\
&\quad + \frac{\exp(\int_t^s a(v)dv}{\exp(\int_0^T a(v)dv-1}\Big)\Big]ds \\
&\quad + \int_t^{t+T}\Big(-b(t)G(t,s)+\frac{\exp(\int_t^s a(v)dv}{\exp(\int_0^T a(v)dv-1}\Big) \\
&\quad \times f(s,h(\varphi_n(s)),g(\varphi_n(s-\tau(s))))ds. \\
&= \frac{\exp\big(\int_t^{t+T} b(v)dv-1\big)}{\exp\big(\int_0^T b(v)dv\big)-1}c(t,\varphi_n(t-\tau(t))) \\
&\quad + \int_t^{t+T} c(s,\varphi_n(s-\tau(s)))\Big[-b(t)E(t,s) \\
&\quad - a(s)\frac{\exp(\int_t^s a(v)dv}{\exp(\int_0^T a(v)dv-1}\Big]ds \\
&\quad + \int_t^{t+T}\frac{\exp(\int_t^s a(v)dv}{\exp(\int_0^T a(v)dv-1}f(s,h(\varphi_n(s)),g(\varphi_n(s-\tau(s))))ds \\
&\quad + \int_t^{t+T} -b(t)G(t,s)\Big[f(s,h(\varphi_n(s)),g(\varphi_n(s-\tau(s)))) \\
&\quad - a(s)c(s,\varphi_n(s-\tau(s)))\Big]ds.
\end{aligned}$$

Consequently, by invoking (6.11), and (6.24), we obtain

$$\begin{aligned}|\frac{d}{dt}(\mathcal{A}\varphi)(t)| &\leq \beta\mu + T\mu[||b||\beta + ||a||\beta] + T\beta\Big(\frac{L-\beta\mu T}{\alpha T}\Big) + ||b||\alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big)\\ &\leq M,\end{aligned}$$

for some positive constant M. Hence $(\mathcal{A}\varphi)$ is equicontinuous. Then by the Ascoli-Arzela theorem we obtain that $\mathcal{A}$ is a compact map. Due to the continuity of all the terms in (6.30), we have that $\mathcal{A}$ is continuous. This completes the proof.

Theorem 6.4.5. Suppose that conditions (6.2)-(6.4), (6.10), (6.21), (6.22)-(6.26) hold, then Equation (6.20) has a positive periodic solution z satisfying $K \leq z \leq L$.

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (6.30) and (6.31) we obtain

$$\begin{aligned}&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t)\\ &= \int_t^{t+T} G(t,s)q(s)[\psi(s) - h(\psi(s))]ds\\ &\quad + \int_t^{t+T}\Big[c(s,\varphi(s-\tau(s)))[E(t,s) - a(s)G(t,s)]\\ &\quad + G(t,s)f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))\Big]ds\\ &= \int_t^{t+T} c(s,\varphi(s-\tau(s)))E(t,s)ds\\ &\quad + \int_t^{t+T} G(t,s)\Big(q(s)[\psi(s) - h(\psi(s))]\\ &\quad + f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) - a(s)c(s,\varphi(s-\tau(s)))\Big)ds\\ &\leq \beta\mu T + \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) = L.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t)\\ &= \int_t^{t+T} c(s,\varphi(s-\tau(s)))E(t,s)ds\\ &\quad + \int_t^{t+T} G(t,s)\Big(q(s)[\psi(s) - h(\psi(s))]\\ &\quad + f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) - a(s)c(s,\varphi(s-\tau(s)))\Big)ds\\ &\geq c^*\sigma T + \gamma T\Big(\frac{K-c^*\sigma T)}{\gamma T}\Big) = K.\end{aligned}$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Thus all the hypotheses of Theorem 2.3.7 are satisfied and therefore equation (6.20) has a periodic solution in $\mathbb{D}$. This completes the proof.

CHAPTER SEVEN

PERIODICITY IN A SYSTEM OF DIFFERENTIAL EQUATIONS

7.1 Introduction

In this Chapter we study the existence and uniqueness of a periodic solution of the system of equations

$$\frac{d}{dt}x(t) = A(t)x(t-\tau), \tag{7.1}$$

where $A(t)$ is an $n \times n$ matrix with continuous real-valued functions as its elements and τ is a positive constant. A number of researchers have studied the existence of periodic solutions of systems of equations of the form of (7.1) where the delay $\tau = 0$. Floquet theory offers a lot of results on the periodicity of the system (7.1) when $\tau = 0$. Weikard (2000) extended the theory to nonautonomous linear systems of the form $z' = A(x)z$, where $A : \mathbb{C} \to \mathbb{C}$ is an $\omega-$ periodic function in the complex variable x, whose solutions are meromorphic. There are however no corresponding results for system (7.1). Therefore, we prove the existence and uniqueness of solutions of system (7.1) using the notion of the fundamental solution coupled with Floquet theory in this Chapter.

7.2 Preliminaries

We assume throughout this Chapter that there exist a nonsingular $n \times n$ matrix $G(t)$ with continuous real-valued functions as its elements such that

$$\frac{d}{dt}x(t) = G(t)x(t) - \frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds + [A(t) - G(t-\tau)]x(t-\tau). \tag{7.2}$$

Lemma 7.2.1. Equation (7.1) is equivalent to (7.2).

Proof. By differentiating the integral term in (7.2) we obtain

$$\frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds = G(t)x(t) - G(t-\tau)x(t-\tau). \tag{7.3}$$

Substituting this into (7.2) gives

$$\begin{aligned}\frac{d}{dt}x(t) &= G(t)x(t) - G(t)x(t) + G(t-\tau)x(t-\tau) + [A(t) - G(t-\tau)]x(t-\tau)\\ &= A(t)x(t-\tau).\end{aligned}$$

For $T > 0$ let P_T be the set of all n-vector valued functions $x(t)$, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x(.)\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^n$. Also, if A is an $n \times n$ real matrix, then we define the norm of A by $|A| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$.

Definition 7.2.2. If the matrix $G(t)$ is periodic of period T, then the linear system

$$y' = G(t)y \tag{7.4}$$

is said to be noncritical with respect to T if it has no periodic solution of period T except the trivial solution $y = 0$.

In this Chapter we assume that

$$A(t+T) = A(t), G(t+T) = G(t) \tag{7.5}$$

Throughout this Chapter it is assumed that the system (7.4) is noncritical. We next state some known results about system (7.4) which will be useful in the rest of the Chapter.

Let $K(t)$ represent the fundamental matrix of the system (7.4) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then:

(i) $\det K(t) \neq 0$.

(ii) There exists a constant matrix B such that $K(t+T) = K(t)e^{BT}$, by Floquet theory.

(iii) System (7.4) is noncritical if and only if $det(I - K(T)) \neq 0$.

7.3 Existence and Uniqueness

We begin this section by stating the following lemma.

Lemma 7.3.1. Suppose (7.5) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of (7.2) if and only if

$$\begin{aligned} x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{ \\ &\quad + \int_{t}^{t+T} K^{-1}(u)[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds]du\Big\} \end{aligned} \tag{7.6}$$

Proof. Let $x(t) \in P_T$ be a solution of (7.2) and $K(t)$ is a fundamental system of solutions of (7.4). We first rewrite (7.2) as

$$\begin{aligned} \frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] &= G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] - G(t)\int_{t-\tau}^{t} G(s)x(s)ds \\ &\quad + A(t)x(t-\tau) - G(t-\tau)x(t-\tau). \end{aligned}$$

Since $K(t)K^{-1}(t) = I$, it follows that

$$\begin{aligned} 0 &= \frac{d}{dt}(K(t)K^{-1}(t)) = \frac{d}{dt}(K(t))K^{-1}(t) + K(t)\frac{d}{dt}(K^{-1}(t)) \\ &= (G(t)K(t))K^{-1}(t) + K(t)\frac{d}{dt}(K^{-1}(t)) \\ &= G(t) + K(t)\frac{d}{dt}(K^{-1}(t)). \end{aligned}$$

This implies

$$\frac{d}{dt}(K^{-1}(t)) = -K^{-1}(t)G(t).$$

If $x(t)$ is a solution of (7.2) with $x(0) = x_0$, then

$$\begin{aligned} &\frac{d}{dt}\Big[K^{-1}(t)\Big(x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big)\Big] \\ &= \frac{d}{dt}K^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] + K^{-1}(t)\frac{d}{dt}\Big(x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big) \end{aligned}$$

$$= -K^{-1}(t)G(t)\Big[x(t)+\int_{t-\tau}^{t} G(s)x(s)ds\Big] + K^{-1}(t)\Big\{G(t)\Big[x(t)+\int_{t-\tau}^{t} G(s)x(s)ds\Big]$$
$$- G(t)\int_{t-\tau}^{t} G(s)x(s)ds + A(t)x(t-\tau) - G(t-\tau)x(t-\tau)\Big\}$$

$$= K^{-1}(t)A(t)x(t-\tau) - K^{-1}(t)G(t-\tau)x(t-\tau) - K^{-1}(t)G(t)\int_{t-\tau}^{t} G(s)x(s)ds.$$

An integration of the above equation from 0 to t yields

$$\begin{aligned} x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)\Big[x_0 + \int_{-\tau}^{0} G(s)x(s)ds\Big] \\ &\quad + K(t)\int_0^t K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \end{aligned} \tag{7.7}$$

Since $x(T) = x_0 = x(0)$, we obtain from (7.7) that

$$\begin{aligned} x_0 + \int_{-\tau}^{0} G(s)x(s)ds &= (I-K(T))^{-1}\int_0^T K(T)K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \end{aligned} \tag{7.8}$$

Substituting (7.8) into (7.7) yields

$$\begin{aligned} x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)\Big((I-K(T))^{-1}\int_0^T K(T)K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big) \\ &\quad + K(t)\int_0^t K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \end{aligned} \tag{7.9}$$

We will now show that (7.9) is equivalent to (7.6).

Since

$$(I-K(T))^{-1} = (K(T)(K^{-1}(T)-I))^{-1} = (K^{-1}(T)-I)^{-1}K^{-1}(T),$$

equation (7.9) becomes

$$\begin{aligned} x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K(T))^{-1} - I)^{-1}\int_0^T K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\ &\quad + K(t)\int_0^t K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \end{aligned}$$

$$\begin{aligned} &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T)) - I)^{-1}\Big\{\int_0^T K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\ &\quad + \int_0^t K^{-1}(T)K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \\ &\quad - \int_0^t K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big\}. \\ &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T)) - I)^{-1}\Big\{\int_t^T K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\ &\quad + \int_0^t K^{-1}(T)K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\ &\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du.\Big\}. \end{aligned}$$

By letting $u = i - T$, the above expression implies

$$\begin{aligned} x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T)) - I)^{-1}\Big\{\int_t^T K^{-1}(u)\Big[A(u)x(u-\tau) \\ &\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\ &\quad + \int_T^{t+T} K^{-1}(T)K^{-1}(i-T)\Big[A(i-T)x(i-T-\tau) \\ &\quad - G(i-T-\tau)x(i-T-\tau) - G(i-T)\int_{i-T-\tau}^{i-T} G(s)x(s)ds\Big]di\Big\} \qquad (7.10) \end{aligned}$$

By (ii) we have $K(t-T) = K(t)e^{-BT}$ and $K(T) = e^{BT}$. Hence, $K^{-1}(T)K^{-1}(i-T) =$

$K^{-1}(i)$. Consequently, (7.10) becomes

$$
\begin{aligned}
x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T)) - I)^{-1}\Big\{\int_{t}^{T} K^{-1}(u)\Big[A(u)x(u-\tau) \\
&\quad - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
&\quad + \int_{T}^{t+T} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big\}.
\end{aligned}
$$

Combining the two integrals in the above equation gives equation (7.6). This completes the proof.

Define a mapping H by

$$
\begin{aligned}
(H\varphi)(t) &= -\int_{t-\tau}^{t} G(s)\varphi(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{\int_{t}^{t+T} K^{-1}(u)[A(u)\varphi(u-\tau) \\
&\quad - G(u-\tau)\varphi(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds]du\Big\}. \qquad (7.11)
\end{aligned}
$$

It is clear from (7.11) that $H : P_T \to P_T$ by the way it was constructed in Lemma 7.2.1.

To apply Theorem 2.3.6 we need to construct two mappings of which one is a contraction and the other is compact. Therefore we express equation (7.11) as

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t),$$

where $C, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = -\int_{t-\tau}^{t} G(s)\varphi(s)ds \qquad (7.12)$$

$$
\begin{aligned}
(C\varphi)(t) &= K(t)(K^{-1}(T) - I)^{-1}\int_{t}^{t+T} K^{-1}(u)[A(u)\varphi(u-\tau) - G(u-\tau)\varphi(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds]du. \qquad (7.13)
\end{aligned}
$$

Lemma 7.3.2. Suppose the assumptions of Lemma 7.3.1 hold. If C is defined by (7.13) then C is continuous and the image of C is contained in a compact set.

Proof. Let $\varphi, \psi \in P_T$. Given $\varepsilon > 0$, take $\delta = \varepsilon/N$ with $N = rT(|A| + |G| + |G|^2|\tau)$, where

$$r = \sup_{t\in[0,T]} \Big(\sup_{t\le u\le t+T} |[K(u)(K^{-1}(T)-I)K^{-1}(t)]^{-1}| \Big). \tag{7.14}$$

Now for $\|\varphi - \psi\| < \delta$, we have that

$$\begin{aligned} \|C\varphi(.) - C\psi(.)\| &\le r\int_0^T \Big[|A|\|\varphi-\psi\| + |G|\|\varphi-\psi\| + |G|^2\tau\|\varphi-\psi\|\Big]du \\ &\le N\|\varphi-\psi\| < \varepsilon. \end{aligned}$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \le R\}$, where R is a fixed positive constant. Let $\varphi_n \in D$ where n is a positive integer. Thus,

$$\begin{aligned} \|C\varphi_n(.)\| &\le r\int_0^T \Big[|A|R + |G|R + |G|^2\tau R\Big]du \\ &\le rT\Big[|A|R + |G|R + |G|^2\tau R\Big] \\ &\le L, \end{aligned}$$

for some positive constant L. Next we calculate $(C\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (7.5) we obtain by taking the derivative in (7.13) that

$$\begin{aligned} (C\varphi_n)'(t) &= K'(t)(K^{-1}(T)-I)^{-1}\int_t^{t+T} K^{-1}(u)\Big[A(u)\varphi_n(u-\tau) \\ &\quad - G(u-\tau)\varphi_n(u-\tau) - G(u)\int_{u-\tau}^u G(s)\varphi_n(s)ds\Big]du \\ &\quad + K(t)(K^{-1}(T)-I)^{-1}K^{-1}(t+T)\Big[A(t)\varphi_n(t-\tau) \\ &\quad - G(t-\tau)\varphi_n(t-\tau) - G(t)\int_{t-\tau}^t G(s)\varphi_n(s)ds\Big] \\ &\quad - K(t)(K^{-1}(T)-I)^{-1}K^{-1}(t)\Big[A(t)\varphi_n(t-\tau) - G(t-\tau)\varphi_n(t-\tau) \\ &\quad - G(t)\int_{t-\tau}^t G(s)\varphi_n(s)ds\Big] \\ &= G(t)(C\varphi_n)(t) + K(t)(K^{-1}(T)-I)^{-1}\Big[K^{-1}(t+T) - K^{-1}(t)\Big] \\ &\quad \times \Big(A(t)\varphi_n(t-\tau) - G(t-\tau)\varphi_n(t-\tau) - G(t)\int_{t-\tau}^t G(s)\varphi_n(s)ds\Big) \end{aligned}$$

By noting that $K^{-1}(t+T) = e^{-BT}K^{-1}(t)$, we have

$$K^{-1}(t+T) - K^{-1}(t) = (e^{-BT}-I)K^{-1}(t) = (K^{-1}(T)-I)K^{-1}(t).$$

Using this in the last expression, yields

$$\begin{aligned}
(C\varphi_n)'(t) &= G(t)(C\varphi_n)(t) \\
&\quad + \Big(A(t)\varphi_n(t-\tau) - G(t-\tau)\varphi_n(t-\tau) - G(t)\int_{t-\tau}^{t} G(s)\varphi_n(s)ds\Big) \\
&\leq |G|L + |A|R + |G|R + |G|^2 R\tau.
\end{aligned}$$

Thus the sequence (φ_n) is uniformly bounded and equi-continuous. Hence by Arzela-Ascoli theorem $C(D)$ is compact. The proof is complete.

Lemma 7.3.3. Suppose that

$$|G|\tau < 1, \tag{7.15}$$

then B is a contraction.

Proof. Let B be defined by (7.12). Then for $\varphi, \psi \in P_T$ we have

$$\begin{aligned}
\|B\varphi(.) - B\psi(.)\| &= \sup_{t\in[0,T]} |B\varphi(t) - B\psi(t)| \\
&\leq \tau|G|\|\varphi - \psi\|.
\end{aligned}$$

Hence B defines a contraction mapping with contraction constant $\tau|G|$.

Lemma 7.3.4. Suppose the hypothesis of Lemma 7.2.3 holds. Let r be given by (7.14). Suppose further that (7.5) hold. Let J be a positive constant satisfying the inequality

$$rT\Big[|A| + |G| + |G|^2\tau\Big]J + \tau|G|J \leq J. \tag{7.16}$$

Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then (7.1) has a solution in $\mathbb{M}$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq J\}$. By Lemma 7.3.2, C is continuous and $C\mathbb{M}$ is contained in a compact set. Also, from Lemma 7.3.3, the mapping B is a contraction and it is clear that $C, B : P_T \to P_T$. Next we show that if $\varphi, \psi \in \mathbb{M}$, we have $||C\varphi + B\psi|| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $||\varphi||, ||\psi|| \leq J$. Then

$$\begin{aligned}
&||C\varphi(.) + B\psi(.)|| \\
&r\int_0^T \Big[|A|||\varphi|| + |G|||\varphi|| + |G|^2\tau||\varphi||\Big]du + \int_{t-\tau}^{t} |G|||\psi||ds \\
&rT\Big[|A| + |G| + |G|^2\tau\Big]J + \tau|G|J \leq J.
\end{aligned}$$

We now see that all conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 7.3.1, this fixed point is a solution of (7.1). Hence (7.1) has a T-periodic solution.

Theorem 7.3.5. Suppose (7.5) hold. If

$$\tau|G| + rT\left[|A| + |G| + |G|^2\tau\right] < 1, \tag{7.17}$$

then (7.1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (7.11). For $\varphi, \psi \in P_T$, we have that

$$\begin{aligned} \|H\varphi(.) - H\psi(.)\| &\leq \int_{t-\tau}^{t} |G|\|\varphi-\psi\|ds + r\int_0^T \left[|A| + |G| + |G|^2\tau\right]\|\varphi-\psi\|ds \\ &\leq \left(\tau|G| + rT\left[|A| + |G| + |G|^2\tau\right]\right)\|\varphi-\psi\| \\ &< \|\varphi-\psi\|. \end{aligned}$$

Thus, H is a contraction. Thus by the contraction mapping principle, (7.1) has a unique T-periodic solution. This completes the proof.

CHAPTER EIGHT

POSITIVE SOLUTIONS FOR A SYSTEM OF PERIODIC NEUTRAL DIFFERENCE EQUATIONS AND EQUATIONS WITH ASYMPTOTICALLY CONSTANT OR PERIODIC SOLUTIONS

8.1 Existence of positive solutions for a system of periodic difference equations

8.1.1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_-$ the negative integers, and $\mathbb{Z}^+$ the non-negative integers. In this section we consider the system of neutral difference equations

$$x(n+1) = A(n)x(n) + C(n)\Delta x(n-\tau(n)) + g(n, x(n-\tau(n))), \tag{8.1}$$

where $A(n) = \text{diag}[a_1(n), a_2(n), ..., a_k(n)]$, a_j is T-periodic, $C(n) = \text{diag}[c_1(n), c_2(n), ..., c_k(n)]$, c_j is T-periodic, $g : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x and $g(n,x)$ is T-periodic in n and x, whenever x is T-periodic, $T \geq 1$ is an integer. Let P_T be the set of all real T-periodic sequences $\phi : \mathbb{Z} \to \mathbb{R}^k$. Endowed with the maximum norm $||\phi|| = \max_{\theta \in \mathbb{Z}} \sum_{j=1}^{k} |\phi_j(\theta)|$ where $\phi = (\phi_1, \phi_2, ..., \phi_k)^t$, P_T is a Banach space. Here t stands for the transpose.

Let $\mathbb{R}_+ = [0, +\infty)$, for each $x = (x_1, x_2, ..., x_k)^t \in \mathbb{R}^k$, the norm of x is defined as $|x| = \sum_{j=1}^{k} |x_j|$. $\mathbb{R}^k_+ = \{(x_1, x_2, ..., x_k)^t \in \mathbb{R}^k : \ x_j \geq 0, \ j = 1, 2, ..., k\}$. Also, we denote $g = (g_1, g_2, ..., g_k)^t$, where t stands for transpose. We say that x is "positive" whenever $x \in \mathbb{R}^k_+$.

Raffoul and Yankson (2010) proved the existence of positive periodic solutions of the scalar version of (8.1). Motivated by this work, we obtain sufficient conditions for the existence of positive periodic solutions of system (8.1).

Remark 8.1.1.1 The content of this Chapter has been published as: E. Yankson, "Positive solutions for a system of periodic neutral delay difference equations", African Diaspora Journal of Mathematics, Volume 11, Number 2, pp. 90-97(2011).

8.1.2 Preliminaries

In this section, we make the following assumptions.

(H1) There exist a constant $\sigma_j > 0$ such that $\sigma_j < c_j(n),\ j = 1,...,k,$ for all $n \in [0, T-1]$.

(H2) $0 < a_j(n) < 1$ for all $n \in [0, T-1]$, $j = 1,...,k$.

(H3) There exist constants α_j, such that $||c_j|| \leq \alpha_j \leq 1,\ j = 1, 2, ..., k$.

Let

$$G_j(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a_j(s)}{1 - \prod_{s=n}^{n+T-1} a_j(s)},\ u \in [n, n+T-1]. \tag{8.2}$$

Note that the denominator in $G_j(n,u)$ is not zero since $0 < a_j(n) < 1$ for $n \in [0, T-1]$.

Define

$$G(n,u) = \mathrm{diag}[G_1(n,u), G_2(n,u), ..., G_k(n,u)]. \tag{8.3}$$

It is clear that $G(n,u) = G(n+T, u+T)$ for all $(n,u) \in \mathbb{Z}^2$. Also, let

$$q_j := \min\{G_j(n,u) : n \geq 0,\ u \leq T\} = G_j(n,n) > 0,\ j = 1,...,k. \tag{8.4}$$

$$\begin{aligned} Q_j := \max\{G_j(n,u) : n \geq 0,\ u \leq T\} &= G_j(n, n+T-1) \\ &= G_j(0, T-1) > 0, j = 1,...,k. \end{aligned} \tag{8.5}$$

Set $q = \min\{q_1, q_2, ..., q_k\}$ and $Q = \max\{Q_1, Q_2, ..., Q_k\}$.

It must be noted that the scalar equations making up the system of equations in (8.1) are of the form

$$x_j(n+1) = a_j(n)x_j(n) + c_j(n)\Delta x_j(n-\tau(n)) + g_j(n, x_j(n-\tau(n))),\ j = 1, ..., k. \tag{8.6}$$

Lemma 8.1.2.1. Suppose (H2) holds. Then $x_j(n) \in P_T$ is a solution of (8.6) if and only if

$$\begin{aligned} x_j(n) &= c_j(n-1)x_j(n-g(n)) + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u, x_j(u-\tau(u))) \\ &\quad -x_j(u-g(u))\phi_j(u)a_j(u)\Big]. \end{aligned} \tag{8.7}$$

where $\phi_j(u) = c_j(u) - c_j(u-1)$.

Proof. Rewrite (8.6) as

$$\Delta\Big[x_j(n)\prod_{s=0}^{n-1} a_j^{-1}(s)\Big] = \Big[c_j(n)\Delta x_j(n-\tau(n)) + g_j(n, x_j(n-\tau(n)))\Big]\prod_{s=0}^{n} a_j^{-1}(s). \tag{8.8}$$

Summing equation (8.8) from n to $n+T-1$ we obtain

$$\begin{aligned} &\sum_{u=n}^{n+T-1} \Delta\Big[x_j(u)\prod_{s=0}^{u-1} a_j^{-1}(s)\Big] \\ &= \sum_{u=n}^{n+T-1} \Big[c_j(u)\Delta x_j(u-\tau(u)) + g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned}$$

Thus,

$$\begin{aligned} x(n+T)\prod_{s=0}^{n+T-1} a_j^{-1}(s) - x(n)\prod_{s=0}^{n-1} a_j^{-1}(s) &= \sum_{u=n}^{n+T-1} \Big[c_j(u)\Delta x_j(u-\tau(u)) \\ &\quad + g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned}$$

Since $x(n+T) = x(n)$, we obtain

$$\begin{aligned} x(n)\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big] &= \sum_{u=n}^{n+T-1} \Big[c_j(u)\Delta x_j(u-\tau(u)) \\ &\quad + g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned} \tag{8.9}$$

But

$$\sum^{n+T-1} c_j(u)\Delta x_j(u-\tau(u))\prod_{s=0}^{u} a_j^{-1}(s)$$
$$= c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$$
$$-\sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Delta\Big[c_j(u-1)\prod_{s=0}^{u-1} a_j^{-1}(s)\Big]$$
$$= c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$$
$$-\sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Big[c_j(u)-c_j(u-1)a_j(u)\Big]\prod_{s=0}^{u} a_j^{-1}(s) \tag{8.10}$$

Substituting (8.10) into (8.9) gives

$$x(n)\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$$
$$= c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$$
$$-\sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Big[c_j(u)-c_j(u-1)a_j(u)\Big]\prod_{s=0}^{u} a_j^{-1}(s)$$
$$+g_j(u,x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \tag{8.11}$$

Dividing through by $\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$ gives the desired result.

8.1.3 Positive periodic solutions

In this section we obtain sufficient conditions for the existence of positive periodic solutions for (8.1). For some nonnegative constant L and a positive constant J we define the set

$$\mathbb{M} = \{\phi \in P_T \ : \ L \le ||\phi|| \le J, \text{ with } \frac{L}{k} \le \phi_j \le \frac{J}{k}, \ j = 1,2,...,k.\}, \tag{8.12}$$

which is a closed convex and bounded subset of the Banach space P_T. We also assume that for all $u \in \mathbb{Z}$ and $\rho \in \mathbb{M}$,

$$\frac{(1-\sigma_j)L}{Tq_jk} \leq g_j(u,\rho_j,\rho_j) - \rho_j\phi_j(u)a_j(u) \leq \frac{(1-\alpha_j)J}{TQ_jk}. \tag{8.13}$$

Define a mapping $H : \mathbb{M} \to P_T$ by

$$\begin{aligned}(Hx)(n) &= C(n-1)x(n-\tau(n)) \\ &\quad + \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u,x(u),x(u-\tau(u))) - \Phi(u)A(u)x(u-\tau(u))\Big].\end{aligned}$$

We denote

$$(Hx) = (H_1x_1, H_2x_2, ..., H_kx_k)^t. \tag{8.14}$$

It is clear that $(Hx)(n+T) = (Hx)(n)$. In order to apply Thoerem 2.3.6 we will construct two mappings of which one is a contraction and the other is compact. Thus we define the map $D : \mathbb{M} \to P_T$ by

$$(D\varphi)(n) = C(n-1)\varphi(n-\tau(n)). \tag{8.15}$$

We also define the map $F : \mathbb{M} \to P_T$ by

$$(F\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u,\varphi(u),\varphi(u-\tau(u))) - \Phi(u)A(u)\varphi(u-\tau(u))\Big]. \tag{8.16}$$

Lemma 8.1.3.1. Suppose (H3) hold. Then the operator D defined by (8.15) is a contraction.

Proof. Let $\varphi, \psi \in \mathbb{M}$ and $\alpha = \max_{1\leq j\leq k}\alpha_j$. Then

$$||(D\varphi) - (D\psi)|| = \max_{n\in[0,T-1]} \sum_{j=1}^{k} |(D_j\varphi_j)(n) - (D_j\psi_j)(n)|$$

But,

$$\begin{aligned}|(D_j\varphi_j)(n) - (D_j\psi_j)(n)| &= |c_j(n-1)\varphi_j(n) - c_j(n-1)\psi_j(n)| \\ &\leq \alpha_j||\varphi_j - \psi_j||.\end{aligned}$$

Thus,

$$\begin{aligned}
||(D\varphi)-(D\psi)|| &\leq \sum_{j=1}^{k}\alpha_j||\varphi_j-\psi_j|| \\
&\leq \alpha||\varphi-\psi||.
\end{aligned}$$

This completes the proof.

Lemma 8.1.3.2. Suppose that (H1), (H2), (H3) and (8.13) hold. Then the operator F defined by (8.16) is completely continuous on $\mathbb{M}$.

Proof. For $n \in [0, T-1]$ and for $\varphi \in \mathbb{M}$, we have by (8.13) that

$$\begin{aligned}
|(F_j\varphi_j)(n)| &\leq \Big|\sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\varphi_j(u-\tau(u)))-\varphi_j(u-g(u))\phi_j(u)a_j(u)\Big]\Big| \\
&\leq Q_jT\frac{(1-\alpha)J}{TQ_jk} \\
&\leq \frac{(1-\alpha)J}{k}
\end{aligned}$$

Thus,

$$||(F\varphi)|| \leq \sum_{j=1}^{k}\frac{(1-\alpha)J}{k}=(1-\alpha)J.$$

It therefore follows that

$$||(F\varphi)|| \leq J.$$

This shows that $F(\mathbb{M})$ is uniformly bounded. Due to the continuity of all terms, we have that F is continuous.

Next we show that F maps bounded subsets into compact sets. Let $S = \{\ \varphi \in P_T\ :\ ||\varphi|| \leq \mu\}$ and $Q = \{\ (F\varphi)(n)\ :\ \varphi \in S\}$, then S is a subset of $\mathbb{R}^{Tk}$ which is closed and bounded and thus compact. As F is continuous in φ, it maps compact sets into compact sets. Therefore $Q = F(S)$ is compact. This completes the proof.

Theorem 8.1.3.3. Suppose that (H1),(H2), (H3) and (3.2) hold. Also suppose that the hypothesis of Lemma 8.1.3.2 also hold. Then equation (8.1) has a positive periodic solution.

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then we have that

$$\begin{aligned}
(D_j\varphi_j)(n) + (F_j\psi_j)(n) &= c_j(n-1)\varphi_j(n-\tau(n)) \\
&\quad + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\leq \frac{\alpha_j J}{k} + Q_j \sum_{u=n}^{n+T-1}\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\leq \frac{\alpha_j J}{k} + \frac{Q_j T(1-\alpha_j)J}{TQ_j k} = \frac{J}{k}
\end{aligned}$$

Thus,

$$||(D\varphi)(n) + (F\psi)(n)|| \leq \sum_{j=1}^{k} \frac{J}{k} = J$$

On the other hand,

$$\begin{aligned}
(D_j\varphi_j)(n) + (F_j\psi_j)(n) &= c_j(n-1)\varphi_j(n-\tau(n)) \\
&\quad + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\geq \frac{\sigma_j L}{k} + q_j \sum_{u=n}^{n+T-1}\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\geq \frac{\sigma_j L}{k} + \frac{q_j T(1-\sigma_j)L}{Tq_j k} = \frac{L}{k}
\end{aligned}$$

Thus,

$$||(D\varphi)(n) + (F\psi)(n)|| \geq \sum_{j=1}^{k} \frac{L}{k} = L.$$

This shows that $(D_j\varphi_j)(n)+(F_j\psi_j)(n)\in\mathbb{M}$. Therefore by Theorem 2.3.6 equation (8.1) has a positive periodic solution in $\mathbb{M}$.

8.2 Neutral functional difference equations with asymptotically constant or periodic solutions

8.2.1 Introduction

In this section we consider a special class of neutral difference equations with the property that every constant is a solution. In particular we consider neutral difference equations of the form

$$\Delta\Big(x(n)-h(x(n-L_1))\Big)=g(x(n))-g(x(n-L_2)),\ \ n\in\mathbb{Z}, \tag{8.17}$$

where $\mathbb{Z}$ is the set of integers. Clearly, any constant function is a solution. We suppose that $g,h:\mathbb{R}\to\mathbb{R}$ and their continuous in x with $\mathbb{R}$ denoting the set of all real numbers.

In addition to (8.17) we will also consider

$$\Delta\Big(x(n)-h(x(n-L_1-L_2))\Big)=g(x(n-L_1))-g(x(n-L_1-L_2)), \tag{8.18}$$

and

$$\begin{aligned}\Delta\Big(x(n)-h(x(n-L))\Big) &= g(n,x(n))-g(n,x(n-L)), \\ g(n+L,x)=g(n,x). &\end{aligned} \tag{8.19}$$

Let $L=\max(L_1,L_2)$ and let $\psi:[-L,0]\to\mathbb{R}$ be a given bounded initial function. We say $x(n,0,\psi)$ is a solution of (8.17) if $x(n,0,\psi)=\psi$ on $[-L,0]$ and $x(n,0,\psi)$ satisfies (8.17) for $n\geq 0$. This section is motivated by the work of Raffoul (2011) where (8.17) with $h(x(n-L_1))=0$ was considered.

8.2.2 Convergence and Stability

In this subsection we determine the constant that all solutions of (8.17) converge to. In particular we prove a theorem which for any given initial function gives explicitly the limit to which the solution converges to. Thus in this subsection we make the following assumptions.

The functions g and h are globally Lipschitz. That is, there exist positive constants K_1, K_2 such that for all $x, y \in \mathbb{R}$ we have

$$|h(x) - h(y)| \leq K_1|x - y| \tag{8.20}$$

and

$$|g(x) - g(y)| \leq K_2|x - y| \tag{8.21}$$

with an $\alpha < 1$ such that

$$K_1 + L_2K_2 \leq \alpha. \tag{8.22}$$

Let $\psi : [-L, 0] \to \mathbb{R}$ be a given initial function. Then for any constant c, equation (8.17) can be written as

$$x(n) = \sum_{s=n-L_2}^{n-1} g(x(s)) + h(x(n - L_1)) + c. \tag{8.23}$$

Substituting $n = 0$ in (8.23) gives

$$\psi(0) = \sum_{s=-L_2}^{-1} g(x(s)) + h(x(-L_1)) + c. \tag{8.24}$$

From (8.24),

$$c = \psi(0) - \sum_{s=-L_2}^{-1} g(x(s)) - h(x(-L_1)). \tag{8.25}$$

Substituting (8.25) into (8.23) gives

$$x(n) = \sum_{s=n-L_2}^{n-1} g(x(s)) + h(x(n - L_1)) + \psi(0) - \sum_{s=-L_2}^{-1} g(\psi(s)) - h(\psi(-L_1)). \tag{8.26}$$

Theorem 8.2.2.1. Suppose that (8.20)-(8.22) hold and let $\psi : [-L, 0] \to \mathbb{R}$ be a given initial function. Then, the unique solution $x(n, 0, \psi)$ of (8.17) satisfies $x(n, 0, \psi) \to r$, where r is unique and is given by

$$r = g(r)L_2 + h(r) + \psi(0) - \sum_{s=-L_2}^{-1} g(\psi(s)) - h(\psi(-L_1)). \tag{8.27}$$

Proof. Let $|.|$ denote the absolute value then the metric space $(\mathbb{R},|.|)$, is complete. Define the mapping $H:\mathbb{R}\to\mathbb{R}$, by

$$Hr = g(r)L_2 + h(r) + \psi(0) - \sum_{s=-L_2}^{-1} g(\psi(s)) - h(\psi(-L_1)).$$

For $a,b\in\mathbb{R}$, we have

$$\begin{aligned}|Ha-Hb| &\leq L_2|g(a)-g(b)| + |h(a)-h(b)|\\ &\leq L_2K_2|a-b| + K_1|a-b|\\ &= (L_2K_2+K_1)|a-b|\\ &\leq \alpha|a-b|.\end{aligned}$$

Thus showing that H is a contraction on the complete metric space $(\mathbb{R},|.|)$. It therefore follows from the Banach principle that H has a unique fixed point r, and this implies that (8.27) has a unique solution. We next show that (8.17) has a unique solution and that it converges to the constant r.

Let $\|.\|$ denote the maximum norm and let $\mathbb{B}$ be the set of bounded functions $\phi:[-L,\infty)\to\mathbb{R}$ with $\phi(n)=\psi(n)$ on $[-L,0]$, $\phi(n)\to r$ as $n\to\infty$. Then $(\mathbb{B},\|.\|)$ defines a complete metric space. For $\phi\in\mathbb{B}$, define $P:\mathbb{B}\to\mathbb{B}$ by

$$(P\phi)(n) = \psi(n), \text{ for } -L\leq n\leq 0,$$

and

$$\begin{aligned}(P\phi)(n) &= \sum_{s=n-L_2}^{n-1} g(\phi(s)) + h(\phi(n-L_1)) + \psi(0) - \sum_{s=-L_2}^{-1} g(\psi(s))\\ &- h(\psi(-L_1)),\ n\geq 0.\end{aligned} \tag{8.28}$$

For $\phi\in\mathbb{B}$ with $\phi(n)\to r$, we have $\sum_{s=n-L_2}^{n-1} g(\phi(s))\to g(r)L_2$ and $h(\phi(n-L_1))\to h(r)$ as $n\to\infty$. Hence,

$$(P\phi)(n)\to g(r)L_2 + h(r) + \psi(0) - \sum_{s=-L_2}^{-1} g(\psi(s)) - h(\psi(-L_1)) = r.$$

This shows that P maps from $\mathbb{B}$ into itself. Finally, we show that P is a contraction.

Let $a, b \in \mathbb{B}$, then we have

$$\begin{aligned}
|(Pa)(n)-(Pb)(n)| &\leq \sum_{s=n-L_2}^{n-1} |g(a)-g(b)| + |h(a(n-L_1)-h(b(n-L_1)))| \\
&\leq L_2K_2||a-b|| + K_1||a-b|| \\
&\leq (L_2K_2+K_1)||a-b|| \leq \alpha||a-b||.
\end{aligned}$$

Thus, P is a contraction and by the Banach's theorem P has a unique fixed point $\phi \in \mathbb{B}$ which satisfies (8.17) due to how the mapping P was constructed.

Remark 8.2.2.2. For any arbitrary initial function, say $\eta : [-L, 0] \to \mathbb{R}$, Theorem 8.2.2.1 shows that $x(n, 0, \eta) \to r$. Thus, we may think of r as being a global attractor.

Theorem 8.2.2.1 may also be thought of as a stability result in the following sense. We know in general that solutions depend continuously on initial functions. That is, solutions which start close remain close on finite intervals. However, under the conditions in Theorem 8.2.2.1 such solutions remain close forever, and their asymptotic respective constants remain close too. This fact is verified in the following theorem.

Theorem 8.2.2.3. Suppose the hypotheses of Theorem 8.2.2.1 hold. Then every initial function is stable. Moreover, if ψ_1 and ψ_2 are two initial functions with $x(n, 0, \psi_1) \to r_1$, and $x(n, 0, \psi_2) \to r_2$, then $|r_1 - r_2| < \varepsilon$ for positive ε.

Proof. Denote by $||\psi||_{[-L,0]}$ the supremum norm of ψ on the interval $[-L, 0]$. Let ψ_1 be a fixed initial function and ψ_2 any other initial function. Let $P_i, i = 1, 2$ be the mapping defined by (8.28). From Theorem 8.2.2.1 there are unique functions θ_1, θ_2 and unique constants r_1 and r_2 such that

$$P_1\theta_1 \to \theta_1, \;\; P_2\theta_2 \to \theta_2, \;\; \theta_1(n) \to r_1 \;\; \theta_2(n) \to r_2.$$

Let $\varepsilon > 0$ be any given positive number and set $\delta = \frac{\varepsilon(1-k_1-L_2K_2)}{1+k_1+L_2K_2}$. Then

$$\begin{aligned}
|\theta_1(n)-\theta_2(n)| &= |(P_1\theta_1)(n)-(P_2\theta_2)(n)| \\
&\leq |\psi_1(0)-\psi_2(0)| + \sum_{s=-L_2}^{-1} |g(\psi_1(s))-g(\psi_2(s))|
\end{aligned}$$

$$
\begin{aligned}
& + |h(\psi_1(-L_1)) - h(\psi_2(-L_1))| + |h(\theta_1(n-L_1)) \\
& - h(\theta_2(n-L_1))| + \sum_{s=n-L_2}^{n-1} |g(\theta_1(s)) - g(\theta_2(s))| \\
\leq \;& |\psi_1(0) - \psi_2(0)| + L_2K_2||\psi_1 - \psi_2|| \\
& + K_1||\psi_1 - \psi_2|| + K_2||\theta_1 - \theta_2|| \\
& + L_2K_2||\theta_1 - \theta_2||
\end{aligned}
$$

This yields

$$
||\theta_1 - \theta_2|| < \frac{1 + k_1 + L_2K_2}{1 - k_1 - L_2K_2} ||\psi_1 - \psi_2||_{[-L,0]} < \varepsilon,
$$

provided that

$$
||\psi_1 - \psi_2||_{[-L,0]} < \frac{\varepsilon(1 - k_1 - L_2K_2)}{1 + k_1 + L_2K_2} := \delta.
$$

This shows that

$$
|x(n,0,\psi_1) - x(n,0,\psi_2)| < \varepsilon, \text{ whenever } ||\psi_1 - \psi_2||_{[-L,0]} < \delta.
$$

To complete the proof we note that $|\theta_i(t) - r_i| \to 0$, as $n \to \infty$ implies that

$$
\begin{aligned}
|r_1 - r_2| &= |r_1 - \theta_1(n) + \theta_1(n) - \theta_2(n) + \theta_2(n) - r_2| \\
&\leq |r_1 - \theta_1(n)| + ||\theta_1 - \theta_2|| + |\theta_2(n) - r_2| \to ||\theta_1 - \theta_2||, \;\; (\text{as } n \to \infty) \\
&< \varepsilon.
\end{aligned}
$$

This completes the proof.

Finally, we study the periodicity of solutions of (8.19). Thus we consider the equation

$$
\Delta\Big(x(n) - h(x(n-L))\Big) \;=\; g(n,x(n)) - g(n,x(n-L)),
$$

where

$$
g(n+L,x) = g(n,x) \tag{8.29}
$$

for all x. In view of condition (8.29) we rewrite (8.19) in the form

$$\begin{aligned}\Delta\Big(x(n)-h(x(n-L))\Big) &= g(n,x(n))-g(n-L,x(n-L)) \\ &= \Delta\sum_{s=n-L}^{n-1} g(s,x(s)). \end{aligned} \tag{8.30}$$

We assume that the function $g(n,x)$ is globally Lipschitz. That is, there exists a constant $k_3 > 0$ such that

$$|g(n,x)-g(n,y)| \le K_3|x-y|. \tag{8.31}$$

We also assume that

$$K_3 L \le \rho, \text{ for some } 0<\rho<1. \tag{8.32}$$

Theorem 8.2.2.4. Suppose that conditions (8.29), (8.31) and (8.32) hold. If (8.19) has an L-periodic solution, then that solution is constant.

Proof. The existence of a unique solution of (8.19) is given by conditions (8.29), (8.31) and (8.32). Summing (8.30) from $u=0$ to $u=n-1$ gives

$$\begin{aligned} x(n) &= h(x(n-L))+\sum_{s=n-L}^{n-1} g(s,x(s))+x(0)-h(x(-L)) \\ &\quad -\sum_{s=-L}^{-1} g(s,x(s)). \end{aligned} \tag{8.33}$$

Set

$$G(n) := g(n,x(n)).$$

Since $x(n+L)=x(n)$ we have by (8.29) that $G(n)$ satisfies $G(n+L)=G(n)$. This implies that the sum of G over any interval of length L is constant. In other words,

$$\sum_{s=n-L}^{n-1} g(s,x(s)) = \sum_{s=n-L}^{n-1} G(s) = \sum_{s=-L}^{-1} G(s) = \sum_{s=-L}^{-1} g(s,x(s)).$$

Moreover, since $x(n-L)=x(n)$ we have that $x(-L)=x(0)$ thus $h(x(n-L)) = h(x(-L))$. It then follows from (8.33) that $x(n)=x(0)$ for $n\ge -L$. Hence x is constant.

8.2.3 Convergence and Stability for (8.18)

In this subsection we consider (8.18). In particular we will show that the behaviour of solutions of (8.18) is the same as Equation (8.17) and that it has no periodic solutions except constants.

Let $\psi : [-L_1 - L_2, 0] \to \mathbb{R}$ be a given initial function. Then for any constant c, equation (8.18) can be written as

$$x(n) = \sum_{s=n-L_1-L_2}^{n-L_1-1} g(x(s)) + h(x(n-L_1-L_2)) + c. \tag{8.34}$$

Substituting $n = 0$ in (8.34) gives

$$\psi(0) = \sum_{s=-L_1-L_2}^{-L_1-1} g(x(s)) + h(x(-L_1-L_2)) + c. \tag{8.35}$$

From (8.35),

$$c = \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(x(s)) - h(x(-L_1-L_2)). \tag{8.36}$$

Substituting (8.36) into (8.34) gives

$$\begin{aligned} x(n) &= \sum_{s=n-L_1-L_2}^{n-L_1-1} g(x(s)) + h(x(n-L_1-L_2)) \\ &\quad + \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(\psi(s)) - h(\psi(-L_1-L_2)). \end{aligned} \tag{8.37}$$

Theorem 8.2.3.1. Suppose that (8.20)-(8.22) hold and let $\psi : [-L_1 - L_2, 0] \to \mathbb{R}$ be a given initial function. Then, the unique solution $x(n, 0, \psi)$ of (8.18) satisfies $x(n, 0, \psi) \to r$, where r is unique and satisfies

$$\begin{aligned} r &= g(r)L_2 + h(r) \\ &\quad + \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(\psi(s)) - h(\psi(-L_1-L_2)). \end{aligned} \tag{8.38}$$

Proof. Let $|.|$ denote the absolute value then the metric space $(\mathbb{R}, |.|)$, is complete. Define the mapping $J : \mathbb{R} \to \mathbb{R}$, by

$$\begin{aligned} Jr &= g(r)L_2 + h(r) \\ &\quad + \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(\psi(s)) - h(\psi(-L_1-L_2)). \end{aligned}$$

For $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
|Ja - Jb| &\leq L_2|g(a) - g(b)| + |h(a) - h(b)| \\
&\leq L_2K_2|a-b| + K_1|a-b| \\
&= (L_2K_2 + K_1)|a-b| \\
&\leq \alpha|a-b|.
\end{aligned}$$

Thus showing that J is a contraction on the complete metric space $(\mathbb{R}, |.|)$. It therefore follows from the Banach principle that J has a unique fixed point r, and this implies that (8.38) has a unique solution. We next show that (8.18) has a unique solution and that it converges to the constant r.

Let $||.||$ denote the maximum norm and let $\mathbb{B}$ be the set of bounded functions $\phi : [-L_1 - L_2, \infty) \to \mathbb{R}$ with $\phi(n) = \psi(n)$ on $[-L_1 - L_2, 0]$, $\phi(n) \to r$ as $n \to \infty$. Then $(\mathbb{B}, ||.||)$ defines a complete metric space. For $\phi \in \mathbb{B}$, define $P : \mathbb{B} \to \mathbb{B}$ by

$$(P\phi)(n) = \psi(n), \text{ for } -L_1 - L_2 \leq n \leq 0,$$

and

$$\begin{aligned}
(P\phi)(n) &= \sum_{s=n-L_1-L_2}^{n-L_1-1} g(\phi(s)) + h(\phi(n - L_1 - L_2)) \\
&\quad + \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(\psi(s)) - h(\psi(-L_1 - L_2)), \; n \geq 0. \quad (8.39)
\end{aligned}$$

For $\phi \in \mathbb{B}$ with $\phi(n) \to r$, we have $\sum_{s=n-L_1-L_2}^{n-L_1-1} g(\phi(s)) \to g(r)L_2$ and $h(\phi(n - L_1 - L_2)) \to h(r)$ as $n \to \infty$. Hence,

$$(P\phi)(n) \to g(r)L_2 + h(r) + \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g(\psi(s)) - h(\psi(-L_1 - L_2)) = r.$$

This shows that P maps from $\mathbb{B}$ into itself. Finally, we show that P is a contraction.

Let $a, b \in \mathbb{B}$, then we have

$$\begin{aligned}
|(Pa)(n) - (Pb)(n)| &\leq \sum_{s=n-L_2}^{n-1} |g(a) - g(b)| + |h(a(n - L_1 - L_2) \\
&\quad - h(b(n - L_1 - L_2)))| \\
&\leq L_2K_2||a-b|| + K_1||a-b|| \\
&\leq (L_2K_2 + K_1)||a-b|| \leq \alpha||a-b||.
\end{aligned}$$

Thus, P is a contraction and by the Banach's theorem P has a unique fixed point $\phi \in \mathbb{B}$ which satisfies (8.18) due to how the mapping P was constructed.

CHAPTER NINE

STABILITY AND PERIODICITY IN NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE DELAYS

9.1 Asymptotic stability of difference equations

9.1.1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+ = [0,\infty)$, $\mathbb{Z}$ the integers, $\mathbb{Z}^-$ the negative integers, and $\mathbb{Z}^+ = \{x \in \mathbb{Z} \ : \ x \geq 0\}$. In this section we consider the scalar delay equation

$$\Delta x(n) = -a(n)x(n-\tau(n)) \tag{9.1}$$

and its generalization

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)(x(n-\tau_j(n)), \tag{9.2}$$

where $a, a_j : \mathbb{Z}^+ \to \mathbb{R}$ and $\tau, \tau_j : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n-\tau(n) \to \infty$ as $n \to \infty$. For each n_0, define $m_j(n_0) = \inf\{s-\tau_j(s) : \ s \geq n_0\}, m(n_0) = \min\{m_j(n_0) \ : \ 1 \leq j \leq N\}$. Note that (9.2) becomes (9.1) for $N = 1$.

Islam and Yankson (2005) showed that the zero solution of the equation

$$x(n+1) = b(n)x(n) + a(n)x(n-\tau(n)) \tag{9.3}$$

is asymptotically stable with one of the assumptions being that

$$\prod_{s=0}^{n-1} b(s) \to 0 \ \text{ as } \ n \to \infty. \tag{9.4}$$

Condition (9.4) cannot hold for (9.2) since $b(n) = 1$, for all $n \in \mathbb{Z}$. Raffoul (2006) obtained results to overcome the requirement of (9.4) for (9.1) when the delay $\tau(n) =$

c, where c is a positive constant. Our objective in this Chapter is to obtain stability results for (9.2) that will also overcome requirement of (9.4).

Remark 9.1.1.1 The content of this Chapter has been published as:

E. Yankson, " Stability in discrete equations with variable delays," Electronic Journal of Qualitative Theory of Differential Equations, No. 8, 2009.

Let $D(n_0)$ denote the set of bounded sequences $\psi : [m(n_0), n_0] \to \mathbb{R}$ with the maximum norm $||.||$. Also, let $(B, ||.||)$ be the Banach space of bounded sequences $\varphi : [m(n_0), \infty) \to \mathbb{R}$ with the maximum norm. Define the inverse of $n - \tau_i(n)$ by $g_i(n)$ if it exists and then set

$$Q(n) = \sum_{j=1}^{N} b(g_j(n)),$$

where

$$\sum_{j=1}^{N} b(g_j(n)) = 1 - \sum_{j=1}^{N} a(g_j(n)).$$

For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, a solution of (9.2) through (n_0, ψ) is a function $x : [m(n_0), n_0 + \alpha) \to \mathbb{R}$ for some positive constant $\alpha > 0$ such that $x(n)$ satisfies (9.2) on $[n_0, n_0 + \alpha)$ and $x(n) = \psi(n)$ for $n \in [m(n_0), n_0]$. We denote such a solution by $x(n) = x(n, n_0, \psi)$. For a fixed n_0, we define

$$||\psi|| = \max\{|\psi(n)| \, : \, m(n_0) \le n \le n_0\}.$$

9.1.2 Asymptotic Stability

In this subsection we obtain conditions for the zero solution of (9.2) to be asymptotically stable.

We begin by rewriting (9.2) as

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(g_j(n))x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s), \tag{9.5}$$

where Δ_n represents that the difference is with respect to n. But (9.5) implies that

$$\begin{aligned} x(n+1) - x(n) &= -\sum_{j=1}^{N} a_j(g_j(n))x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s) \\ x(n+1) &= \Big(1 - \sum_{j=1}^{N} a_j(g_j(n))\Big)x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s). \end{aligned}$$

If we let

$$\sum_{j=1}^{N} b_j(g_j(n)) = 1 - \sum_{j=1}^{N} a_j(g_j(n)),$$

then (9.5) is equivalent to

$$x(n+1) = \sum_{j=1}^{N} b_j(g_j(n))x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s). \tag{9.6}$$

Lemma 9.1.2.1. Suppose that $Q(n) \neq 0$ for all $n \in \mathbb{Z}^+$ and the inverse function $g_j(n)$ of $n - \tau_j(n)$ exists. Then $x(n)$ is a solution of (9.2) if and only if

$$\begin{aligned} x(n) &= \Big(x(n_0) - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))x(s)\Big) \prod_{s=n_0}^{n-1} Q(s) \\ &\quad + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s) \\ &\quad - \sum_{s=n_0}^{n-1} \Big([1-Q(s)] \prod_{k=s+1}^{n-1} Q(s) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))x(u)\Big), \ \ n \geq n_0. \end{aligned}$$

Proof. By the variation of parameters formula we obtain

$$\begin{aligned} x(n) &= x(n_0) \prod_{s=n_0}^{n-1} Q(s) \\ &\quad + \sum_{k=0}^{n-1} \Big(\prod_{s=k}^{n-1} Q(s) \Delta_k \sum_{j=1}^{N} \sum_{s=k-\tau_j(k)}^{k-1} a_j(g_j(s))x(s)\Big). \end{aligned} \tag{9.7}$$

Using the summation by parts formula we obtain

$$\begin{aligned} &\sum_{k=0}^{n-1} \Big(\prod_{s=k}^{n-1} Q(s) \Delta_k \sum_{j=1}^{N} \sum_{s=k-\tau_j(k)}^{k-1} a_j(g_j(s))x(s)\Big) \\ &= \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s) \\ &\quad - \prod_{s=n_0}^{n-1} Q(s) \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))x(s) \\ &\quad - \sum_{s=n_0}^{n-1} \Big([1-Q(s)] \prod_{k=s+1}^{n-1} Q(k) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))x(u)\Big). \end{aligned} \tag{9.8}$$

Substituting (9.8) into (9.7) gives the desired result. This completes the proof.

We next state and prove the main results in this section.

Theorem 9.1.2.2. Suppose that the inverse function $g_j(n)$ of $n-\tau_j(n)$ exists, and assume there exists a constant $\alpha \in (0,1)$ such that

$$\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|a_j(g_j(s))| + \sum_{s=n_0}^{n-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(k)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big) \le \alpha. \qquad (9.9)$$

Moreover, assume that there exists a positive constant M such that

$$\Big|\prod_{s=n_0}^{n-1}Q(s)\Big| \le M.$$

Then the zero solution of (9.2) is stable.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$(M + M\alpha)\delta + \alpha\varepsilon \le \varepsilon.$$

Let $\psi \in D(n_0)$ such that $|\ \psi(n)\ | \le \delta$. Define $S = \{\varphi \in B\ :\ \varphi(n) = \psi(n) \text{ if } n \in [m(n_0), n_0], \|\ \varphi\ \| \le \varepsilon\}$. Then $(S, \|\ .\ \|)$ is a complete metric space where, $\|\ .\ \|$ is the maximum norm.

Define the mapping $P : S \to S$ by

$$(P\varphi)(n) = \psi(n) \text{ for } n \in [m(n_0), n_0],$$

and

$$\begin{aligned}(P\varphi)(n) \;=\; & \Big(\psi(n_0) - \sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}a_j(g_j(s))\psi(s)\Big)\prod_{s=n_0}^{n-1}Q(s) \\ & + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}a_j(g_j(s))\varphi(s) \\ & - \sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(s)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}a_j(g_j(u))\varphi(u)\Big)\end{aligned} \qquad (9.10)$$

We first show that P maps from S to S. By (9.10),

$$\begin{aligned}
|\,(P\varphi)(n)\,| &\leq M\delta + M\alpha\delta + \Big\{\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s)) \\
&\quad + \sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(k)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\Big\}\,\|\,\varphi\,\| \\
&\leq (M+M\alpha)\delta + \alpha\varepsilon \\
&\leq \varepsilon.
\end{aligned}$$

Thus P maps from S into itself. We next show that $P\varphi$ is continuous.

Let $\varphi,\phi \in S$. Given any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{\alpha}$ such that $||\varphi-\phi|| < \delta$. Then,

$$\begin{aligned}
||(P\varphi)-(P\phi)|| &\leq \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} |a_j(g_j(s))|||\varphi-\phi|| \\
&\quad - \sum_{s=n_0}^{n-1}\Big([1-Q(s)]\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big) \\
&\quad \times ||\varphi-\phi|| \\
&\leq \alpha||\varphi-\phi|| \\
&\leq \varepsilon.
\end{aligned}$$

Thus showing that $P\varphi$ is continuous. Finally we show that P is a contraction.

Let $\varphi,\eta \in S$. Then

$$\begin{aligned}
&|(P\varphi)(n)-(P\eta)(n)| \\
= &\Big|\Big(\psi(n_0) - \sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))\psi(s)\Big)\prod_{s=n_0}^{n-1}Q(s) \\
&+ \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))\varphi(s) \\
&- \sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(s)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\varphi(u)\Big)| \\
&- \Big(\psi(n_0) - \sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))\psi(s)\Big)\prod_{s=n_0}^{n-1}Q(s) \\
&- \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))\eta(s) \\
&+ \sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(s)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\eta(u)\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))\|\varphi-\eta\| \\
&\quad + \sum_{s=n_0}^{n-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big)\|\varphi-\eta\| \\
&\le \Big\{\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s)) \\
&\quad + \sum_{s=n_0}^{n-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big)\Big\}\|\varphi-\eta\| \\
&\le \alpha\|\varphi-\eta\|.
\end{aligned}
$$

This shows that P is a contraction. Thus, by the contraction mapping principle, P has a unique fixed point in P which solves (9.2) and for any $\varphi \in S$, $\|P\| \le \varepsilon$. This proves that the zero solution of (9.2) is stable.

Theorem 9.1.2.3. Assume that the hypotheses of Theorem 9.1.2.2 hold. Also assume that

$$\prod_{k=n_0}^{n-1} Q(k) \to 0 \ \text{ as } \ n \to \infty. \tag{9.11}$$

Then the zero solution of (9.2) is asymptotically stable.

Proof. We have already proved that the zero solution of (9.2) is stable. Let $\psi \in D(n_0)$ such that $|\psi(n)| \le \delta$ and define

$$
\begin{aligned}
S^* \;=\; &\Big\{\varphi \in B \mid \varphi(n) = \psi(n) \text{ if } n \in [m(n_0), n_0],\ ||\varphi|| \le \varepsilon \text{ and} \\
&\varphi(n) \to 0, \text{ as } n \to \infty\Big\}.
\end{aligned}
$$

Define $P : S^* \to P^*$ by (9.10). From the proof of Theorem 9.1.1.2, the map P is a contraction and for every $\varphi \in S^*$, $||(P\varphi)|| \le \varepsilon$.

We next show that $(P\varphi)(n) \to 0$ as $n \to \infty$. The first term on the right hand side of (9.10) goes to zero because of condition (9.11). It is clear from (9.9) and the fact that $\varphi(n) \to 0$ as $n \to \infty$ that

$$\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}\Big|a_j(g_j(s))\Big||\varphi(s)| \to 0 \text{ as } n \to \infty.$$

Now we show that the last term on the right hand side of (9.10) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, for each $\varepsilon_1 > 0$, there exists an $N_1 > n_0$ such that $s \geq N_1$ implies $|\varphi(s-\tau_j(s))| < \varepsilon_1$ for $j = 1,2,3,...,N$. Thus for $n \geq N_1$, the last term, I_3 in (9.10) satisfies

$$\begin{aligned}
|I_3| &= \Big|\sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(s)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}a_j(g_j(u))\varphi(u)\Big)\Big| \\
&\leq \sum_{s=n_0}^{N_1-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))||\varphi(u)|\Big) \\
&\quad + \sum_{s=N_1}^{n}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))||\varphi(u)|\Big) \\
&\leq \max_{\sigma\geq m(n_0)}|\varphi(\sigma)|\sum_{s=n_0}^{N_1-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))| \\
&\quad + \varepsilon_1\sum_{s=N_1}^{n}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big)
\end{aligned}$$

By (9.11), there exists $N_2 > N_1$ such that $n \geq N_2$ implies

$$\max_{\sigma\geq m(n_0)}|\varphi(\sigma)|\sum_{s=n_0}^{N_1-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(s)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))| < \varepsilon_1.$$

Applying (9.9) gives $|I_3| \leq \varepsilon_1 + \varepsilon_1\alpha < 2\varepsilon_1$. Thus, $I_3 \to 0$ as $n \to \infty$. Hence $(P\varphi)(n) \to 0$ as $n \to \infty$, and so $P\varphi \in S^*$.

By the contraction mapping principle, P has a unique fixed point that solves (9.2) and goes to zero as n goes to infinity. Therefore the zero solution of (9.2) is asymptotically stable.

9.2 Stability of totally nonlinear difference equations

In this section we turn our attention to the totally nonlinear delay difference equation

$$\Delta x(n) = -a(n)f(x(n-\tau(n))), \tag{9.12}$$

where Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $a : \mathbb{Z}^+ \to \mathbb{R}$, $\tau : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(0) = 0$, f is continuous, locally Lipschitz, and odd, while $x - f(x)$ is nondecreasing and $f(x)$ is increasing on an interval $[0,L]$ for some

$L > 0$.

For each n_0 define $m(n_0) = \inf\{s - \tau(s) : s \geq n_0\}$, $\mathbb{Z}^+ = [0,\infty) \cap \mathbb{Z}$, and $\mathbb{Z}_{m(n_0)} = [m(n_0),\infty) \cap \mathbb{Z}$. Note that if $0 < L_1 < L$, then the conditions on f given with the (9.12) hold on $[-L_1, L_1]$. Also note that if $\phi : \mathbb{Z}_{m(n_0)} \to \mathbb{R}$ with $\phi_{m(n_0)} = \psi$, if $|\phi| \leq L$, then for $n \in \mathbb{Z}^+$ we have

$$|\phi(n) - f(\phi(n))| \leq L - f(L),$$

since $x - f(x)$ is odd and nondecreasing on $[0, L]$. The symbol $\phi_{m(n_0)}$ denotes a segment of ϕ on $[m(n_0), n_0] \cap \mathbb{Z}$.

Let $(\mathbb{C}, ||.||)$ be the Banach space of real sequences $\phi : \mathbb{Z}_{m(n_0)} \to \mathbb{R}$ with supremum norm $||.||$. For any sequence ψ with $|\psi| < L$ we define

$$\mathbb{M} = \{\phi : \mathbb{Z}_{m(n_0)} \to \mathbb{R} \mid \phi_{m(n_0)} = \psi,\ \phi \in \mathbb{C},\ |\phi(n)| \leq L\}.$$

We will also use $||.||$ to denote the supremum norm of an initial sequence. It will be obvious from the sequence to which it is applied whether the norm denotes the supremum norm on $[m(n_0), n_0] \cap \mathbb{Z}$ or on $\mathbb{Z}_{m(n_0)}$. Finally, note that $(\mathbb{M}, ||.||)$ is a Banach space since $\mathbb{M}$ is a closed subset of $\mathbb{C}$.

Also define the inverse of $n - \tau(n)$ by $g(n)$ if it exists.

9.2.1 Stability

In this section we obtain sufficient conditions for the zero solution of (9.12) to be stable.

We begin by writing (9.12) as

$$\begin{aligned}
\Delta x(n) &= -a(g(n))f(x(n)) + \Delta_n \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)) \\
&= -a(g(n))x(n) + a(g(n))[x(n) - f(x(n))] + \Delta_n \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)).
\end{aligned} \tag{9.13}$$

Lemma 9.2.1.1. Suppose the inverse function $g(n)$ of $n-\tau(n)$ exists. Also, suppose that $a(g(n)) \neq 1$ for all $n \in \mathbb{Z}^+$. Then $x(n)$ is a solution of (9.13) if and only if

$$\begin{aligned} x(n) &= x(n_0)\prod_{s=n_0}^{n-1}[1-a(g(s))] - \prod_{u=n_0}^{n-1}[1-a(g(u))]\sum_{s=n_0-\tau(n_0)}^{n_0-1} a(g(s))f(x(s)) \\ &\quad + \sum_{s=n_0}^{n-1} a(g(s))\prod_{k=s}^{n-1}[1-a(g(k))][x(s)-f(x(s))] \\ &\quad - \sum_{s=n_0}^{n-1} a(g(s))\prod_{k=s+1}^{n-1}[1-a(g(k))]\sum_{u=s-\tau(s)}^{s-1} a(g(u))f(x(u)) \\ &\quad + \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)),\ n \geq n_0. \end{aligned} \tag{9.14}$$

Proof. Applying the variation of parameters formula to (9.13) gives

$$\begin{aligned} x(n) &= x(n_0)\prod_{s=n_0}^{n-1}[1-a(g(s))] + \sum_{k=n_0}^{n-1}\Big(\prod_{s=k}^{n-1}[1-a(g(s))]\Big[a(g(k))[x(k)-f(x(k))] \\ &\quad + \Delta_k \sum_{s=k-\tau(k)}^{k-1} a(g(s))f(x(s))\Big]\Big). \end{aligned} \tag{9.15}$$

Using the summation by parts formula we obtain

$$\begin{aligned} &\sum_{k=n_0}^{n-1}\Big(\prod_{s=k}^{n-1}[1-a(g(s))]\Delta_k \sum_{s=k-\tau(k)}^{k-1} a(g(s))f(x(s))\Big) \\ &= \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)) - \prod_{s=n_0}^{n-1}[1-a(g(s))]\sum_{s=n_0-\tau(n_0)}^{n_0-1} a(g(s))f(x(s)) \\ &\quad - \sum_{s=n_0}^{n-1}\Big(a(g(s))\prod_{k=s+1}^{n-1}[1-a(g(k))]\sum_{u=s-\tau(s)}^{n-1} a(g(u))f(x(u))\Big) \end{aligned} \tag{9.16}$$

Substituting (9.16) into (9.15) gives the desired results. This completes the proof.

We next state and prove our main results.

Theorem 9.2.1.2. Suppose the inverse function $g(n)$ of $n-\tau(n)$ exists. Let f be odd, increasing on $[0,L]$, satisfy a Lipschitz condition, and let $x-f(x)$ be nondecreasing on $[0,L]$. Suppose also that $|a(n)| < 1$ and for each $L_1 \in (0,L]$ we have

$$\begin{aligned} &|L_1 - f(L_1)| \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s}^{n-1} [1-a(g(k))] \\ &+ f(L_1) \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1-a(g(k))] \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \\ &+ f(L_1) \sup_{n\in\mathbb{Z}^+} \sum_{s=n-\tau(n)}^{n-1} |a(g(s))| < L_1. \end{aligned} \tag{9.17}$$

Then the zero solution of (9.12) is stable.

Proof. Let $L > 0$ be given. Choose $\delta > 0$ such that

$$\delta + K\delta \sum_{s=n_0-\tau(n_0)}^{n_0-1} |a(g(s)| + \alpha L \leq L.$$

Let $\psi : [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ be a bounded initial function such that $|\psi(n)| < \delta$. Define a mapping $H : \mathbb{M} \to \mathbb{M}$ using (9.21) so that for $\phi \in \mathbb{M}$ we have

$$(H\phi)(n) = \psi(n),\ n \in [m(n_0), n_0] \cap \mathbb{Z},$$

and for $n \in \mathbb{Z}^+$

$$\begin{aligned} (H\phi)(n) \;=\; & \psi(n_0) \prod_{s=n_0}^{n-1} [1-a(g(s))] - \prod_{u=n_0}^{n-1} [1-a(g(u))] \sum_{s=n_0-\tau(n_0)}^{n_0-1} a(g(s)) f(\psi(s)) \\ &+ \sum_{s=n_0}^{n-1} a(g(s)) \prod_{k=s}^{n-1} [1-a(g(k))][\phi(s) - f(\phi(s))] \\ &- \sum_{s=n_0}^{n-1} a(g(s)) \prod_{k=s+1}^{n-1} [1-a(g(k))] \sum_{u=s-\tau(s)}^{s-1} a(g(u)) f(\phi(u)) \\ &+ \sum_{s=n-\tau(n)}^{n-1} a(g(s)) f(\phi(s)). \end{aligned}$$

We first show that P maps from $\mathbb{M}$ to $\mathbb{M}$. By (9.17) there is $\alpha < 1$ such that if $\phi \in \mathbb{M}$, then

$$\begin{aligned} |(H\phi)(n)| \;\leq\; & ||\psi|| + ||f(\psi)|| \sum_{s=n_0-\tau(n_0)}^{n_0-1} |a(g(s))| \\ &+ |L - f(L)| \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s}^{n-1} [1-a(g(k))] \\ &+ f(L) \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1-a(g(k))] \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \end{aligned}$$

$$
\begin{aligned}
& + f(L)\sup_{n\in\mathbb{Z}^+}\sum_{s=n-\tau(n)}^{n-1}|a(g(s))| \\
\leq\ & ||\psi|| + ||f(\psi)||\sum_{s=n_0-\tau(n_0)}^{n_0-1}|a(g(s))| + \alpha L \\
\leq\ & ||\psi|| + K||\psi||\sum_{s=n_0-\tau(n_0)}^{n_0-1}|a(g(s))| + \alpha L \\
\leq\ & \delta + K\delta\sum_{s=n_0-\tau(n_0)}^{n_0-1}|a(g(s))| + \alpha L \\
\leq\ & L.
\end{aligned}
$$

Thus showing that H maps from $\mathbb{M}$ to $\mathbb{M}$. We will next show that H is a contraction. Let $d > \max\{3, \frac{1}{K}\}$, where K is the common Lipschitz constant for $f(x)$ and $x - f(x)$ on $[-L, L]$. Define a metric ρ on $\mathbb{M}$ as follows:

$$
\rho(\phi,\eta) = |\phi-\eta|_K := \sup_{n\in\mathbb{Z}^+}\prod_{j=0}^{n-1}\frac{1-|a(g(j))|}{dK[1+|a(g(j))|]}|\phi(n)-\eta(n)|.
$$

Then $(\mathbb{M}, \rho)$ is a complete metric space.

We now use this metric to show that H is a contraction with constant $\frac{3}{d}$. For $\phi, \eta \in \mathbb{M}$ we have

$$
\begin{aligned}
& |(H\phi)(n) - (H\eta)(n)| \\
\leq\ & \sum_{s=n_0}^{n-1}|a(g(s))|\prod_{k=s}^{n-1}[1-a(g(k))]|\phi(s) - f(\phi(s)) - \eta(s) + f(\eta(s))| \\
& + \sum_{s=n_0}^{n-1}|a(g(s))|\prod_{k=s+1}^{n-1}[1-a(g(k))]\sum_{u=s-\tau(s)}^{s-1}|a(g(u))|f(\phi(u)) - f(\eta(u))| \\
& + \sum_{s=n-\tau(n)}^{n-1}|a(g(s))||f(\phi(s)) - f(\eta(u))|. \qquad (9.18)
\end{aligned}
$$

Since $f(x)$ and $w(x) = x - f(x)$ both satisfy a Lipschitz condition with the same constant K, then the first term on the right hand side of (9.18) satisfies

$$
\begin{aligned}
& \prod_{j=0}^{n-1}\frac{1-|a(g(j))|}{dK[1+|a(g(j))|]}\sum_{s=n_0}^{n-1}|a(g(s))|\prod_{k=s}^{n-1}[1-a(g(k))]|w(\phi(s)) - w(\eta(s))| \\
\leq\ & K\sum_{s=n_0}^{n-1}|a(g(s))|\prod_{j=0}^{s-1}\frac{1-|a(g(j))|}{dK[1+|a(g(j))|]}|\phi(s)-\eta(s)| \\
& \times\prod_{k=s}^{n-1}\frac{[1-a(g(k))][1-|a(g(k))|]}{dK[1+|a(g(k))|]}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{d}|\phi-\eta|_K \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{j=0}^{s-1}[1-|a(g(j))|] \\
&\le \frac{1}{d}|\phi-\eta|_K.
\end{aligned}
$$

Similarly, the second term on the right hand side of (9.18) satisfies

$$
\begin{aligned}
&\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1}[1-a(g(k))] \\
&\times \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| f(\phi(u)) - f(\eta(u))|
\end{aligned}
$$

$$
\begin{aligned}
&\le K \sum_{s=n_0}^{n-1} \frac{|a(g(s))|[1-|a(g(s))|]}{dK[1+|a(g(s))|]} \prod_{k=s+1}^{n-1} \frac{[1-a(g(k))][1-|a(g(k))|]}{dK[1+|a(g(k))|]} \\
&\quad \times \sum_{u=s-\tau(s)}^{s-1} |a(g(u))|\phi(u)-\eta(u)| \prod_{j=0}^{u-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \\
&\quad \times \prod_{j=u}^{s-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \\
&\le (\frac{1}{d})|\phi-\eta|_K \sum_{s=n_0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1}[1-|a(g(k))|] \\
&\quad \times \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \prod_{j=u}^{s-1}[1-|a(g(j))|] \\
&\le (\frac{1}{d})|\phi-\eta|_K.
\end{aligned}
$$

Finally, the third term on the right hand side of (9.18) satisfies

$$
\begin{aligned}
&\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \sum_{s=n-\tau(n)}^{n-1} |a(g(s))||f(\phi(s))-f(\eta(u))| \\
&\le K \sum_{s=n-\tau(n)}^{n-1} |a(g(s))||\phi(s)-\eta(u)| \prod_{j=0}^{s-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \\
&\quad \times \prod_{j=s}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \\
&\le (\frac{1}{d})|\phi-\eta|_K \sum_{s=n-\tau(n)}^{n-1} |a(g(s))| \prod_{j=s}^{n-1}[1-|a(g(j))|] \\
&\le (\frac{1}{d})|\phi-\eta|_K.
\end{aligned}
$$

Thus $|H\phi - H\eta|_K \le (\frac{3}{d})|\phi-\eta|_K$. Therefore, by the contraction mapping principle H has a unique fixed point in $\mathbb{M}$. This completes the proof.

In the next theorem, we consider a situation when the coefficient $a(n)$ of equation (9.12) does not satisfy theorem 9.2.1.2. In that case, we can remove a portion of $a(n)$ which has a sufficiently small average.

Theorem 9.2.1.3. Let f satisfy the conditions in Theorem 9.2.1.2 and suppose the inverse function $g(n)$ of $n-\tau(n)$ exists. Suppose also that

$$a(n) = c(n) - b(n), \tag{9.19}$$

where $0 \le c(n) < 1$, $|b(n)| < 1$, while

$$\sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |b(s)| \prod_{k=s}^{n-1} [1-c(g(k))] + 2 \sup_{n\in\mathbb{Z}^+} \sum_{u=n-\tau(n)}^{n-1} c(g(u)) < 1. \tag{9.20}$$

Then the zero solution of (9.12) is stable.

Proof. Rewrite equation (9.12) as

$$\begin{aligned}
\Delta x(n) &= -c(g(n))f(x(n)) + \Delta_n \sum_{s=n-\tau(n)}^{n-1} c(g(s))f(x(s)) + b(n)f(x(n-\tau(n))) \\
&= -c(g(n))x(n) + c(g(n))[x(n) - f(x(n))] \\
&\quad + \Delta_n \sum_{s=n-\tau(n)}^{n-1} c(g(s))f(x(s)) + b(n)f(x(n-\tau(n))).
\end{aligned}$$

Using the variation of parameters formula followed by summation by parts as we have done before we obtain

$$\begin{aligned}
x(n) &= x(n_0) \prod_{s=n_0}^{n-1} [1-c(g(s))] - \prod_{u=n_0}^{n-1} [1-c(g(u))] \sum_{s=n_0-\tau(n_0)}^{n_0-1} c(g(s))f(x(s)) \\
&\quad + \sum_{s=n_0}^{n-1} c(g(s)) \prod_{k=s}^{n-1} [1-c(g(k))][x(s) - f(x(s))] \\
&\quad + \sum_{s=n_0}^{n-1} b(s) \prod_{k=s}^{n-1} [1-c(g(k))] f(x(s-\tau(s))) \\
&\quad - \sum_{s=n_0}^{n-1} c(g(s)) \prod_{k=s+1}^{n-1} [1-c(g(k))] \sum_{u=s-\tau(s)}^{s-1} c(g(u))f(x(u)) \\
&\quad + \sum_{s=n-\tau(n)}^{n-1} c(g(s))f(x(s)), \; n \ge n_0.
\end{aligned}$$

Note that $w(x) = x - f(x)$ has a maximum on $[0,L]$ at L. Define

$$\mathbb{M} = \{\phi : \mathbb{Z}_{m(n_0)} \to \mathbb{R} \mid \phi_{m(n_0)} = \psi,\ \phi \in \mathbb{C},\ |\phi(n)| \le L\}.$$

Define a mapping H on $\mathbb{M}$ using the last equation at x as before. Thus

$$\begin{aligned}
|(Hx(n)| &\leq ||\psi|| + f(||\psi||) \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0-\tau(n_0)}^{n_0-1} c(g(s)) \\
&\quad + L - f(L) + f(L) \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |b(s)| \prod_{k=s}^{n-1} [1 - c(g(k))] \\
&\quad + 2f(L) \sup_{n\in\mathbb{Z}^+} \sum_{s=n-\tau(n)}^{n-1} c(g(s)).
\end{aligned}$$

In order to say that $H : \mathbb{M} \to \mathbb{M}$ we need

$$L - f(L) + f(L) \sup_{n\in\mathbb{Z}^+} \sum_{s=n_0}^{n-1} |b(s)| \prod_{k=s}^{n-1} [1 - c(g(k))] + 2f(L) \sup_{n\in\mathbb{Z}^+} \sum_{s=n-\tau(n)}^{n-1} c(g(s)) < L.$$

Subtracting L from each side and dividing by $f(L)$, we arrive at (9.20). The contraction argument parallel to Theorem 9.2.1.2 uses the metric

$$\rho(\phi,\eta) = |\phi - \eta|_K := \sup_{n\in\mathbb{Z}^+} \prod_{j=0}^{n-1} \frac{[1 - |b(j)|][1 - c(g(j))]}{dK[1 + |b(j)|][1 + |c(g(j))|]} |\phi(n) - \eta(n)|,$$

where $d > \max\{4, 1/K\}$. The rest of the proof is exactly as before and so we omit it.

9.3 Periodic Solutions

In this section we study the existence of periodic solutions of the equation

$$\begin{aligned}
\Delta x(n) &= -a(n)h(x(n+1)) + c(n)\Delta x(n - \tau(n)) \\
&\quad + G(n, x(n), x(n - \tau(n))),\ \forall n \in \mathbb{Z}, \qquad (9.21)
\end{aligned}$$

where

$$G : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ and $\mathbb{R}$ being the set of integers and real numbers respectively.

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z},\mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z},\mathbb{R})$ is the space of all real valued functions. Then $(P_T, ||.||)$ is a Banach space with the maximum norm

$$||\varphi|| = \max_{n\in[0,T-1]} |\varphi(n)|.$$

Also, for any $L > 0$, define

$$\mathbb{M} = \{\varphi \in P_T \ : \ ||\varphi|| \le L\}.$$

We make the following assumptions in this section.

$$a(n+T) = a(n),\ c(n+T) = c(n),\ \tau(n+T) = \tau(n),\ \tau(n) \ge \tau^* > 0, \tag{9.22}$$

for some constant τ^*. Suppose further that

$$a(n) > 0, \tag{9.23}$$

and

$$G(n+T,x,y) = G(n,x,y). \tag{9.24}$$

Moreover, we also assume that G is Lipschitz continuous in x and y. That is, there are positive constants k_1, k_2 such that

$$|G(n,x,y) - G(n,z,w)| \le k_1||x-z|| + k_2||y-w||, \text{ for } x,y,z,w \in \mathbb{R}. \tag{9.25}$$

Lemma 9.3.1. Suppose that (9.22) and (9.23) hold. If $x \in P_T$, then x is a solution of equation (9.21) if and only if

$$\begin{aligned} x(n) \;=\; & \frac{c(n-1)}{1+a(n-1)}x(n-\tau(n)) + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ & \times \Big[\sum_{r=n-T}^{n-1} a(r)(x(r+1) - h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1} \\ & + \sum_{r=n-T}^{n-1}\{x(r-\tau(r))\phi(r) + G(r,x(r),x(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big], \end{aligned} \tag{9.26}$$

where

$$\phi(r) = \frac{c(r-1)}{1+a(r-1)} - c(r). \tag{9.27}$$

Proof. Let $x \in P_T$ be a solution of (9.21). Rewrite (9.21) as

$$\begin{aligned} &\Delta x(n) + a(n)x(n+1) \\ &= a(n)x(n+1) - a(n)h(x(n+1)) + c(n)\Delta x(n-\tau(n)) + G(n,x(n),x(n-\tau(n))). \end{aligned}$$

We consider two cases; $n \geq 1$ and $n \leq 0$. Considering first the case when $n \geq 1$ by multiplying both sides of the above equation by $\prod_{s=0}^{n-1}(1+a(s))$ and summing from $(n-T)$ to $(n-1)$ we obtain

$$\begin{aligned}
&\sum_{r=n-T}^{n-1} \Delta\Big[\prod_{s=0}^{r-1}(1+a(s))x(r)\Big] \\
&= \sum_{r=n-T}^{n-1} a(r)\{x(r+1)-h(x(r+1))\}\prod_{s=0}^{r-1}(1+a(s)) \\
&+ \sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r))+G(r,x(r),x(r-\tau(r)))\}\prod_{s=0}^{r-1}(1+a(s)).
\end{aligned}$$

Which gives

$$\begin{aligned}
&\prod_{s=0}^{n-1}(1+a(s))x(n) - \prod_{s=0}^{n-T-1}(1+a(s))x(n-T) \\
&= \sum_{r=n-T}^{n-1} a(r)\{x(r+1)-h(x(r+1))\}\prod_{s=0}^{r-1}(1+a(s)) \\
&+ \sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r))+G(n,x(r),x(r-\tau(r)))\}\prod_{s=0}^{r-1}(1+a(s)).
\end{aligned}$$

By dividing both sides of the above expression by $\prod_{s=0}^{n-1}(1+a(s))$ and the fact that $x(n) = x(n-T)$, we obtain

$$\begin{aligned}
x(n) &= \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \qquad (9.28)\\
&\quad\times\Big[\sum_{r=n-T}^{n-1} a(r)(x(r+1)-h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\quad+\sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r))+G(r,x(r),x(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big].
\end{aligned}$$

But,

$$\begin{aligned}
&\sum_{r=n-T}^{n-1} c(r)\Delta x(r-\tau(r))\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&= \sum_{r=n-T}^{n-1} c(r)\prod_{s=r}^{n-1}(1+a(s))^{-1}\Delta x(r-\tau(r)).
\end{aligned}$$

By considering $z = x(r-\tau(r))$ and $Ey = c(r)\prod_{s=r}^{n-1}(1+a(s))^{-1}$ we get $y = c(r-1)\prod_{s=r-1}^{n-1}(1+a(s))^{-1}$. Thus, by performing a summation by parts on the above equation using the summation by parts formula

$$\sum Ey\Delta z = yz - \sum z\Delta y,$$

we obtain

$$
\begin{aligned}
&\sum_{r=n-T}^{n-1} c(r)\Delta x(r-\tau(r)) \prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&= \Big[c(r-1) \prod_{s=r-1}^{n-1} (1+a(s))^{-1} x(r-\tau(r))\Big]_{n-T}^{n} \\
&\quad - \sum_{r=n-T}^{n-1} x(r-\tau(r))\Delta\Big(c(r-1) \prod_{s=r-1}^{n-1} (1+a(s))^{-1}\Big) \\
&= c(n-1) \prod_{s=n-1}^{n-1} (1+a(s))^{-1} x(n-\tau(n)) \\
&\quad - c(n-T-1) \prod_{s=n-T-1}^{n-1} (1+a(s))^{-1} x(n-T-\tau(n-T)) \\
&\quad - \sum_{r=n-T}^{n-1} x(r-\tau(r))\Delta\Big(c(r-1) \prod_{s=r-1}^{n-1} (1+a(s))^{-1}\Big) \\
&= \frac{c(n-1)}{1+a(n-1)} x(n-\tau(n)) - c(n-1) \prod_{s=n-T-1}^{n-1} (1+a(s))^{-1} x(n-\tau(n)) \\
&\quad - \sum_{r=n-T}^{n-1} x(r-\tau(r))\Big\{c(r) \prod_{s=r}^{n-1}(1+a(s))^{-1} - c(r-1) \prod_{s=r-1}^{n-1} (1+a(s))^{-1}\Big\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{r=n-T}^{n-1} c(r)\Delta x(r-\tau(r)) \prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&= \frac{c(n-1)}{1+a(n-1)} x(n-\tau(n))\Big(1 - \prod_{s=n-T}^{n-1} (1+a(s))^{-1}\Big) \\
&\quad + \sum_{r=n-T}^{n-1} x(r-\tau(r))\phi(r) \prod_{s=r}^{n-1}(1+a(s))^{-1},
\end{aligned} \tag{9.29}
$$

where ϕ is given by (9.27). Finally, substituting (9.29) into (9.28) completes the proof.

Now for $n \leq 0$, equation (9.21) is equivalent to

$$
\begin{aligned}
&\Delta\Big[\prod_{s=n-1}^{0} (1+a(s))x(n)\Big] \\
&= a(t)\{x(n+1) - h(x(n+1))\} \prod_{s=n-1}^{0} (1+a(s)) \\
&\quad + \{c(n)\Delta x(n-\tau(n)) + G(n, x(n), x(n-\tau(n)))\} \prod_{s=n-1}^{0} (1+a(s)).
\end{aligned}
$$

Summing the above equation from $(n-T)$ to $n-1$ we obtain (9.26).

For the next lemma we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$.

(H2) h is strictly increasing on U_L.

(H3) $\sup_{s \in U_L \cap \mathbb{Z}} \Delta h(s) \leq 1$.

(H4) $(s-r)\Big\{ \sup_{i \in U_L \cap \mathbb{Z}} \Delta h(i) \Big\} \geq h(s) - h(r) \geq (s-r)\Big\{ \inf_{i \in U_L \cap \mathbb{Z}} \Delta h(i) \Big\} \geq 0$ for $s, r \in U_L$ with $s \geq r$.

Lemma 9.3.2. Let L be a positive constant and $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1)-(H4)$. If $(H\varphi)(n) = \varphi(n+1) - h(\varphi(n+1))$, then H is a large contraction on the set $\mathbb{M}$.

Proof. Let $\phi, \varphi \in \mathbb{M}$ with $\phi \neq \varphi$. Then $\phi(n+1) \neq \varphi(n+1)$ for some $n \in \mathbb{Z}$. Define the set

$$D(\phi, \varphi) = \Big\{ n \in \mathbb{Z} : \phi(n+1) \neq \varphi(n+1) \Big\}.$$

Note that $\varphi(n+1) \in U_L$ for all $n \in \mathbb{Z}$ whenever $\varphi \in \mathbb{M}$. Since h is strictly increasing

$$\frac{h(\varphi(n+1)) - h(\phi(n+1))}{\varphi(n+1) - \phi(n+1)} = \frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)} > 0 \tag{9.30}$$

holds for all $n \in D(\phi, \varphi)$. By (H3) we have

$$1 \geq \sup_{i \in U_L \cap \mathbb{Z}} \Delta h(i) \geq \inf_{s \in U_L \cap \mathbb{Z}} \Delta h(s) \geq 0. \tag{9.31}$$

Define the set $U_n \subset U_L$ by $U_n = [\varphi(n+1), \phi(n+1)] \cap U_L$ if $\phi(n+1) > \varphi(n+1)$, and $U_n = [\phi(n+1), \varphi(n+1)] \cap U_L$ if $\phi(n+1) < \varphi(n+1)$, for $n \in D(\phi, \varphi)$. Hence, for a fixed $n_0 \in D(\phi, \varphi)$ we get by (H4) and (9.30) that

$$\sup\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \frac{h(\phi(n_0+1)) - h(\varphi(n_0+1))}{\phi(n_0+1) - \varphi(n_0+1)} \geq \inf\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\}.$$

Since $U_n \subset U_L$ for every $n \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq \sup\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \inf\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq \frac{h(\varphi(n+1)) - h(\phi(n+1))}{\varphi(n+1) - \phi(n+1)} \geq \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq 0 \qquad (9.32)$$

for all $n \in D(\phi, \varphi)$. So, (9.32) yields

$$\begin{aligned} |(H\phi)(n) - (H\varphi)(n)| &= |\phi(n+1) - h(\phi(n+1)) - \varphi(n+1) + h(\varphi(n+1))| \\ &= |\phi(n+1) - \varphi(n+1)| \Big| 1 - \Big(\frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)} \Big) \Big| \\ &\leq |\phi(n+1) - \varphi(n+1)| \Big(1 - \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \Big) \qquad (9.33) \end{aligned}$$

for all $n \in D(\phi, \varphi)$. Thus, (9.32) and (9.33) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}$ satisfying

$$\|\phi - \varphi\| = \sup_{n \in [-L, L] \cap \mathbb{Z}} |\phi(n+1) - \varphi(n+1)| \geq \varepsilon.$$

If $|\phi(n+1) - \varphi(n+1)| \leq \varepsilon/2$ for some $n \in D(\phi, \varphi)$, then from (9.33)

$$|(H\phi)(n) - (H\varphi)(n)| \leq |\phi(n+1) - \varphi(n+1)| \leq \frac{1}{2}\|\phi - \varphi\|. \qquad (9.34)$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(n+1) - \varphi(n+1)|$ for some $n \in D(\phi, \varphi)$, then from (9.32) and (H3) we conclude that

$$1 \geq \frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(n) - (H\varphi)(n)| &\leq |\phi(n+1) - \varphi(n+1)| \Big\{ 1 - \frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)} \Big\} \\ &\leq (1 - \lambda)\|\phi(n+1) - \varphi(n+1)\|, \qquad (9.35) \end{aligned}$$

where

$$\lambda := \frac{1}{2L} \min \Big\{ h(u + \frac{\varepsilon}{2}) - h(u), u \in [-L, L] \Big\} > 0.$$

Consequently, it follows from (9.34) and (9.35) that

$$|(H\phi(n) - (H\varphi)(n)| \leq \delta \|\phi - \varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1$. The proof is complete.

Define the maps $A, B : \mathbb{M} \to \mathbb{M}$ as follows

$$\begin{aligned}(A\varphi)(n) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n)) + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1}\{\varphi(r-\tau(r))\phi(r) + G(r, \varphi(r), \varphi(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1},\end{aligned} \tag{9.36}$$

and

$$\begin{aligned}(B\varphi)(n) &= \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)(x(r+1) - h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1}.\end{aligned} \tag{9.37}$$

For the rest of the section we make the following assumptions.

$$(k_1+k_2)L + |G(n,0,0)| \leq \beta L a(n), \tag{9.38}$$

$$|\phi(n)| \leq \delta a(n), \tag{9.39}$$

$$\max_{n\in[0,T-1]}\left|\frac{c(n-1)}{1+a(n-1)}\right| = \alpha, \tag{9.40}$$

$$J(\beta+\alpha+\delta) \leq 1, \tag{9.41}$$

where α, β, δ and J are positive constants with $J \geq 3$.

Lemma 9.3.3. Suppose (9.22)-(9.25) and (9.38)-(9.41) hold. Then the mapping $A : \mathbb{M} \to \mathbb{M}$ defined in (9.36) is continuous in the maximum norm and maps $\mathbb{M}$ into compact subsets of $\mathbb{M}$.

Proof. We first show that $A:\mathbb{M}\to\mathbb{M}$. Let $\varphi\in\mathbb{M}$. Then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n+T-1)}{1+a(n+T-1)}\varphi(n+T-\tau(n+T))+\Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1}\\ &\quad\times\sum_{r=n}^{n+T-1}\{\varphi(r-\tau(r))\phi(r)+G(r,\varphi(r),\varphi(r-\tau(r)))\}\prod_{s=r}^{n+T-1}(1+a(s))^{-1}\\ &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n))+\Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1}\\ &\quad\times\sum_{r=n}^{n+T-1}\{\varphi(r-\tau(r))\phi(r)+G(r,\varphi(r),\varphi(r-\tau(r)))\}\prod_{s=r}^{n+T-1}(1+a(s))^{-1}\end{aligned}$$

Let $j=r-T$, then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n))+\Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1}\\ &\quad\times\sum_{j=n-T}^{n+T-1}\{\varphi(j+T-\tau(j+T))\phi(j+T)\\ &\quad+G(j+T,\varphi(j+T),\varphi(j+T-\tau(j+T)))\}\prod_{s=j+T}^{n+T-1}(1+a(s))^{-1}.\end{aligned}$$

Now let $k=s-T$, then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n))+\Big(1-\prod_{k=n-T}^{n-1}(1+a(k))^{-1}\Big)^{-1}\\ &\quad\times\sum_{j=n-T}^{n-1}\{\varphi(j-\tau(j))\phi(j)\\ &\quad+G(j,\varphi(j),\varphi(j-\tau(j)))\}\prod_{k=j}^{n-1}(1+a(k))^{-1}.\\ &\quad=(A\varphi)(n).\end{aligned}$$

Consequently, $A:P_T\to P_T$.

In view of (9.25) we have that

$$\begin{aligned}|G(n,x,y)| &= |G(n,x,y)-G(n,0,0)+G(n,0,0)|\\ &\le |G(n,x,y)-G(n,0,0)|+|G(n,0,0)|\\ &\le k_1\|k_1\|+k_2\|y\|+|G(n,0,0)|.\end{aligned}$$

Also it follows from (9.23) that $1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in\mathbb{M}$, we obtain

$$
\begin{aligned}
|(A\varphi)(n)| &\le \Big|\frac{c(n-1)}{1+a(n-1)}\Big||\varphi(n-\tau(n))| + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{|\varphi(r-\tau(r))||\phi(r)| + |G(r,\varphi(r),\varphi(r-\tau(r)))|\} \\
&\quad \times \prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\le \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{\delta L a(r) + (k_1+k_2)L + |G(r,0,0)|\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\le \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{(\delta+\beta)L a(r)\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\le \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times (\delta+\beta)L\sum_{r=n-T}^{n-1}\Delta_r\Big[\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big] \\
&= (\alpha+\delta+\beta)L \le \frac{L}{J} < L.
\end{aligned}
$$

Thus $A\varphi\in\mathbb{M}$. Consequently, we have $A:\mathbb{M}\to\mathbb{M}$.

We next show that A is continuous in the maximum norm. Let $\varphi,\psi\in\mathbb{M}$, and let

$$
\begin{aligned}
&\mu_1 = \max_{n\in[0,T-1]}\Big|\frac{c(n-1)}{1+a(n-1)}\Big|,\ \ \mu_2 = \max_{n\in[0,T-1]}\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}, \\
&\mu_3 = \max_{r\in[n-T,T-1]}|\phi(r)|.
\end{aligned}
$$

Let $\varepsilon>0$ be given. Choose $\eta=\varepsilon/\rho$ where $\rho=\mu_1+\mu_2T(\mu_3+k_1+k_2)$ such that $\|\varphi-\psi\|<\eta$. Note that from (9.23), we have $\max_{r\in[n-T,T-1]}\prod_{s=r}^{n-1}(1+a(s))^{-1}\le 1$. Thus,

$$
\begin{aligned}
&|(A\varphi)(n)-(A\psi)(n)| \\
&\le \Big|\frac{c(n-1)}{1+a(n-1)}\Big|\|\varphi-\psi\|
\end{aligned}
$$

$$+\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}\sum_{r=n-T}^{n-1}\Big\{\|\varphi-\psi\||\phi(r)|$$
$$+|G(r,\varphi(r),\varphi(r-\tau(r)))-G(r,\psi(r),\psi(r-\tau(r)))|\Big\}\prod_{s=r}^{n-1}(1+a(s))^{-1}$$
$$\leq \mu_1\|\varphi-\psi\|+\mu_2\sum_{r=n-T}^{n-1}\Big\{\mu_3\|\varphi-\psi\|+(k_1+k_2)\|\varphi-\psi\|\Big\}$$
$$\leq \Big\{\mu_1+\mu_2 T(\mu_3+k_1+k_2)\Big\}\|\varphi-\psi\|<\varepsilon.$$

Therefore showing that A is continuous.

Next, we show that A maps bounded subsets into compact sets. Since M is bounded and A is continuous, $A\mathbb{M}$ is a subset of $\mathbb{R}^T$ which is bounded. So, $A\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$. The proof is complete.

Lemma 9.3.4. Suppose (9.22)-(9.25) and (9.38) hold. Also, suppose that

$$\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}$$
$$\times\sum_{r=n-T}^{n-1}|a(r)||H(\varphi(r+1))|\prod_{s=r}^{n-1}(1+a(s))^{-1}\leq\frac{(J-1)L}{J}. \qquad (9.42)$$

For A,B defined by (9.36) and (9.37) respectively, if $\varphi,\psi\in\mathbb{M}$ are arbitrary, then

$$A\varphi+B\psi:\mathbb{M}\to\mathbb{M}.$$

Proof. Let $\varphi,\psi\in\mathbb{M}$ be arbitrary. Using the result of Lemma 3.1 we obtain

$$|(A\varphi)(n)+(B\psi)(n)|$$
$$\leq\Big|\frac{c(n-1)}{1+a(n-1)}\Big||\varphi(n-\tau(n))|+\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}$$
$$\times\sum_{r=n-T}^{n-1}\{|\varphi(r-\tau(r))||\phi(r)|+|G(r,\varphi(r),\varphi(r-\tau(r)))|\}\prod_{s=r}^{n-1}(1+a(s))^{-1}$$
$$+\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}$$
$$\times\sum_{r=n-T}^{n-1}|a(r)||H(\varphi(r+1))|\prod_{s=r}^{n-1}(1+a(s))^{-1}$$
$$\leq\frac{L}{J}+\frac{(J-1)L}{J}=L.$$

Thus $A\varphi+B\psi\in\mathbb{M}$. This completes the proof.

The next result gives a relationship between the mappings H and B in the sense of a large contraction.

Lemma 9.3.5. Let B be defined by (9.37) and assume that (9.22)-(9.23) and (9.42) hold. If H is a large contraction on $\mathbb{M}$ then so is the mapping $B : \mathbb{M} \to \mathbb{M}$.

Proof. We will first show that B maps $\mathbb{M}$ into itself. Let $\varphi \in \mathbb{M}$ then

$$\begin{aligned}(B\varphi)(n+T) &= \Big(1 - \prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n}^{n+T-1} a(r)(x(r+1) - h(x(r+1))) \prod_{s=r}^{n+T-1}(1+a(s))^{-1}\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}(B\varphi)(n+T) &= \Big(1 - \prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n}^{n+T-1} a(j+T)(x(j+T+1) - h(x(j+T+1))) \\ &\quad \times \prod_{s=j+T}^{n+T-1}(1+a(s))^{-1}.\end{aligned}$$

Now let $k = s - T$, then

$$\begin{aligned}(B\varphi)(n+T) &= \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(j)(x(j+1) - h(x(j+1))) \prod_{k=j}^{n-1}(1+a(s))^{-1} \\ &= (B\varphi)(n).\end{aligned}$$

That is, $B : P_T \to P_T$.

In view of (9.42), we have

$$\begin{aligned}|(B\varphi)(n)| &\leq \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} |a(r)||H(\varphi(r+1))| \prod_{s=r}^{n-1}(1+a(s))^{-1} \qquad (9.43) \\ &\leq \frac{(J-1)L}{J} < L. \qquad (9.44)\end{aligned}$$

That is $B\varphi \in \mathbb{M}$ and consequently we have $B : \mathbb{M} \to \mathbb{M}$.

We next show that B is a large contraction. If H is a large contraction on $\mathbb{M}$, for $x, y \in \mathbb{M}$, with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. Thus, it follows from the equality

$$a(r)\prod_{s=r}^{n-1}(1+a(s))^{-1} = \Delta\Big[\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big]$$

that

$$\begin{aligned} |Bx(n) - By(n)| &\leq \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)|H(x(r+1)) - H(y(r+1))| \prod_{s=r}^{n-1}(1+a(s))^{-1} \\ &\leq \|x-y\|\Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)\prod_{s=r}^{n-1}(1+a(s))^{-1} = \|x-y\|. \end{aligned}$$

Thus

$$\|Bx - By\| \leq \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \leq \delta\|x - y\|$$

holds if we know the existence of a $\delta \in (0,1)$ and that for all $\varepsilon > 0$

$$[x, y \in \mathbb{M}, \|x-y\| > 0] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete.

Theorem 9.3.6. Let $(P_T, \|.\|)$ be the Banach space of T-periodic real valued functions and $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where L is a positive constant. Suppose that (9.22)-(9.25) and (9.38)-(9.41) hold. Then equation (9.21) has a T-periodic solution φ in $\mathbb{M}$.

Proof. By Lemma 9.3.1, φ is a solution of (9.21) if

$$\varphi = A\varphi + B\varphi,$$

where A and B are given by (9.36) and (9.37) respectively. By Lemma 9.3.3, $A : \mathbb{M} \to \mathbb{M}$ is completely continuous. By Lemma 9.3.4, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in$

$\mathbb{M}$. Moreover, $B : \mathbb{M} \to \mathbb{M}$ is a large contraction by Lemma 9.3.5. Thus all the hypotheses of Theorem 2.3.7 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (9.21) has a $T-$ periodic solution.

CHAPTER TEN

PERIODICITY AND STABILITY OF DYNAMIC EQUATIONS ON TIME SCALE

10.1 Introduction

In this Chapter, we obtain sufficient conditions for solutions of nonlinear neutral dynamic equations to be periodic on time scales. We further prove that the zero solution of Volterra dynamic equations are asymptotically stable on time scales. The concept of time scale analysis is a fairly new idea. It combines the traditional areas of continuous and discrete analysis into one theory. The study of dynamic equations brings together the traditional research areas of differential and difference equations. In the first section, we obtain sufficient conditions for the existence of periodic solutions of the nonlinear neutral dynamic equation

$$\begin{aligned} x^{\Delta}(t) &= -a(t)h(x(\sigma(t)) + (Q(t,x(t),x(t-g(t)))))^{\Delta} \\ &\quad + G\big(t,x(t),x(t-g(t))\big), t \in \mathbb{T}, \end{aligned}$$

on the time scale $\mathbb{T}$. Adivar and Raffoul (2009) considered the above equation when $Q(t,x(t),x(t-g(t)))) = 0$.

Motivated by the work of Wong and Soh (2005), Wong and Boey (2004) and Kulik and Tisdell (2008) on the theory of Fredholm-type and Volterra-type equations on time scales we consider the nonlinear dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(t-r(t)) + \int_{t-r(t)}^{t} k(t,s)h(x(s))\,\Delta s,\ t \in \mathbb{T},$$

on the time scale $\mathbb{T}$. In particular, we prove that the zero solution of the above equation is asymptotically stable.

10.2 Periodic solutions of totally nonlinear neutral dynamic equations on time scale

We begin this section by giving some definitions.

Definition 10.2.1. We say that a time scale $\mathbb{T}$ is *periodic* if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

For example, the following time scales are periodic.

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[2(i-1)h, 2ih]$, $h > 0$ has period $p = 2h$.

2. $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.

3. $\mathbb{T} = \mathbb{R}$.

4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $p = 1$.

Remark 10.2.2. All periodic time scales are unbounded above and below.

Definition 10.2.3. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 10.2.4. If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

In this section we show that the neutral dynamic equation

$$\begin{aligned} x^{\Delta}(t) &= -a(t)h(x(\sigma(t)) + (Q(t, x(t), x(t-g(t)))))^{\Delta} \\ &\quad + G\big(t, x(t), x(t-g(t))\big), t \in \mathbb{T}, \end{aligned} \tag{10.1}$$

has a periodic solution.

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a,b), (a,b]$, and (a,b) are defined similarly. Define $P_T = \{\varphi \in C(\mathbb{T},R) : \varphi(t+T) = \varphi(t)\}$ where, $C(\mathbb{T},R)$ is the space of all real valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

Lemma 10.2.5.[Kaufmann and Raffoul (2006)] Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.

In this section we assume that $a \in \mathcal{R}^+$ is continuous, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t+T) = a(t), \quad \big(id - g\big)(t+T) = \big(id - g\big)(t), \tag{10.2}$$

where, *id* is the identity function on $\mathbb{T}$. We also require that $Q(t,x)$ and $G(t,x,y)$ are continuous and periodic in t and Lipschitz continuous in x and y. That is,

$$Q(t+T,x) = Q(t,x),\ G(t+T,x,y) = G(t,x,y), \tag{10.3}$$

and there are positive constants E_1, E_2, E_3 such that

$$|Q(t,x) - Q(t,y)| \leq E_1\|x-y\|, \text{ for } x,y \in \mathbb{R}, \tag{10.4}$$

and

$$|G(t,x,y) - G(t,z,w)| \leq E_2\|x-z\| + E_3\|y-w\|, \text{ for } x,y,z,w \in \mathbb{R}. \tag{10.5}$$

Lemma 10.2.6. Suppose (10.2), (10.3) hold. If $x \in P_T$, then x is a solution of equation (10.1) if, and only if,

$$\begin{aligned} x(t) &= Q\big(t, x(t-g(t))\big) + \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[a(s)[x^\sigma(s) - h(x(\sigma(s)))] - a(s)Q^\sigma\big(s, x(s-g(s))\big) \\ &\quad + G\big(s, x(s), x(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s. \end{aligned} \tag{10.6}$$

Proof. Let $x \in P_T$ be a solution of (10.1). First we write (10.1) as

$$
\begin{aligned}
\{x(t) - Q\big(t, x(t-g(t))\big)\}^{\Delta} = -a(t)\{x^{\sigma}(t) - Q^{\sigma}\big(t, x(t-g(t))\big)\} \\
+ a(t)[x^{\sigma}(t) - h(x(\sigma(t)))] \\
- a(t)Q^{\sigma}\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big).
\end{aligned}
$$

Multiply both sides by $e_a(t,0)$ and then integrate from $t-T$ to t to obtain

$$
\begin{aligned}
&\int_{t-T}^{t} \big[e_a(s,0)\{x(s) - Q\big(s, x(s-g(s))\big)\}\big]^{\Delta} \Delta s \\
&= \int_{t-T}^{t} \Big[a(s)[x^{\sigma}(s) - h(x(\sigma(s)))] - a(s)Q^{\sigma}\big(s, x(s-g(s))\big) \\
&\quad + G\big(s, x(s), x(s-g(s))\big)\Big] e_a(s,0)\,\Delta s.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&e_a(t,0)\Big(x(t) - a(t)Q\big(t, x(t-g(t))\big)\Big) \\
&- e_a(t-T,0)\Big(x(t-T) - a(t-T)Q\big(t-T, x(t-T-g(t-T))\big)\Big) \\
&= \int_{t-T}^{t} \big[a(s)[x^{\sigma}(s) - h(x(\sigma(s)))] - a(s)Q^{\sigma}\big(s, x(s-g(s))\big) \\
&\quad + G\big(s, x(s), x(s-g(s))\big)\Big] e_a(s,0)\,\Delta s.
\end{aligned}
$$

After making use of (10.2), (10.3) and $x \in P_T$, we divide both sides of the above equation by $e_a(t,0)$ to obtain

$$
\begin{aligned}
x(t) &= Q\big(t, x(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t} \Big[a(s)[x^{\sigma}(s) - h(x(\sigma(s)))] - a(s)Q^{\sigma}\big(s, x(s-g(s))\big) \\
&\quad + G\big(s, x(s), x(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s.
\end{aligned}
$$

Since each step is reversible, the converse follows. This completes the proof.

Define the mapping $H : P_T \to P_T$ by

$$
\begin{aligned}
&(H\varphi)(t) \\
&= Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t} \Big[a(s)[x^{\sigma}(s) - h(x(\sigma(s)))] - a(s)Q^{\sigma}\big(s, \varphi(s-g(s))\big) \\
&\quad + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s.
\end{aligned}
\tag{10.7}
$$

We express equation (10.7) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t)$$

where, A, B are given by

$$(B\varphi)(t) = \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \int_{t-T}^{t} a(s)[x^{\sigma}(s) - h(x(\sigma(s)))] e_{\ominus a}(t,s)\,\Delta s. \quad (10.8)$$

and

$$\begin{aligned} &(A\varphi)(t) \\ &= Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[-a(s)Q^{\sigma}\big(s, \varphi(s-g(s))\big) \\ &\quad + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s. \end{aligned} \quad (10.9)$$

In the rest of the section we require the following conditions.

$$E_1 L + |Q(t,0)| \le \alpha L, \quad (10.10)$$

$$E_2 L + E_2 L + |G(t,0,0)| \le L\gamma a(t), \quad (10.11)$$

and

$$J(2\alpha + \gamma) \le 1, \quad (10.12)$$

where α, γ, L and J are constants with $J \ge 3$.

Lemma 10.2.7. Suppose (10.2)–(10.5) and (10.10)–(10.12) hold. Then $A : \mathbb{M} \to \mathbb{M}$, as defined by (10.9), is continuous in the supremum norm and maps $\mathbb{M}$ into a compact subset of $\mathbb{M}$.

Proof. We first show that $A : \mathbb{M} \to \mathbb{M}$. Evaluate (10.9) at $t+T$.

$$\begin{aligned} (A\varphi)(t+T) &= Q\big(t+T, \varphi(t+T-g(t+T))\big) \\ &\quad + \big(1 - e_{\ominus a}(t+T, t)\big)^{-1} \times \int_{t}^{t+T} \Big[-a(s)Q^{\sigma}\big(s, \varphi(s-g(s))\big) \\ &\quad + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t+T, s)\,\Delta s. \end{aligned} \quad (10.13)$$

With $u = s - T$ and conditions (10.2) – (10.3) to get

$$\begin{aligned}(A\varphi)(t+T) = {} & Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t+T,t)\big)^{-1} \\ & \times \int_{t-T}^{t} \Big[-a(u+T)Q^{\sigma}\big(u-T, \varphi(u-T-g(u-T))\big) \\ & + G\big(s, \varphi(u-T), \varphi(u-T-g(u-T))\big)\Big] e_{\ominus a}(t+T, u+T)\,\Delta u.\end{aligned}$$

But we have that $e_{\ominus a}(t+T, u+T) = e_{\ominus a}(t,u)$ and $e_{\ominus a}(t+T,t) = e_{\ominus a}(t, t-T)$. Thus (10.13) becomes

$$\begin{aligned}(A\varphi)(t+T) &= Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ & \quad \times \int_{t-T}^{t} \Big[-a(u)Q^{\sigma}\big(u, \varphi(u-g(u))\big) \\ & \quad + G\big(u, \varphi(u), \varphi(u-g(u))\big)\Big] e_{\ominus a}(t+T, u)\,\Delta u \\ &= (A\varphi)(t).\end{aligned}$$

Note that in view of (10.4) and (10.5) we have

$$\begin{aligned}|Q(t,x)| &= |Q(t,x) - Q(t,0) + Q(t,0)| \\ &\le |Q(t,x) - Q(t,0)| + |Q(t,0)| \\ &\le E_1\|x\| + |Q(t,0)|.\end{aligned}$$

Similarly,

$$\begin{aligned}|G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\le |G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \\ &\le E_2\|x\| + E_3\|y\| + |G(t,0,0)|.\end{aligned}$$

Thus, for any $\varphi \in \mathbb{M}$ we have

$$\begin{aligned} & |(A\varphi)(t)| \\ = \; & \Big| Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ & \times \int_{t-T}^{t} \Big[-a(s)Q^{\sigma}\big(s, \varphi(s-g(s))\big) \\ & + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s \Big|\end{aligned}$$

$$\begin{aligned}
&\leq \; |Q\big(t,\varphi(t-g(t))\big)| + \big(1-e_{\ominus a}(t,t-T)\big)^{-1}\int_{t-T}^{t}\big|-a(s)\big|\,\big|Q^{\sigma}\big(s,\varphi(s-g(s))\big)\big| \\
&\quad + \big|G\big(s,\varphi(s),\varphi(s-g(s))\big)\big| e_{\ominus a}(t,s)\,\Delta s \\
&\leq \; E_1L + |Q(t,0)| + \big(1-e_{\ominus a}(t,t-T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t}\Big[a(s)(E_1L+|Q(s,0)|) + (E_2+E_3)L + |G(s,0,0)|\Big] e_{\ominus a}(t,s)\,\Delta s \\
&\leq \; \alpha L + \big(1-e_{\ominus a}(t,t-T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t}\Big[\alpha L + \gamma L\Big] a(s) e_{\ominus a}(t,s)\,\Delta s \\
&\leq \; (2\alpha+\gamma)L \leq \frac{L}{J} < L.
\end{aligned}$$

Thus showing that A maps $\mathbb{M}$ into itself.

To see that A is continuous, we let $\varphi,\psi \in \mathbb{M}$ and define

$$\begin{aligned}
&\eta := \max_{t\in[0,T]} \big|\big(1-e_{\ominus a}(t,t-T)\big)^{-1}\big|, \;\; \rho := \max_{t\in[0,T]} |a(t)|, \\
&\gamma := \max_{u\in[t-T,t]} e_{\ominus a}(t,u), \;\; \mu := \max_{t\in[0,T]} |\left(Q(t,x(t),x(t-g(t))))\right)^{\Delta}|, \\
&\alpha := \sup_{t\in[0,T]} |Q(t,0)|, \;\; \beta := \sup_{t\in[0,T]} |G(t,0,0)|.
\end{aligned} \tag{10.14}$$

Given $\varepsilon > 0$, take $\delta = \varepsilon/M$ with $M = E_1 + \eta\,\gamma\,T(\rho\,E_1 + E_2 + E_3)$ where, E_1, E_2 and E_3 are given by (10.4) and (10.5) such that $\|\varphi-\psi\| < \delta$. Using (10.9) we get

$$\begin{aligned}
\big\|A\varphi - A\psi\big\| &\leq E_1\|\varphi-\psi\| + \eta\gamma\int_0^T \Big[\rho\,E_1\|\varphi-\psi\| + (E_2+E_3)\|\varphi-\psi\|\Big]\Delta u \\
&\leq M\|\varphi-\psi\| < \varepsilon.
\end{aligned}$$

This proves that A is continuous.

We next show that A is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset \mathbb{M}$. Thus as before we have that

$$\|A(\varphi_n)\| \leq L,$$

showing that the sequence $\{A\varphi_n\}$ is uniformly bounded. Now, it can be easily checked that

$$\begin{aligned}
(A\varphi_n)^{\Delta}(t) \;&=\; (Q(t,x(t),x(t-g(t)))))^{\Delta} - a(t)Q^{\sigma}\big(t,\varphi(t-g(t))\big) \\
&\quad + G\big(t,\varphi(t),\varphi(t-g(t))\big) - a(t)\Big\{\big(1-e_{\ominus a}(t,t-T)\big)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \int_{t-T}^{t} \big[-a(s)Q^{\sigma}\big(s,\varphi(s-g(s))\big) + G\big(s,\varphi(s),\varphi(s-g(s))\big)\big] \\
&\quad \times e_{\ominus a}(t,s)\,\Delta s \Big\} \\
&= (Q(t,x(t),x(t-g(t)))))^{\Delta} - a(t)Q^{\sigma}\big(t,\varphi(t-g(t))\big) \\
&\quad + G\big(t,\varphi(t),\varphi(t-g(t))\big) - a(t)\Big\{ \big(1-e_{\ominus a}(t,t-T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t} \big[-a(s)Q^{\sigma}\big(s,\varphi(s-g(s))\big) + G\big(s,\varphi(s),\varphi(s-g(s))\big)\big] \\
&\quad \times e_{\ominus a}(t,s)\,\Delta s + Q(t,\varphi(t-g(t)))\Big\} + a(t)Q(t,\varphi(t-g(t))).
\end{aligned}
$$

$$
\begin{aligned}
(A\varphi_n)^{\Delta}(t) &= (Q(t,x(t),x(t-g(t)))))^{\Delta} - a(t)(A\varphi_n)^{\sigma}(t) - a(t)Q^{\sigma}\big(t,\varphi(t-g(t))\big) \\
&\quad + G\big(t,\varphi(t),\varphi(t-g(t))\big) + a(t)Q(t,\varphi(t-g(t))).
\end{aligned}
$$

Consequently,

$$
|(A\varphi_n)^{\Delta}(t)| \le \mu + L\rho + 2\rho(E_1L+\alpha) + (E_2+E_3)L + \beta
$$

for all n. That is $\|(A\varphi_n)^{\Delta}\| \le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equi-continuous. The Arzel$\grave{a}$-Ascoli theorem implies that there is a subsequence $\{A\varphi_{n_k}\}$ which converges uniformly to a continuous T-periodic function φ^*. Thus A is compact.

Lemma 10.2.8. Suppose $g : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. Suppose U is a compact interval with enpoints $r,s \in \mathbb{T}$ and $g^{\Delta}(t) \ge 0$ for all $t \in U^{\kappa} \cap D$. Then we have

$$
g(s) - g(r) \ge |s-r| \Big\{ \inf_{t \in U^{\kappa} \cap D} g^{\Delta}(t) \Big\}. \tag{10.15}
$$

Proof. Let the function $f : \mathbb{T} \to \mathbb{R}$ be defined by

$$
f(t) = (t-r)\Big\{ \inf_{t \in U^{\kappa} \cap D} g^{\Delta}(t) \Big\} \text{ for } t \in \mathbb{T}.
$$

Evidently, f is pre-differentiable with D and

$$
|f^{\Delta}(t)| = f^{\Delta}(t) = \Big\{ \inf_{t \in U^{\kappa} \cap D} g^{\Delta}(t) \Big\} \le g^{\Delta}(t).
$$

From (2.29) we derive

$$
g(s) - g(r) \ge |f(s) - f(r)| = |s-r| \Big\{ \inf_{t \in U^{\kappa} \cap D} g^{\Delta}(t) \Big\}.
$$

as desired. The proof is complete.

Corollary 10.2.9. Suppose $g : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. Suppose U is a compact interval with enpoints $r,s \in \mathbb{T}$ and $g^{\Delta}(t) \geq 0$ for all $t \in U^{\kappa} \cap D$ if and only if g is non-decreasing on U.

Proof. If $g^{\Delta}(t) \geq 0$ for all $t \in U^{\kappa} \cap D$, then from (10.15), we have

$$g(s) - g(r) \geq |s-r| \Big\{ \inf_{t \in U^{\kappa} \cap D} g^{\Delta}(t) \Big\} \geq 0$$

for $s,r \in U$ with $s \geq r$. Conversely, let g be non-decreasing on U. For a $t \in U^{\kappa} \cap D$, there are two possible cases:

$$\mu(t) = 0 \text{ or } \mu(t) > 0.$$

If $\mu(t) = \sigma(t) - t > 0$, then we have

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)} > 0.$$

If $\mu(t) = 0$, then we obtain

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{s - t} \geq 0.$$

This completes the proof.

In the next lemma we prove that H is a large contraction on $\mathbb{M}$. To this end we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$ and differentiable on U_L^{κ}.

(H2) h is strictly increasing on U_L.

(H3) $\sup_{s \in U_L^{\kappa}} h^{\Delta}(s) \leq 1$.

Lemma 10.2.10. Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H3)$. Then the mapping H is a large contraction on the set $\mathbb{M}$.

Proof. The function h satisfies the assumptions of Lemma 10.2.8 on the compact interval $U_L = [-L, L] \cap \mathbb{T}$. Thus it follows from (2.30) and (10.15) that

$$(s-r) \Big\{ \sup_{t \in U_L^{\kappa}} h^{\Delta}(t) \Big\} \geq h(s) - h(r) \geq (s-r) \Big\{ \inf_{t \in U_L^{\kappa}} h^{\Delta}(t) \Big\} \geq 0 \tag{10.16}$$

Let $\phi, \varphi \in \mathbb{M}$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{T}$. Define the set

$$D(\phi, \varphi) = \Big\{ t \in \mathbb{T} : \phi(t) \neq \varphi(t) \Big\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{T}$ whenever $\varphi \in \mathbb{M}$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \tag{10.17}$$

holds for all $t \in D(\phi, \varphi)$. By (H3) we have

$$1 \geq \sup_{t \in U_L^\kappa} h^\Delta(t) \geq \inf_{s \in U_L^\kappa} h^\Delta(s) \geq 0. \tag{10.18}$$

Define the set $U_t \subset U_L$ by $U_t = [\varphi(t), \phi(t)] \cap U_L$ if $\phi(t) > \varphi(t)$, and $U_t = [\phi(t), \varphi(t)] \cap U_L$ if $\phi(t) < \varphi(t)$, for $t \in D(\phi, \varphi)$. Hence, for a fixed $t_0 \in D(\phi, \varphi)$ we get by (10.16) and (10.17) that

$$\sup\{h^\Delta(u) : u \in U_{t_0}^\kappa\} \geq \frac{h(\phi(t_0)) - h(\varphi(t_0))}{\phi(t_0) - \varphi(t_0)} \geq \inf\{h^\Delta(u) : u \in U_{t_0}^\kappa\}.$$

Since $U_t \subset U_L$ for every $t \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_L^\kappa} h^\Delta(u) \geq \sup\{h^\Delta(u) : u \in U_{t_0}^\kappa\} \geq \inf\{h^\Delta(u) : u \in U_{t_0}^\kappa\} \geq \inf_{u \in U_L^\kappa} h^\Delta(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L^\kappa} h^\Delta(u) \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \geq \inf_{u \in U_L^\kappa} h^\Delta(u) \geq 0 \tag{10.19}$$

for all $t \in D(\phi, \varphi)$. So, (10.19) yields

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \Big| 1 - \Big(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \Big) \Big| \\ &\leq |\phi(t) - \varphi(t)| \Big(1 - \inf_{u \in U_L^\kappa} h^\Delta(u) \Big) \end{aligned} \tag{10.20}$$

for all $t \in D(\phi, \varphi)$. Thus, (10.19) and (10.20) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\varepsilon \in (0, 1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_L$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [-L, L]} |\phi(t) - \varphi(t)| \geq \varepsilon.$$

If $|\phi(t)-\varphi(t)| \le \varepsilon/2$ for some $t \in D(\phi,\varphi)$, then from (10.20)

$$|(H\phi)(t)-(H\varphi)(t)| \le |\phi(t)-\varphi(t)| \le \frac{1}{2}\|\phi-\varphi\|. \tag{10.21}$$

Since h is continuous and strictly increasing, the function $h(u+\frac{\varepsilon}{2})-h(u)$ attains its minimum on the closed and bounded interval $[-L,L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t)-\varphi(t)|$ for some $t \in D(\phi,\varphi)$, then from (10.19) and (H3) we conclude that

$$1 \ge \frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(t)-(H\varphi)(t)| &\le |\phi(t)-\varphi(t)|\Big\{1-\frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)}\Big\} \\ &\le (1-\lambda)\|\phi(t)-\varphi(t)\|, \end{aligned} \tag{10.22}$$

where

$$\lambda := \frac{1}{2L}\min\Big\{h(u+\frac{\varepsilon}{2})-h(u), u \in [-L,L]\Big\} > 0.$$

Consequently, it follows from (10.21) and (10.22) that

$$|(H\phi(t)-(H\varphi)(t)| \le \delta\|\phi-\varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1$. The proof is complete.

The next result gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 10.2.11. If H is a large contraction on $\mathbb{M}$, then so is the mapping B.

Proof. If H is a large contraction on $\mathbb{M}$, then for $x,y \in \mathbb{M}$, with $x \neq y$, we have $\|Hx-Hy\| \le \|x-y\|$. Then it follows from the equality

$$a(u)e_{\ominus a}(t+T,\sigma(u)) = [e_{\ominus a}(t+T,u)]^{\Delta_s},$$

where Δ_s indicates the delta derivative with respect to s ( ) that

$$\begin{aligned} |Bx(t)-By(t)| &\le \int_t^{t+T}\frac{e_{\ominus a}(t+T,\sigma(u))}{1-e_{\ominus a}(t,t+T)}a(u)\big|H(x(u))-H(y(u))\big|\Delta u \\ &\le \frac{\|x-y\|}{1-e_{\ominus a}(t,t+T)}\int_t^{t+T} a(u)e_{\ominus a}(t+T,\sigma(u))\Delta u \\ &= \|x-y\|. \end{aligned}$$

Thus,

$$\|Bx - By\| \leq \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \leq \delta\|x - y\|$$

holds if we know the existence of a $0 < \delta < 1$, such that for all $\varepsilon > 0$

$$\big[x, y \in \mathbb{M},\ \|x - y\| \geq \varepsilon\big] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete.

Lemma 10.2.12. Suppose (10.2)-(10.5), and (10.10)-(10.12) hold. Suppose also that

$$\big(1 - e_{\ominus a}(t, t - T)\big)^{-1} \int_{t-T}^{t} a(s)|H(x(\sigma(s)))|e_{\ominus a}(t, s)\,\Delta s \leq \frac{(J-1)L}{J}. \tag{10.23}$$

For B, A defined by (10.8) and (10.9), if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then

$$A\varphi + B\psi : \mathbb{M} \to \mathbb{M}.$$

Proof. Let $\varphi, \psi \in \mathbb{M}$ be arbitrary. Using the definition of B and the result of Lemma 1.6 we obtain

$$\begin{aligned}
\|A(\varphi) + B(\psi)\| &\leq Q\big(t, \varphi(t - g(t))\big) + \big(1 - e_{\ominus a}(t, t - T)\big)^{-1} \\
&\quad \times \int_{t-T}^{t} \big[-a(s)Q^{\sigma}\big(s, \varphi(s - g(s))\big) \\
&\quad + G\big(s, \varphi(s), \varphi(s - g(s))\big)\big] e_{\ominus a}(t, s)\,\Delta s \\
&\quad + \big(1 - e_{\ominus a}(t, t - T)\big)^{-1} \int_{t-T}^{t} a(s)[\psi^{\sigma}(s) - h(\psi(\sigma(s)))] \\
&\quad \times e_{\ominus a}(t, s)\,\Delta s \\
&\leq \frac{L}{J} + \frac{(J-1)L}{J} = L.
\end{aligned}$$

Theorem 10.2.13. Let $(S, \|.\|)$ be the Banach space of rd-continuous T-periodic real functions. Suppose (10.2)-(10.5) and (10.10)-(10.12) hold. Then equation (10.1) has a periodic solution in the subset $\mathbb{M}$.

Proof. By Lemma 10.2.6, φ is a solution of (10.1) if

$$\varphi = A\varphi + B\varphi,$$

where B and A are given by (10.8) and (10.9) respectively. By Lemma 10.2.7, $A : \mathbb{M} \to \mathbb{M}$ is completely continuous. By Lemma 10.2.12, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in \mathbb{M}$. Moreover, $B : \mathbb{M} \to \mathbb{M}$ is a large contraction by Lemma 10.2.11. Thus all the hypotheses of Theorem 2.3.7 are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (10.1) has a $T-$ periodic solution.

10.3 Stability of dynamic equations on time scale

Let $\mathbb{T}$ be a time scale which is unbounded above and below with $0 \in \mathbb{T}$. Also, let $id - r : \mathbb{T} \to \mathbb{T}$ be such that $(id - r)(\mathbb{T})$ is a time scale. We consider the neutral nonlinear Volterra dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(t - r(t)) + \int_{t-r(t)}^{t} k(t,s)h(x(s))\,\Delta s,\ t \in \mathbb{T}, \tag{10.24}$$

where, $a : \mathbb{T} \to \mathbb{R}$, $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are continuous and that $c : \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(t - r(t))$ to be well-defined and differentiable over $\mathbb{T}$, we assume that $r : \mathbb{T} \to \mathbb{R}$ is positive and twice continuously delta-differentiable, and that $id - r : \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - r)(\mathbb{T})$ is closed where id is the identity function on the time scale $\mathbb{T}$. Here we assume that h is locally Lipschitz continuous in x. That is, there is an $L > 0$ so that if $|x| \leq L$, then

$$|h(x) - h(z)| \leq L_1|x - z| \tag{10.25}$$

for some positive constant L_1. Also, we assume that

$$h(0) = 0. \tag{10.26}$$

We assume further that

$$r^{\Delta}(t) \neq 1, \text{ for all } t \in \mathbb{T}. \tag{10.27}$$

Lemma 10.2.1. Suppose (10.27) hold. If $x(t)$ is a solution of (10.24) on an interval $[0,T)_{\mathbb{T}}$,$(T>0)$ satisfying the initial condition $x(t)=\psi(t)$ for $t\in(-\infty,0]_{\mathbb{T}}$, then $x(t)$ is a solution of the integral equation

$$\begin{aligned} x(t) &= \Big(\psi(0)-\frac{c(0)}{1-r^{\Delta}(0)}x(-r(0))\Big)e_{\ominus a}(t,0)+\frac{c(t)}{1-r^{\Delta}(t)}x(t-r(t)) \\ &\quad -\int_0^t[\phi(u)x^{\sigma}(u-r(u))-\int_{u-r(u)}^{u}k(u,s)h(x(s))\Delta s]e_{\ominus a}(t,u)\Delta u, \end{aligned} \tag{10.28}$$

where

$$\phi(u)=\frac{\big(c^{\Delta}(u)+c^{\sigma}(u)a(u)\big)(1-r^{\Delta}(u))+r^{\Delta\Delta}(u)c(u)}{(1-r^{\Delta}(u))(1-r^{\Delta}(\sigma(u)))}. \tag{10.29}$$

Conversely, if rd-continuous function $x(t)$ satisfies $x(t)=\psi(t)$ for $t\in(-\infty,0]_{\mathbb{T}}$ and is a solution of (10.28) on some interval $[0,T)_{\mathbb{T}}$, $(T>0)$, then $x(t)$ is a solution of equation (10.24) on $[0,T)_{\mathbb{T}}$.

Proof. Rewrite (10.24) as

$$x^{\Delta}(t)+a(t)x^{\sigma}(t)=c(t)x^{\tilde{\Delta}}(t-g(t))+\int_{t-r(t)}^{t}k(t,s)h(x(s))\,\Delta s,\ t\in\mathbb{T},$$

Multiply both sides of the above equation by $e_a(t,0)$ and then integrate from 0 to t to obtain

$$\begin{aligned} &\int_0^t(e_a(u,0)x(u))^{\Delta}\Delta u \\ &=\int_0^t\Big[c(u)x^{\tilde{\Delta}}(u-g(u))+\int_{u-r(u)}^{u}k(u,s)h(x(s))\Big]e_a(u,0)\Delta u. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &e_a(t,0)x(t)-x(0) \\ &=\int_0^t\Big[c(u)x^{\tilde{\Delta}}(u-g(u))+\int_{u-r(u)}^{u}k(u,s)h(x(s))\Big]e_a(u,0)\Delta u. \end{aligned}$$

Multiply both sides of the above equation to get

$$\begin{aligned} &x(t) \\ &=x(0)e_{\ominus a}(t,0)+\int_0^t\Big[c(u)x^{\tilde{\Delta}}(u-g(u))+\int_{u-r(u)}^{u}k(u,s)h(x(s))\Big]e_{\ominus a}(t,u)\Delta u. \end{aligned} \tag{10.30}$$

But,

$$\begin{aligned}&\int_0^t c(u)x^{\tilde{\Delta}}(u-g(u))e_{\ominus a}(t,u)\Delta u\\&=\int_0^t x^{\tilde{\Delta}}(u-g(u))(1-r^{\Delta}(u))\frac{c(u)}{(1-r^{\Delta}(u))}e_{\ominus a}(t,u)\Delta u.\end{aligned}$$

Using the integration by parts formula we obtain

$$\begin{aligned}&\int_0^t c(u)x^{\tilde{\Delta}}(u-g(u))e_{\ominus a}(t,u)\Delta u\\&=\frac{c(t)}{1-r^{\Delta}(t)}x(t-r(t))-\frac{c(0)}{1-r^{\Delta}(0)}x(-r(0))e_{\ominus a}(t,0)\\&\quad-\int_0^t \phi(u)x^{\sigma}(u-r(u))e_{\ominus a}(t,u)\Delta u,\end{aligned}\tag{10.31}$$

where ϕ is given by (10.29). Substituting the right hand side of (10.31) into (10.30) we obtain (10.28).

Conversely, suppose that a *rd*-continuous function $x(t)$ satisfying $x(t)=\psi(t)$ for $t\in(-\infty,0]_{\mathbb{T}}$ and is a solution of (10.28) on an interval $[0,T)_{\mathbb{T}}$. Then it is $\Delta-$ differentiable on $[0,T)_{\mathbb{T}}$. By $\Delta-$ differentiating (10.29) we obtain (10.24).

Let $\psi:(-\infty,0]_{\mathbb{T}}\to\mathbb{R}$ be a given Δ-differentiable bounded initial function. We say $x(t):=x(t,0,\psi)$ is a solution of (10.24) if $x(t)=\psi(t)$ for $t\le 0$ and satisfies (10.24) for $t\ge 0$. We say the zero solution of (10.24) is stable at t_0 if for each $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$ such that $\left[\psi:(-\infty,t_0]_{\mathbb{T}}\to\mathbb{R}\text{ with }\|\psi\|<\delta\right]$ implies $|x(t,t_0,\psi)|<\varepsilon$.

Let $C_{rd}=C_{rd}(\mathbb{T},\mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T}\to\mathbb{R}$ and define the set S by

$$S=\{\varphi\in C_{rd}:||\varphi||\le E,\,\varphi(t)=\psi(t)\text{ if }t\le 0,\,\varphi(t)\to 0\text{ as }\ t\to\infty\}.$$

Then, $\left(S,\|\cdot\|\right)$ is a complete metric space where, $\|\cdot\|$ is the supremum norm. For the next theorem we impose the following conditions.

$$e_{\ominus a}(t,0)\to 0,\text{ as }t\to\infty,\tag{10.32}$$

there is an $\alpha > 0$ such that

$$\Big|\frac{c(t)}{1-r^{\Delta}(t)}\Big| + \int_0^t \Big[|\phi(u)| + L_1 \int_{u-r(u)}^u |k(u,s)|\Delta s\Big] e_{\ominus a}(t,u)\,\Delta u \le \alpha < 1, t \ge 0, \tag{10.33}$$

$$t - r(t) \to \infty, \text{as } t \to \infty. \tag{10.34}$$

Theorem 10.2.2. If (10.25)–(10.27) and (10.32)–(10.34) hold, then every solution $x(t,0,\psi)$ of (10.24) with small continuous initial function $\psi(t)$, is bounded and goes to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

Proof. Let $E > 0$ be given. Choose $\delta > 0$ such that

$$\Big|1 - \frac{c(0)}{1-r^{\Delta}(0)}\Big|\delta + \alpha E \le E.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $||\psi|| < \delta$. Define the mapping $P : S \to S$ by

$$\big(P\varphi\big)(t) = \psi(t) \quad \text{if } t \le 0$$

and

$$\begin{aligned}\big(P\varphi)(t)\big) \;=\;& \Big(\psi(0) - \frac{c(0)}{1-r^{\Delta}(0)}\psi(-r(0))\Big)e_{\ominus a}(t,0) + \frac{c(t)}{1-r^{\Delta}(t)}\varphi(t-r(t)) \\ &- \int_0^t [\phi(u)\varphi^{\sigma}(u-r(u)) - \int_{u-r(u)}^u k(u,s)h(\varphi(s))\Delta s]e_{\ominus a}(t,u)\Delta u.\end{aligned}$$

It is clear that for $\varphi \in S$, $P\varphi$ is continuous. Let $\varphi \in S$, then

$$\begin{aligned}|\big(P\varphi)(t)\big)| \;\le\;& \Big|\psi(0) - \frac{c(0)}{1-r^{\Delta}(0)}\psi(-r(0))\Big| + \Big|\frac{c(t)}{1-r^{\Delta}(t)}\varphi(t-r(t))\Big| \\ &+ \int_0^t \Big(\Big|\phi(u)\varphi^{\sigma}(u-r(u))\Big| + \int_{u-r(u)}^u \Big|k(u,s)h(\varphi(s))\Big|\Delta s\Big)e_{\ominus a}(t,u)\Delta u. \\ \le\;& \Big|1 - \frac{c(0)}{1-r^{\Delta}(0)}\Big|\delta + \Big|\frac{c(t)}{1-r^{\Delta}(t)}\Big|E \\ &+ \int_0^t \Big(\Big|\phi(u)\Big|E + L_1 E\int_{u-r(u)}^u \Big|k(u,s)\Big|\Delta s\Big)e_{\ominus a}(t,u)\Delta u \\ \le\;& \Big|1 - \frac{c(0)}{1-r^{\Delta}(0)}\Big|\delta + \Big\{\Big|\frac{c(t)}{1-r^{\Delta}(t)}\Big| \\ &+ \int_0^t \Big(\Big|\phi(u)\Big| + L_1\int_{u-r(u)}^u \Big|k(u,s)\Big|\Delta s\Big)e_{\ominus a}(t,u)\Delta u\Big\}E \\ \le\;& \Big|1 - \frac{c(0)}{1-r^{\Delta}(0)}\Big|\delta + \alpha E,\end{aligned}$$

which implies that $|(P\varphi)(t)| \leq E$ for the chosen δ. Thus we have $||P\varphi|| \leq E$.

Next we show that $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. The first term on the right side of $\big(P\varphi\big)(t)$ tends to zero, by condition (10.32). Also, the second term on the right side tends to zero, because of (10.34) and the fact that $\varphi \in S$. We next show that the integral term goes to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given and $\varphi \in S$ with $\|\varphi\| \leq E$, $E > 0$. Then, there exists a $t_1 > 0$ so that for $t > t_1$, $|\varphi(t - g(t))| < \varepsilon$. Due to condition (10.32), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that $e_{\ominus a}(t, t_1) < \frac{\varepsilon}{\alpha E}$.

Thus for $t > t_2$, we have

$$
\begin{aligned}
&\Big| - \int_0^t \Big[\phi(u)\varphi^\sigma(u - r(u)) - \int_{u-r(u)}^u k(u,s)h(\varphi(s))\Delta s\Big] e_{\ominus a}(t,u)\Delta u\Big| \\
&\leq E \int_0^{t_1} \Big[|\phi(u)| + L_1 \int_{u-r(u)}^u |k(u,s)|\Delta s\Big] e_{\ominus a}(t,u)\, \Delta u \\
&\quad + \varepsilon \int_{t_1}^t \Big[|\phi(u)| + L_1 \int_{u-r(u)}^u |k(u,s)|\Delta s\Big] e_{\ominus a}(t,u)\, \Delta u \\
&\leq E e_{\ominus a}(t,t_1) \int_0^{t_1} \Big[|\phi(u)| + L_1 \int_{u-r(u)}^u |k(u,s)|\Delta s\Big] e_{\ominus a}(t_1,u)\, \Delta u + \alpha\varepsilon \\
&\leq \alpha E e_{\ominus a}(t,t_1) + \alpha\varepsilon \\
&\leq \varepsilon + \alpha\varepsilon.
\end{aligned}
$$

Hence, $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$.

Finally, we show that P is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$
\begin{aligned}
\Big|(P\zeta)(t) - (P\eta)(t)\Big| &\leq \Big|\frac{c(t)}{1 - r^\Delta(t)}\Big| ||\zeta - \eta|| \\
&\quad + \int_0^t \Big(\Big|\phi(u)(\zeta^\sigma(u - r(u)) - \eta^\sigma(u - r(u)))\Big| \\
&\quad + \int_{u-r(u)}^u \Big|k(u,s)(h(\zeta(s)) - h(\eta(s)))\Big|\Delta s\Big) e_{\ominus a}(t,u)\Delta u. \\
&\leq \Big\{\Big|\frac{c(t)}{1 - r^\Delta(t)}\Big| + \int_0^t \Big[|\phi(u)| \\
&\quad + L_1 \int_{u-r(u)}^u |k(u,s)|\Delta s\Big] e_{\ominus a}(t,u)\, \Delta u\Big\} \|\zeta - \eta\| \\
&\leq \alpha\|\zeta - \eta\|.
\end{aligned}
$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (10.24), is bounded and tends to zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing E by ε. This completes the proof.

CHAPTER ELEVEN

SUMMARY, CONCLUSION AND FUTURE DIRECTIONS

11.1 Summary

In this thesis, as set out in the objectives of the research, we investigated the qualitative properties of solutions of certain classes of neutral functional differential and difference equations. We also studied the qualitative properties of neutral dynamic equations on time scale. The fixed point theory was used to investigate the qualitative behaviour of classes of of first order and second nonlinear functional differential equations. The same method was also used to study the qualitative behavoiur of neutral difference equations and neutral dynamic equations on time scale.

We inverted or transformed the equations into equivalent integral equations in the case of neutral functional differential equations or dynamic equations on time scale. In the case of difference equations however, the inversion resulted into equivalent summation equations. The integral or summation equation was then used to define a mapping that was used for the discussion of the qualitative behaviour of the classes of equations considered. In some situations, the mapping was expressed as a sum of a completely continuous map and a large contraction map. In those cases, the reformulated version of Krasnoselskii's fixed point theorem was used to prove the main results. In particular, the periodicity, stability, and positivity of solutions of totally nonlinear equations were proved with this theorem. Moreover, in some other situations the mapping was expressed as a sum of a completely continuous operator and a contraction. The Krasnoselskii's fixed point theorem was then used to establish the main results. Particularly, this was used to establish periodicity and positivity of neutral equations that are not totally nonlinear. Finally, in the cases when the mappings were contraction mappings, the Banach's fixed point theorem was used. This theorem was mainly used to prove the asymptotic stability of the zero solution of certain classes of difference equations and neutral Volterra dynamic equations on time scale.

11.2 Conclusion

Sufficient conditions for the existence of periodic solutions of both functional neutral first order differential, difference and dynamic equations have been established. It has also been established that solutions of a system of functional differential equations with finite delay are periodic.

Criteria for the existence of positive periodic solutions of functional neutral second order differential equations have been obtained. New results for the existence of positive periodic solutions for a system of neutral difference equations with delay has also been obtained.

The zero solution of a certain class of neutral dynamic equation on time scale has been proved to be asymptotically stable. Moreover, the zero solution of neutral functional differential equations as well as neutral functional difference equations have also been proved to be asymptotically stable.

11.3 Future Directions

In most of the problems studied by means of contractions we have first shown that the mapping maps a bounded set into itself before we ever bring up the topic contractions. We then assume a Lipschitz condition. Many of these problems can fruitfully be studied again by dropping the Lipschitz condition and using a Schauder-type fixed point theorem.

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