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ASYMPTOTIC BEHAVIOUR OF THE WRONSKIAN OF BOUNDARY CONDITION FUNCTIONS FOR A FOURTH ORDER BOUNDARY VALUE ROBLEM (A SPECIAL CASE)

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Abstract: In this paper, we prove that the Wronskian $W(\lambda)$ of the boundary condition functions for the following boundary value problem π:

$$\pi : L\phi \equiv \phi^{(4)}(x) + P_2(x)\,\phi^{(2)}(x) + P_3(x)\,\phi^{(1)}(x) + P_4(x)\,\phi(x) = \lambda\phi(x)$$
$$\phi(a) = \phi^{/}(a) = \phi(b) = \phi^{/}(b) = 0$$

is asymptotically equivalent for large values of $|\lambda|$, to the Wronskian of the boundary condition functions of the corresponding Fourier problem π_F given by

$$\pi_F : \phi^{(4)}(x) = \lambda\phi(x),$$
$$\phi(a) = \phi^{/}(a) = \phi(b) = \phi^{/}(b) = 0.$$

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1. Introduction

Boundary condition functions have been studied widely by many mathemati-

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cians for some years now. The use of boundary condition functions for boundary value problems was first considered by Kodaira in [1]. Since then, quite a number authors including [12] and [13] have worked on Boundary value problems.

In [5], D. N. Offei proved that the boundary condition functions, the Wronskian of the boundary conditions and the Green's function for the boundary-value problem:

$$\begin{aligned} L\phi &= i^3\phi^{(3)} + p_2(x)\,\phi^{(1)}(x) + p_3(x)\,\phi(x) = \lambda\phi(x) \\ \phi(a) &= \phi(b) = \phi^{(1)}(b) = 0, \end{aligned}$$

are asymptotically equivalent, for suitably large values of $|\lambda|$, to the corresponding functions, associated with the corresponding Fourier problem.

In [15] M. Bonsu Osei, Samuel Asiedu-Addo, considered the Asymptotic behaviour of Wronskian of boundary condition functions for a second order boundary value problem.IeJPAM, 1(1), (2010), 93-101.

In [14], E. K. Essel et.al proved that the boundary condition functions of the Fourth order boundary value problem are asymptotically equivalent to the boundary condition functions of the corresponding Fourth order Fourier problem.

2. Notations

In this section we give some properties of the linear differential expression L and some notations used in subsequent sections of this paper.

1. (a) For a suitable set of functions, the symbol $\Phi(x)$ denotes the 4 x 4 Wronskian matrix $\left[\phi_r^{(s-1)}(x)\right], (1 \le r, s, \le 4)$.

$$\Phi(x) = \begin{bmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \phi_4(x) \\ \phi_1^{(1)}(x) & \phi_2^{(1)}(x) & \phi_3^{(1)}(x) & \phi_4^{(1)}(x) \\ \phi_1^{(2)}(x) & \phi_2^{(2)}(x) & \phi_3^{(2)}(x) & \phi_4^{(2)}(x) \\ \phi_1^{(3)}(x) & \phi_2^{(3)}(x) & \phi_3^{(3)}(x) & \phi_4^{(3)}(x) \end{bmatrix}.$$

and $W(\phi_1\phi_2\phi_3\phi_4)(x) \equiv \det\Phi(x)$. A similar notation is used if ϕ is replaced by another symbol; the respective capital always representing the Wronskian matrix.

(b) If $\phi_1(x,\lambda),\ \phi_2(x,\lambda),\ \phi_3(x,\lambda),\ \phi_4(x,\lambda)$ are the solutions of $L\phi = \lambda\phi$ and if $x_0,\ x_1 \in [a,b]$, then

$$W(\phi_1\phi_2\phi_3\phi_4)(x_1) = W(\phi_1\phi_2\phi_3\phi_4)(x_0)\exp\int_{x_0}^{x^1} -\frac{P_1(t)}{P_0(t)}dt, \quad (1)$$

(see chap. 3, [8]). If $P_1(x) = 0$ for $x \in [a,b]$ then it follows from (1) that $W(\phi_1\phi_2\phi_3\phi_4)(x)$ is independent of $x \in [a,b]$.

2. Given the linear expression defined by

$$\begin{aligned} L\phi &\equiv P_0\phi^{(4)}(x) + P_1(x)\phi^{(3)} + P_2(x)\phi^{(2)}(x) \\ &\quad + P_3(x)\phi^{(1)}(x) + P_4(x)\phi(x); \quad (a \le x \le b), \end{aligned}$$

the Lagrange adjoint of L is denoted by L^+ and is defined as

$$\begin{aligned} L^+\psi &\equiv (-1)^4\left(\bar{P}_0\psi\right)^{(4)} + (-1)^3\left(\bar{P}_1\psi\right)^{(3)} + (-1)^2\left(\bar{P}_2\psi\right)^{(2)} \\ &\quad + (-1)\left(\bar{P}_3\psi\right)^{(1)} + \bar{P}_4(x)\psi. \end{aligned}$$

3. (a) For suitable pairs of functions f and g

$$\int_a^b \left\{\bar{g}Lf - f\overline{L^+g}\right\} dx = [fg](b) - [fg](a).$$

Here $[fg](x)$ is a bilinear form in

$$\left(f, f^{(1)}, f^{(2)}, f^{(3)}\right),$$

and

$$\left(\bar{g}, \bar{g}^{(1)}, \bar{g}^{(2)}, \bar{g}^{(3)}\right),$$

given by

$$\begin{aligned} [fg](x) &= \sum_{j=1}^{4}\sum_{k=1}^{4} B_{jk}(x)\,\bar{g}^{(j-1)}(x)\,f^{(k-1)}(x) \\ &= \hat{g}^{\square}(x)\,\mathbf{B}(x)\,\hat{f}(x) \end{aligned}$$

where $\hat{f}(x)$ represents the column vector with components

$$\left(f(x), f^{(1)}(x), ..., f^{(n-1)}(x)\right),$$

and $\hat{g}^{\square}(x)$ denotes the row vector with components

$$\left(\bar{g}(x), \bar{g}^{(1)}(x), ..., \bar{g}^{(n-1)}(x)\right),$$

and

$$\mathbf{B}(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) & P_{14}(x) \\ -P_2 + 2P_1^{(1)} - 2P_0^{(2)} & -P_1 - 2P_0^{(1)} & -P_0 & 0 \\ P_1 - 3P_0^{(1)} & P_0 & 0 & 0 \\ -P_0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} P_{11}(x) &= P_3^{(3)}(x) - P_2^{(1)}(x) + P_1^{(2)}(x) - P_0^{(2)}(x) - P_0^{(3)}(x) \\ P_{12}(x) &= P_2(x) - P_1^{(1)}(x) + P_0^{(2)}(x) \\ P_{13}(x) &= P_1(x) - P_0^{(1)}(x) \\ P_{14}(x) &= P_0(x). \end{aligned}$$

(b) If $P_1(x), P_2(x)$ and $P_3(x)$ are identically zero in some neighbourhood of a and b and P_0 is a constant independent of x then

$$\mathbf{B}(a) = \mathbf{B}(b) = \begin{bmatrix} 0 & 0 & 0 & P_0 \\ 0 & 0 & -P_0 & 0 \\ 0 & P_0 & 0 & 0 \\ -P_0 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

(c) The notation $\langle \phi, \psi \rangle$ is used to denote $\int_a^b \phi(x) \overline{\psi(x)} dx$ and the expression $\int \left\{ \bar{g} L f - f \overline{L^+ g} \right\} dx$ may be written as $\langle Lf, g \rangle - \langle f, L^+ g \rangle$.

(d) The Lagrange adjoint of L^+ is L and for suitable pair of functions g and f

$$\int_a^b \left\{ \bar{f} L^+ g - g \overline{Lf} \right\} dx = [gf](b) - [gf](a)$$

where

$$\begin{aligned} \{gf\}(x) &= \sum_{j=1}^{4} \sum_{k=1}^{4} A_{jk}(x) \bar{f}^{(j-1)}(x) g^{(k-1)}(x) \\ &= \hat{f}^{\square}(x) \mathbf{A}(x) \hat{g}(x). \end{aligned}$$

The A_{jk} are dependent on the coefficients of the differential expression L^+ and $\mathbf{A}(x) = [A_{jk}]$.

4. If $\phi(x,\lambda)$ is a solution of $L\phi = \lambda\phi$ and $\psi(x,\lambda)$ is a solution of $L^{+}\psi = \bar{\lambda}\psi$ then,

$$\begin{aligned}
[\phi\psi](x_2) - [\phi\psi](x_1) &= \int_{x_1}^{x_2} \left\{\bar{\psi}L\phi - \phi\overline{L^{+}\psi}\right\} dx, \quad (a \le x_1 \le x_2 \le b) \\
&= \int_{x_1}^{x_2} \left\{\bar{\psi}\lambda\phi - \phi\lambda\overline{\psi}\right\} dx \\
&= 0
\end{aligned}$$

and hence,

$$[\phi\psi](x_2) = [\phi\psi](x_1).$$

Thus,

$$[\phi(x,\lambda)\,\psi(x,\lambda)](x)$$

is independent of $x \in [a,b]$.

Similarly,

$$\{\psi(x,\lambda)\,\phi(x,\lambda)\}(x)$$

is independent of $x \in [a,b]$.

This implies that $[\phi(x,\lambda)\,\psi(x,\lambda)](x)$ and $\{\psi(x,\lambda)\,\phi(x,\lambda)\}(x)$ may be denoted by $[\phi\psi]$ and $\{\psi\phi\}$, respectively.

(a) If there is a constant K such that $|fx| \le K\phi$ for $x \ge x_0$ we write

$$f = O(\phi).$$

(b) If $\frac{f(x)}{\phi(x)} \to l$, $x \to \infty$ where $l \ne 0$ we write $f \sim l\phi$.

3. Preliminaries

In this paper we consider the boundary value problem

$$\begin{aligned}
\pi \quad : \quad & L\phi \equiv \phi^{(4)}(x) + P_2(x)\,\phi^{(2)}(x) \\
& +P_3(x)\,\phi^{(1)}(x) + P_4(x)\,\phi(x) \\
= \quad & \lambda\phi(x) \qquad (3)
\end{aligned}$$

$$\phi(a) = \phi^{/}(a) = \phi(b) = \phi^{/}(b) = 0 \qquad (4)$$

which is a special case of the boundary value problem

$$\begin{aligned}\pi \quad &: \quad L\phi \equiv \phi^{(4)}(x) + P_2(x)\,\phi^{(2)}(x) \\ &\qquad +P_3(x)\,\phi^{(1)}(x) + P_4(x)\,\phi(x) \\ &= \quad \lambda\phi(x)\end{aligned}$$

$$\begin{aligned}\sum_{s=1}^{4} m_{rs}\phi^{(s-1)}(a) &= 0, \quad (r = 1, 2) \\ \sum_{s=1}^{4} n_{rs}\phi^{(s-1)}(b) &= 0, \quad (r = 3, 4).\end{aligned}$$

The Fourier problem corresponding to (3) - (4) is given by

$$\pi_F \quad : \quad \phi^{(4)}(x) = \lambda\phi(x) \tag{5}$$

$$\phi(a) = \phi^{/}(a) = \phi(b) = \phi^{/}(b) = 0. \tag{6}$$

In this special case where,

$$\phi(a) = \phi^{/}(a) = \phi(b) = \phi^{/}(b) = 0,$$

the matrix $\mathbf{M} = [m_{rs}]$ and $\mathbf{N} = [n_{rs}]$ in

$$\sum_{s=1}^{4} m_{rs}\phi^{(s-1)}(a) = 0, \quad (r = 1, 2)$$

and

$$\sum_{s=1}^{4} n_{rs}\phi^{(s-1)}(b) = 0, \quad (r = 3, 4)$$

are given respectively by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Substituting $P_0 = 1$ from (3) into (2) (i.e., Notation 3(b)) we see that

$$\mathbf{B}(a) = \mathbf{B}(b) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

We now state some Lemmas that will enable us to prove our main result.

Lemma 1.

(i)

$$\psi_{Fr}(a/x, \lambda) = \sum_{s=1}^{4} \bar{m}_{rs} f_s(a/x, \lambda)$$

(ii)

$$\chi_{Fr}(b/x, \lambda) = \sum_{s=1}^{4} \bar{n}_{rs} g_s(b/x, \lambda)$$

(iii) Let $f_s(x) = f_s(a/x, \lambda)$, $g_s(x) = g_s(b/x, \lambda)$. Then,

$f_s(x) = (-1)^{(s-1)} f_1^{(s-1)}(x) \quad 2 \leq s \leq 4$

$g_s(x) = (-1)^{(s-1)} g_1^{(s-1)}(x) \quad 2 \leq s \leq 4$

See [14] for proof.

Lemma 2.

(a) (i) $\psi_r^{(s-1)}(a/x, \lambda) = \psi_{Fr}^{(s-1)}(a/x, \lambda) + O\left(|P|^{s-2} e^{\sigma(x-a)}\right)$ as $|\lambda| \to \infty$

(ii) $\psi_{Fr}^{(s-1)}(a/x, \lambda) = O\left(|P|^{s-1} e^{\sigma(x-a)}\right)$ as $|\lambda| \to \infty$, $(1 \leq r, s \leq 4)$

(iii) (i) and (ii) $\Longrightarrow$

$$\psi_r^{(s-1)}(a/x, \lambda) \sim \psi_{Fr}^{(s-1)}(a/x, \lambda) \ \ as\, |\lambda| \to \infty$$

(b) (i) $\chi_r^{(s-1)}(b/x, \lambda) = \chi_{Fr}^{(s-1)}(b/x, \lambda) + O\left(|P|^{s-2} e^{\sigma(x-a)}\right)$ as $|\lambda| \to \infty$

(ii) $\chi_{Fr}^{(s-1)}(b/x, \lambda) = O\left(|P|^{s-1} e^{\sigma(b-x)}\right)$ as $|\lambda| \to \infty$, $(1 \leq r, s \leq 4)$

(iii) (i) and (ii) $\Longrightarrow$

$$\chi_r^{(s-1)}(b/x, \lambda) \sim \chi_{Fr}^{(s-1)}(b/x, \lambda).$$

See [14] for proof.

4. Main Result

Let

$$\begin{aligned} W(\lambda) &= W(\eta_1\eta_2\eta_3\eta_4)(x), \quad \left(\eta_r = \eta_r\left(x,\bar{\lambda}\right)\right) \\ W_F(\lambda) &= W(\eta_{F1}\eta_{F2}\eta_{F3}\eta_{F4})(x), \quad \left(\eta_{Fr} = \eta_{Fr}\left(x,\bar{\lambda}\right)\right). \end{aligned}$$

Then

$$W(\lambda) \sim W_F(\lambda)$$

for suitably large values of $|\lambda|$.

We prove our main result via two theorems.

4.1. Theorem 1

$$W_F(\lambda) = \left(O\,|P|\,e^{2\sigma(b-a)}\right) \text{ as } \quad |\lambda| \to \infty$$

Proof. Let $\{\psi_{Fr}(a/x,\lambda,\chi_{Fr}(b/x,\lambda))\}$ be the boundary condition function for π_F. Then $\psi_{Fr}(a/x,\lambda), \chi_{Fr}(b/x,\lambda),\ 1 \le r \le 4$ are solutions of $\psi^{(4)}(x) = \bar{\lambda}\psi(x)$ such that

$$\Psi_{Fr}(a) = \mathbf{B}(a)\,M^{\square} \text{ and } \chi_{Fr}(b) = \mathbf{B}(b)\,N^{\square} \tag{9}$$

where $\mathbf{B}(a)$ and $\mathbf{B}(b)$ are as in (8). Substituting (7) and (8) into (9) we have

$$\begin{aligned} &\begin{bmatrix} \psi_{F1}(a) & \psi_{F2}(a) & \psi_{F3}(a) & \psi_{F4}(a) \\ \psi^{(1)}_{F1}(a) & \psi^{(1)}_{F2}(a) & \psi^{(1)}_{F3}(a) & \psi^{(1)}_{F4}(a) \\ \psi^{(2)}_{F1}(a) & \psi^{(2)}_{F2}(a) & \psi^{(2)}_{F3}(a) & \psi^{(2)}_{F4}(a) \\ \psi^{(3)}_{F1}(a) & \psi^{(3)}_{F2}(a) & \psi^{(3)}_{F3}(a) & \psi^{(3)}_{F4}(a) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{10}$$

and

$$\begin{bmatrix} \chi_{F1}(b) & \chi_{F2}(b) & \chi_{F3}(b) & \chi_{F4}(b) \\ \chi_{F1}^{(1)}(b) & \chi_{F2}^{(1)}(b) & \chi_{F3}^{(1)}(b) & \chi_{F4}^{(1)}(b) \\ \chi_{F1}^{(2)}(b) & \chi_{F2}^{(2)}(b) & \chi_{F3}^{(2)}(b) & \chi_{F4}^{(2)}(b) \\ \chi_{F1}^{(3)}(b) & \chi_{F2}^{(3)}(b) & \chi_{F3}^{(3)}(b) & \chi_{F4}^{(3)}(b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{11}$$

From Lemma 1, with the matrices M and N as in (7) we have

$$\begin{aligned} \eta_{F1}(x,\lambda) &= \psi_{F1}(a/x,\lambda) = f_1(a/x,\lambda) \\ \eta_{F2}(x,\lambda) &= \psi_{F2}(a/x,\lambda) = f_2(a/x,\lambda) \\ \eta_{F3}(x,\lambda) &= \chi_{F3}(b/x,\lambda) = g_2(b/x,\lambda) \\ \eta_{F4}(x,\lambda) &= \chi_{F4}(a/x,\lambda) = g_1(a/x,\lambda) \\ \psi_{F3}(a/x,\lambda) &= 0, \psi_{F4}(a/x,\lambda) = 0 \\ \chi_{F1}(b/x,\lambda) &= 0, \chi_{F2}(b/x,\lambda) = 0. \end{aligned} \tag{12}$$

By definition

$$W_F(\lambda) = W(\eta_{F1}\eta_{F2}\eta_{F3}\eta_{F4})(x,\lambda) \tag{13}$$

Substituting (12) into (13) we see that

$$W_F(\lambda) = W(\psi_{F1}\psi_{F2}\chi_{F3}\chi_{F4})(x,\lambda) \tag{14}$$

is independent of $x \in [a,b]$ (see notation1(b)). Comparing corresponding elements on the right and left hand sides of (11) we see that for $x = b$, (14) reduces to

$$\begin{aligned} W_F(\lambda) &= W(\psi_{F1}\psi_{F2}\chi_{F3}\chi_{F4})(b,\lambda) \\ &= \begin{vmatrix} \psi_{F1}(b) & \psi_{F2}(b) & \psi_{F3}(b) & \psi_{F4}(b) \\ \psi_{F1}^{(1)}(b) & \psi_{F2}^{(1)}(b) & \psi_{F3}^{(1)}(b) & \psi_{F4}^{(1)}(b) \\ \psi_{F1}^{(2)}(b) & \psi_{F2}^{(2)}(b) & \psi_{F3}^{(2)}(b) & \psi_{F4}^{(2)}(b) \\ \psi_{F1}^{(3)}(b) & \psi_{F2}^{(3)}(b) & \psi_{F3}^{(3)}(b) & \psi_{F4}^{(3)}(b) \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} \psi_{F1}(b) & \psi_{F2}(b) & 0 & 0 \\ \psi_{F1}^{(1)}(b) & \psi_{F2}^{(1)}(b) & 0 & 0 \\ \psi_{F1}^{(2)}(b) & \psi_{F2}^{(2)}(b) & 1 & 0 \\ \psi_{F1}^{(3)}(b) & \psi_{F2}^{(3)}(b) & 0 & -1 \end{vmatrix}. \tag{15}$$

Evaluating we have

$$W_F(\lambda) = \psi_{F1}^{(1)}(b)\,\psi_{F2}(b) - \psi_{F1}(b)\,\psi_{F2}^{(1)}(b). \tag{16}$$

Using Lemma 2 (ii) we find that

$$\begin{aligned} \psi_{F1}^{(1)}(b) &= O\left(|P|\,e^{\sigma(b-a)}\right) \quad \text{as} \quad |P| \to \infty, \\ \psi_{F2}^{(1)}(b) &= O\left(|P|\,e^{\sigma(b-a)}\right) \quad \text{as} \quad |P| \to \infty, \\ \psi_{F2}(b) &= O\left(e^{\sigma(b-a)}\right) \quad \text{as} \quad |P| \to \infty, \\ \psi_{F1}(b) &= O\left(e^{\sigma(b-a)}\right) \quad \text{as} \quad |P| \to \infty, \end{aligned} \tag{17}$$

Substituting all of (17) in (16) we see that

$$W_F(\lambda) = O\left(|P|\,e^{2\sigma(b-a)}\right) \quad \text{as} \quad |\lambda| \to \infty. \tag{18}$$

□

4.2. Theorem 2

$$W(\lambda) = W_F(\lambda) + O\left(e^{2\sigma(b-a)}\right) \quad \text{as} \quad |\lambda| \to \infty.$$

Proof. Let $\{\psi_r(a/x, \lambda, \chi_r(b/x, \lambda))\}$ be the boundary condition function for π. Then $\psi_r(a/x, \lambda), \chi_r(b/x, \lambda),\ 1 \le r \le 4$ are solutions of $L^+\psi = \bar{\lambda}\psi$ such that

$$\Psi_r(a) = \mathbf{B}(a)\,M^{\square} \text{ and } \chi_r(b) = \mathbf{B}(b)\,N^{\square} \tag{19}$$

where $\mathbf{B}(a)$ and $\mathbf{B}(b)$ are as in (8). Substituting (7) and (8) into (19) we obtain

$$\begin{bmatrix} \psi_1(a) & \psi_2(a) & \psi_3(a) & \psi_4(a) \\ \psi_1^{(1)}(a) & \psi_2^{(1)}(a) & \psi_3^{(1)}(a) & \psi_4^{(1)}(a) \\ \psi_1^{(2)}(a) & \psi_2^{(2)}(a) & \psi_3^{(2)}(a) & \psi_4^{(2)}(a) \\ \psi_1^{(3)}(a) & \psi_2^{(3)}(a) & \psi_3^{(3)}(a) & \psi_4^{(3)}(a) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

and

$$\begin{bmatrix} \chi_1(b) & \chi_2(b) & \chi_3(b) & \chi_4(b) \\ \chi_1^{(1)}(b) & \chi_2^{(1)}(b) & \chi_3^{(1)}(b) & \chi_4^{(1)}(b) \\ \chi_1^{(2)}(b) & \chi_2^{(2)}(b) & \chi_3^{(2)}(b) & \chi_4^{(2)}(b) \\ \chi_1^{(3)}(b) & \chi_2^{(3)}(b) & \chi_3^{(3)}(b) & \chi_4^{(3)}(b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{21}$$

Using similar deductions as in (12) we have

$$\begin{aligned} \eta_1(x,\lambda) &= \psi_1(a/x,\lambda) \\ \eta_2(x,\lambda) &= \psi_2(a/x,\lambda) \\ \eta_3(x,\lambda) &= \chi_3(b/x,\lambda) \\ \eta_4(x,\lambda) &= \chi_4(a/x,\lambda) \\ \psi_3(a/x,\lambda) &= 0,\ \psi_4(a/x,\lambda) = 0 \\ \chi_1(b/x,\lambda) &= 0,\ \chi_2(b/x,\lambda) = 0. \end{aligned} \tag{22}$$

By definition,

$$W(\lambda) = W(\eta_1\eta_2\eta_3\eta_4)(x,\lambda), \tag{23}$$

and by substituting (22) into (23) it reduces to

$$W(\lambda) = W(\psi_1\psi_2\chi_3\chi_4)(b,\lambda).$$

That is,

$$\begin{aligned} W(\lambda) &= W(\psi_1\psi_2\chi_3\chi_4)(b,\lambda) \\ &= \begin{vmatrix} \psi_1(b) & \psi_2(b) & \psi_3(b) & \psi_4(b) \\ \psi_1^{(1)}(b) & \psi_2^{(1)}(b) & \psi_3^{(1)}(b) & \psi_4^{(1)}(b) \\ \psi_1^{(2)}(b) & \psi_2^{(2)}(b) & \psi_3^{(2)}(b) & \psi_4^{(2)}(b) \\ \psi_1^{(3)}(b) & \psi_2^{(3)}(b) & \psi_3^{(3)}(b) & \psi_4^{(3)}(b) \end{vmatrix} \\ &= \begin{vmatrix} \psi_{F1}(b) & \psi_{F2}(b) & 0 & 0 \\ \psi_{F1}^{(1)}(b) & \psi_{F2}^{(1)}(b) & 0 & 0 \\ \psi_{F1}^{(2)}(b) & \psi_{F2}^{(2)}(b) & 1 & 0 \\ \psi_{F1}^{(3)}(b) & \psi_{F2}^{(3)}(b) & 0 & -1 \end{vmatrix} \\ &= \psi_1^{(1)}(b)\,\psi_2(b) - \psi_1(b)\,\psi_2^{(1)}(b). \end{aligned} \tag{24}$$

Using Lemma 2, we find that for $s = 1, 2$ we have

$$\left.\begin{aligned} \psi_r(a/x,\lambda) = \psi_{Fr}(a/b,\lambda) + O\left(|P|^{-1} e^{\sigma(x-a)}\right) \quad \text{as} \quad |\lambda| \to \infty \\ \psi_r^{(1)}(a/x,\lambda) = \psi_{Fr}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(x-a)}\right) \quad \text{as} \quad |\lambda| \to \infty \end{aligned}\right\}. \tag{25}$$

Put $r = 1, 2$ and $x = b$ in (25) and substitute into (24) to obtain

$$\begin{aligned} & W(\lambda) \\ = \; & \left[\psi_{F1}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(b-a)}\right)\right] \\ & \times \left[\psi_{F2}(a/b,\lambda) + O\left(|P|^{-1} e^{\sigma(b-a)}\right)\right] \\ & - \left[\psi_{F1}(a/b,\lambda) + O\left(|P|^{-1} e^{\sigma(b-a)}\right)\right] \\ & \times \left[\psi_{F2}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(b-a)}\right)\right]. \end{aligned} \tag{26}$$

The product of the first two expressions of (26) is obtained as follows:

$$\begin{aligned} & \left[\psi_{F1}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(b-a)}\right)\right] \\ & \times \left[\psi_{F2}(a/b,\lambda) + O\left(|P|^{-1} e^{\sigma(b-a)}\right)\right] \\ = \; & \psi_{F1}^{(1)}(a/b,\lambda)\,\psi_{F2}(a/b,\lambda) + \psi_{F1}^{(1)}(a/b,\lambda)\,O\left(|P|^{-1} e^{\sigma(b-a)}\right) \\ & + \psi_{F2}(a/b,\lambda)\,O\left(e^{\sigma(b-a)}\right) + O\left(e^{\sigma(b-a)}\right) O\left(|P|^{-1} e^{\sigma(b-a)}\right) \end{aligned} \tag{27}$$

Applying Lemma 2a (ii) on the 2nd and 3rd terms on the right hand side of (27) we get

$$= \psi_{F1}^{(1)}(a/b,\lambda)\,\psi_{F2}(a/b,\lambda) + O(\phi_1+\phi_2+\phi_3),$$

where

$$\left.\begin{array}{c} \phi_1 = \left(|P|\,e^{\sigma(b-a)}\right)\left(|P|^{-1}\,e^{\sigma(b-a)}\right) = \left(e^{2\sigma(b-a)}\right) \\ \phi_2 = \left(e^{\sigma(b-a)}\right)\left(e^{\sigma(b-a)}\right) = \left(e^{2\sigma(b-a)}\right) \\ \phi_3 = \left(e^{\sigma(b-a)}\right)\left(|P|^{-1}\,e^{\sigma(b-a)}\right) = \left(|P|^{-1}\,e^{2\sigma(b-a)}\right) \end{array}\right\}. \tag{28}$$

Substituting (28) into (27) we find that

$$\begin{aligned} & \left[\psi_{F1}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(b-a)}\right)\right] \\ & \times\left[\psi_{F2}(a/b,\lambda) + O\left(|P|^{-1}\,e^{\sigma(b-a)}\right)\right] \\ = \;& \psi_{F1}^{(1)}(a/b,\lambda)\,\psi_{F2}(a/b,\lambda) + O(\phi_1+\phi_2+\phi_3) \\ = \;& \psi_{F1}^{(1)}(b,\lambda)\,\psi_{F2}(b,\lambda) + O\left(e^{2\sigma(b-a)}\right) \quad \text{as} \quad |\lambda|\to\infty. \end{aligned} \tag{29}$$

Similarly the product of the last two expressions of (26) is obtained as follows:

$$\begin{aligned} & \left[\psi_{F1}(a/b,\lambda) + O\left(|P|^{-1}\,e^{\sigma(b-a)}\right)\right] \\ & \cdot\left[\psi_{F2}^{(1)}(a/b,\lambda) + O\left(e^{\sigma(b-a)}\right)\right] \\ = \;& \psi_{F1}(b,\lambda)\,\psi_{F2}^{(1)}(b,\lambda) + O\left(e^{2\sigma(b-a)}\right) \quad \text{as} \quad |\lambda|\to\infty. \end{aligned} \tag{30}$$

Substituting (29) and (30) into (26) we obtain

$$\begin{aligned} W(\lambda) \;=\; & \psi_{F1}^{(1)}(b,\lambda)\,\psi_{F2}(b,\lambda) - \psi_{F1}(b,\lambda)\,\psi_{F2}^{(1)}(b,\lambda) \\ & + O\left(e^{2\sigma(b-a)}\right) \end{aligned} \tag{31}$$

Substituting (16) into (31) we get

$$W(\lambda) = W_F(\lambda) + O\left(e^{2\sigma(b-a)}\right) \quad \text{as} \quad |\lambda|\to\infty. \tag{32}$$

If follows from the results of Theorem 1 (i.e., (18)) and Theorem 2 (i.e., (32)) that

$$W(\lambda) \sim W_F(\lambda)$$

□

5. Conclusion

We have succesfully proved through Theorem 1 and Theorem 2, that the Wronskian of the boundary condition functions of the fourth order boundary value problem is asymptotically equivalent to the corresponding Wronskian of the fourth order Fourier problem.

References

[1] K. Kodiara, On Ordinary Differential Equations of Any Even Order and the Corresponding Eigenfunction Expansions, American Journal of Math., 72, (1950), 502-544.

[2] D.N. Offei, The Use of Boundary Condition Functions for Non Self-Adjoint Boundary Value Problems, J. Math Ana and Appl., 31(2), (1970), 369-382.

[3] E.C. Titchmarsh, Eigenfunction Expansions Associated with Second Order Differential Equations. Part 1, 2ed Oxford, 1962.

[4] W.N. Everitt, Self-Adjoint Boundary Value Problems on Finite Intervals, J. London Math Soc., 37, (1962), 372-384.

[5] D.N. Offei, Some Asymptotic Expansions of Third Order Differential Equations, J. London Math Soc., 44, (1969), 71-87.

[6] G.H. Hardy. A Course of Pure Mathematics, Cambridge University Press, (1960).

[7] W.N. Everitt. Some Asymptotic Expansions of a Fourth Order Differential Equation, Quart. J of Math. Oxford, 2(16), (1965), 269-278.

[8] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York/London, Reprint Edition, 1984.

[9] M.B. Osei, Boundary Condition Functions Associated with Boundary Value Problems for a Second Order Differential Equation, Journal of Natural Sciences, 1(2), (2001), 93-106.

[10] David L. Powers, Boundary Value Problems, Academic Press San Diego/London 4th ed, 1999.

[11] C. Henry Edwards and David E. Penny, Differential Equations and Boundary Value Problems, Pearson Education Inc, New Jersey 3rd ed, 2004.

[12] Keyu Zhang, Jiafa Xu and Wei Dong, Positive solutions for a fourth-order p-Laplacian boundary value problem with impulsive effects, Boundary Value Problems, 2013, (2013), 120.

[13] Juan J. Nieto, Existence of a solution for a three-point boundary value for a second-order differential equation at resonance, Boundary Value Problems, (2013), 130.

[14] E.K. Essel, E. Yankson and S.M. Naandam. Asymptotic behaviour of boundary condition functions for a fourth-order boundary value problem.Theoretical Mathematics & Applications (TMA),Vol.3, no.3, 2013, 121-143,ISSN: 1792- 9687 (print), 1792-9709 (online).

[15] Bonsu Osei, Samuel Asiedu-Addo. Asymptotic behaviour of Wronskian of boundary condition functions for a second order boundary value problem.IeJPAM, 1(1), (2010), 93-101.