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Multidimensional Hardy-type inequalities with general kernels

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ABSTRACT

Some new multidimensional Hardy-type inequalities involving arithmetic mean operators with general positive kernels are derived. Our approach is mainly to use a convexity argument and the results obtained improve some known results in the literature and, in particular, some recent results in [S. Kaijser, L. Nikolova, L.-E. Persson, A. Wedestig, Hardy-type inequalities via convexity, Math. Inequal. Appl. 8 (3) (2005) 403–417] are generalized and complemented.

1. Introduction

G.H. Hardy in [2] proved the following classical inequality: For any $p > 1$ and any integrable function $f(x) \geqslant 0$ on $(0, \infty)$, the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx \tag{1.1}$$

holds, where the constant $(\frac{p}{p-1})^p$ is the best possible. Inequality (1.1) is generally known as Hardy's inequality and it has an interesting prehistory and history (see [3,6,7]). Because of the fundamental importance of this inequality in analysis and its applications, many interesting extensions, generalizations, variants and alternative proofs of (1.1) have appeared in the literature (see for instance [3,5–7,9] and the references cited therein). A well-known simple fact is that (1.1) can equivalently (via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$), be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p \frac{dx}{x} \leqslant \int_0^\infty h^p(x)\frac{dx}{x} \tag{1.2}$$

and in this form it even holds with equality when $p = 1$. In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality but this was not discovered in the dramatic period when Hardy discovered in [1] and finally proved his inequality in 1925 (see [2,3,7]). Guided by this fact, in this paper the weighted Hardy-type inequality

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$$\int_0^\infty \dots \int_0^\infty \Phi\big(A_K f(x_1,\dots,x_n)\big)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n} \leqslant \int_0^\infty \dots \int_0^\infty \Phi\big(f(x_1,\dots,x_n)\big)v(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{1.3}$$

will be studied, where Φ is a convex function and A_K and its dual A_{K^*} are general Hardy-type (arithmetic mean) operators defined by

$$A_K f(x_1,\dots,x_n) := \frac{1}{K(x_1,\dots,x_n)} \int_0^{x_1} \dots \int_0^{x_n} k(x_1,\dots,x_n,t_1,\dots,t_n) f(t_1,\dots,t_n)\,dt_1\dots dt_n \tag{1.4}$$

and

$$A_{K^*} f(x_1,\dots,x_n) := \frac{1}{\widetilde{K}(x_1,\dots,x_n)} \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} k(x_1,\dots,x_n,t_1,\dots,t_n) f(t_1,\dots,t_n)\,dt_1\dots dt_n \tag{1.5}$$

with $K(x_1,\dots,x_n)$ and $\widetilde{K}(x_1,\dots,x_n)$ given by

$$K(x_1,\dots,x_n) := \int_0^{x_1} \dots \int_0^{x_n} k(x_1,\dots,x_n,t_1,\dots,t_n)\,dt_1\dots dt_n \tag{1.6}$$

and

$$\widetilde{K}(x_1,\dots,x_n) := \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} k(x_1,\dots,x_n,t_1,\dots,t_n)\,dt_1\dots dt_n, \tag{1.7}$$

respectively (here $k = k(x_1,\dots,x_n,t_1,\dots,t_n)$ is a kernel, i.e. a locally integrable and positive function in $\mathbb{R}^{2n}$).

In this paper we prove and discuss some new multidimensional Hardy-type inequalities involving arithmetic mean operators with general positive kernels. We remark that our condition is only sufficient (but see also Remark 3.4). Our approach is to use a convexity argument, which is completely different from the classical ones used by Hardy and others (see e.g. [3] and [6]). In particular, our results further generalize and complement some recent results in [4].

This paper is organized as follows: The results are presented and proved in Section 2. In Section 3 we present some illustrative examples and remarks.

Notations. All functions in this paper are assumed to be measurable and expressions of the form $0.\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{a}{\infty}$ $(a \in \mathbb{R})$ are taken to be equal to zero. Finally, as usual, by a weight $u = u(x_1,\dots,x_n)$ we mean a nonnegative measurable function on a subset of $\mathbb{R}^n_+$.

2. Results

In this section we state and prove the results of this paper.

Proposition 2.1. *Let $n \in \mathbb{N}_+$, $k(x_1,\dots,x_n,t_1,\dots,t_n)$ and $u(x_1,\dots,x_n)$ be weight functions and assume that*

$$\frac{k(x_1,\dots,x_n,t_1,\dots,t_n)u(x_1,\dots,x_n)}{x_1\dots x_n K(x_1,\dots,x_n)}$$

is locally integrable and for each $(t_1,\dots,t_n)$, $t_i \in (0,b_i)$, define v by

$$v(x_1,\dots,x_n) = t_1\dots t_n \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} \frac{k(x_1,\dots,x_n,t_1,\dots,t_n)u(x_1,\dots,x_n)}{K(x_1,\dots,x_n)} \frac{dx_1\dots dx_n}{x_1\dots x_n} < \infty. \tag{2.1}$$

(i) *If Φ is positive and convex on (a,c), $-\infty \leqslant a < c \leqslant \infty$, then*

$$\int_0^{b_1} \dots \int_0^{b_n} \Phi\big(A_K f(x_1,\dots,x_n)\big)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n} \leqslant \int_0^{b_1} \dots \int_0^{b_n} \Phi\big(f(x_1,\dots,x_n)\big)v(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.2}$$

for all f with $a < f(x_1,\dots,x_n) < c$, $0 \leqslant x_i \leqslant b_i$, $i = 1,2,\dots,n$.

(ii) *If Φ is positive and concave on (a,c), $-\infty \leqslant a < c \leqslant \infty$, then*

$$\int_0^{b_1}\cdots\int_0^{b_n}\Phi\big(A_K f(x_1,\ldots,x_n)\big)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \geqslant \int_0^{b_1}\cdots\int_0^{b_n}\Phi\big(f(x_1,\ldots,x_n)\big)v(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \tag{2.3}$$

for all f with $a < f(x_1,\ldots,x_n) < c$, $0 \leqslant x_i \leqslant b_i$, $i = 1,2,\ldots,n$.

Here A_K and $K(x_1,\ldots,x_n)$ are as defined by (1.4) *and* (1.6), *respectively.*

Remark 2.1. For the case $n = 1$ Proposition 2.1 (i) coincides with Theorem 4.1 in [4]. Moreover, for the case $k \equiv 1$ in Proposition 2.1 we obtain Proposition 2.1 in [8].

Proof. (i) By applying Jensen's inequality and Fubini's theorem to the left-hand side of (2.2) we have that

$$\begin{aligned}
&\int_0^{b_1}\cdots\int_0^{b_n}\Phi\big(A_K f(x_1,\ldots,x_n)\big)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\\
&\quad=\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(\frac{1}{K(x_1,\ldots,x_n)}\int_0^{x_1}\cdots\int_0^{x_n}k(x_1,\ldots,x_n,t_1,\ldots,t_n)f(t_1,\ldots,t_n)\,dt_1\ldots dt_n\right)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\\
&\quad\leqslant\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{K(x_1,\ldots,x_n)}\int_0^{x_1}\cdots\int_0^{x_n}k(x_1,\ldots,x_n,t_1,\ldots,t_n)\Phi(f(t_1,\ldots,t_n))\,dt_1\ldots dt_n\right)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\\
&\quad=\int_0^{b_1}\cdots\int_0^{b_n}\Phi\big(f(t_1,\ldots,t_n)\big)\left(\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\frac{1}{K(x_1,\ldots,x_n)}k(x_1\ldots x_n,t_1,\ldots,t_n)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\right)dt_1\ldots dt_n\\
&\quad=\int_0^{b_1}\cdots\int_0^{b_n}\Phi\big(f(x_1,\ldots,x_n)\big)v(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}.
\end{aligned}$$

(ii) The proof is completely similar to the proof of (i). We only note that in this case, since the function Φ is concave and, hence, the inequality sign is reversed. The proof is complete. □

Proposition 2.2. *Let $n \in \mathbb{N}_+$, $0 \leqslant b_i < x_i$, $t_i \leqslant \infty$, $i = 1,2,\ldots,n$, and let $k(x_1\ldots x_n,t_1,\ldots,t_n)$ and $u(x_1,\ldots,x_n)$ be weight functions such that*

$$\frac{k(x_1,\ldots,x_n,t_1,\ldots,t_n)u(x_1,\ldots,x_n)}{x_1\ldots x_n\widetilde{K}(x_1,\ldots,x_n)}$$

is locally integrable and for each $(t_1,\ldots,t_n)$, $t_i \in (b_i,\infty)$, define v by

$$v(x_1,\ldots,x_n) = t_1\ldots t_1\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\frac{k(x_1,\ldots,x_n,t_1,\ldots,t_n)u(x_1,\ldots,x_n)}{\widetilde{K}(x_1,\ldots,x_n)}\frac{dx_1\ldots dx_n}{x_1\ldots x_n} < \infty. \tag{2.4}$$

(i) *If Φ is positive and convex on (a,c), $-\infty \leqslant a < c \leqslant \infty$ and A_{K^*} is the general dual Hardy operator defined by* (1.5), *then the inequality*

$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\big(A_{K^*}f(x_1,\ldots,x_n)\big)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \leqslant \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\big(f(x_1,\ldots,x_n)\big)v(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \tag{2.5}$$

holds for all f with $a < f(x_1,\ldots,x_n) < c$, $0 \leqslant x_i \leqslant b_i$, $i = 1,2,\ldots,n$.

(ii) *If Φ is positive and concave on (a,c), $-\infty \leqslant a < c \leqslant \infty$ and A_{K^*} is the general dual Hardy operator defined by* (1.5), *then the inequality*

$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\big(A_{K^*}f(x_1,\ldots,x_n)\big)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \geqslant \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\big(f(x_1,\ldots,x_n)\big)v(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \tag{2.6}$$

holds for all f with $a < f(x_1,\ldots,x_n) < c$, $0 \leqslant x_i \leqslant b_i$, $i = 1,2,\ldots,n$, and $\widetilde{K}(x_1,\ldots,x_n)$ is as defined by (1.7).

Proof. (i) By applying Jensen's inequality and Fubini's theorem to the left-hand side of (2.5) we find that

$$\begin{aligned}
&\int_{b_1}^{\infty}\dots\int_{b_n}^{\infty}\Phi\big(A_{K^*}f(x_1,\dots,x_n)\big)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\\
&=\int_{b_1}^{\infty}\dots\int_{b_n}^{\infty}\Phi\left(\frac{1}{\widetilde{K}(x_1,\dots,x_n)}\int_{x_1}^{\infty}\dots\int_{x_n}^{\infty}k(x_1,\dots,x_n,t_1,\dots,t_n)f(t_1,\dots,t_n)\,dt_1\dots dt_n\right)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\\
&\leqslant\int_{b_1}^{\infty}\dots\int_{b_n}^{\infty}\left(\frac{1}{\widetilde{K}(x_1,\dots,x_n)}\int_{x_1}^{\infty}\dots\int_{x_n}^{\infty}k(x_1,\dots,x_n,t_1,\dots,t_n)\Phi\big(f(t_1,\dots,t_n)\big)\,dt_1\dots dt_n\right)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\\
&=\int_{b_1}^{\infty}\dots\int_{b_n}^{\infty}\Phi\big(f(t_1,\dots,t_n)\big)\left(\int_{b_1}^{t_1}\dots\int_{b_n}^{t_n}\frac{1}{\widetilde{K}(x_1,\dots,x_n)}k(x_1,\dots,x_n,t_1,\dots,t_n)u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)dt_1\dots dt_n\\
&=\int_{b_1}^{\infty}\dots\int_{b_n}^{\infty}\Phi\big(f(x_1,\dots,x_n)\big)v(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}.
\end{aligned}$$

(ii) The proof is completely similar to the proof of (i). As before we note that the function Φ is concave and, thus, the inequality sign is reversed. The proof is complete. □

In order to be able to prove the final (and main) result of this paper we need to formulate the following Minkowski type inequality of independent interest:

Lemma 2.1. *Let $p>1$, $-\infty\leqslant a_i<b_i\leqslant\infty$, $k=k(x_1\dots x_n,t_1,\dots,t_n)$ be a locally integrable kernel and Φ and Ψ be positive and measurable functions. Then*

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n}\left(\int_{a_1}^{x_1}\dots\int_{a_n}^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\Psi(y_1,\dots,y_n)\,dy_1\dots dy_n\right)^p\Phi(x_1,\dots,x_n)\,dx_1\dots dx_n\right)^{\frac{1}{p}}\\
&\quad\leqslant\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n}\left(\int_{y_1}^{b_1}\dots\int_{y_n}^{b_n}\Phi(x_1,\dots,x_n)k^p(x_1,\dots,x_n,t_1,\dots,t_n)\,dx_1\dots dx_n\right)^{\frac{1}{p}}\Psi(y_1,\dots,y_n)\,dy_1\dots dy_n.
\end{aligned}\tag{2.7}$$

For the readers convenience we include a simple proof of this lemma in the following a little more general form:

Proposition 2.3. *Let $p>1$, $-\infty\leqslant a_i<b_i\leqslant\infty$, and $-\infty\leqslant c_i<d_i\leqslant\infty$, $n=1,2,\dots,n$ ($n\in\mathbb{Z}_+$). Then*

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n}\left(\int_{c_1}^{d_1}\dots\int_{c_n}^{d_n}k(x_1,\dots,x_n,y_1,\dots,y_n)dy_1\dots dy_n\right)^p dx_1\dots dx_n\right)^{\frac{1}{p}}\\
&\quad\leqslant\int_{c_1}^{d_1}\dots\int_{c_n}^{d_n}\left(\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n}k^p(x_1,\dots,x_n,y_1,\dots,y_n)\,dx_1\dots dx_n\right)^{\frac{1}{p}}dy_1\dots dy_n.
\end{aligned}\tag{2.8}$$

Remark 2.2. We see that Lemma 2.1 is a special case of Proposition 2.3 by choosing $k=k(x_1\dots x_n,y_1,\dots,y_n)$ in the following way

$$k=\begin{cases}k(x_1,\dots,x_n,t_1,\dots,t_n)\Psi(y_1,\dots,y_n)\Phi^{\frac{1}{p}}(x_1,\dots,x_n), & a_i\leqslant y_i\leqslant x_i\leqslant b_i,\\ 0, & \text{elsewhere.}\end{cases}$$

Proof. By using the fact that we can have equality in Hölder's inequality we find that

$$I_0 := \left(\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} \left(\int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n)\, dy_1 \ldots dy_n\right)^p dx_1 \ldots dx_n\right)^{\frac{1}{p}}$$
$$= \sup_{\varphi \geqslant 0} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} \varphi(x_1, \ldots, x_n) \int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n)\, dy_1 \ldots dy_n\, dx_1 \ldots dx_n, \tag{2.9}$$

where the supremum is taken over all φ such that

$$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} \varphi(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = 1.$$

Hence, by Fubini's theorem, a trivial estimate and using (2.9) again, we find that

$$I_0 = \sup_{\varphi \geqslant 0} \int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n)\varphi(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\, dy_1 \ldots dy_n$$
$$\leqslant \int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} \left(\sup_{\varphi \geqslant 0} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n)\varphi(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\right) dy_1 \ldots dy_n$$
$$= \int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} \left(\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} k^p(x_1, \ldots, x_n, y_1, \ldots, y_n)\, dx_1 \ldots dx_n\right)^{\frac{1}{p}} dy_1 \ldots dy_n.$$

The proof is complete. □

Theorem 2.1. *Let $1 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $s_1, \ldots, s_n \in (1, p)$, $i = 1, 2, \ldots, n$, and let Φ be a convex function on (a, c), $-\infty \leqslant a < c \leqslant \infty$. Let A_K be the general Hardy operator defined by* (1.4) *and let $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ be weight functions, where $v(x_1, \ldots, x_n)$ is of product type i.e. $v(x_1, \ldots, x_n) = v(x_1).v(x_2) \ldots v(x_n)$. Then the inequality*

$$\left(\int_0^{b_1} \ldots \int_0^{b_n} \left[\Phi\left(A_K f(x_1, \ldots, x_n)\right)\right]^q u(x_1, \ldots, x_n) \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_0^{b_1} \ldots \int_0^{b_n} \Phi^p\left(f(x_1, \ldots, x_n)\right) v(x_1, \ldots, x_n) \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}\right)^{\frac{1}{p}} \tag{2.10}$$

holds for all functions $f(x_1, \ldots, x_n)$, $a < f(x_1, \ldots, x_n) < c$, if

$$A(s_1, \ldots, s_n) := \sup_{0 < t_1 \ldots t_n < b_1 \ldots b_n} \left(\int_{t_1}^{b_1} \ldots \int_{t_n}^{b_n} \left[\frac{k(x_1, \ldots, x_n, y_1, \ldots, y_n)}{K(x_1, \ldots, x_n)}\right]^q u(x_1, \ldots, x_n)\right.$$
$$\left. \times V_1^{\frac{q(p-s_1)}{p}}(x_1) \ldots V_n^{\frac{q(p-s_n)}{p}}(x_n) \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}\right)^{\frac{1}{q}} V_1^{\frac{s_1-1}{p}}(t_1) \ldots V_n^{\frac{s_n-1}{p}}(t_n) < \infty \tag{2.11}$$

holds, where

$$V_i(x_i) = \int_0^{x_i} v^{1-p'}\, dt_i, \quad i = 1, 2, \ldots, n, \tag{2.12}$$

and $p' = \frac{p}{p-1}$. Furthermore, if C is the best possible constant in (2.10)*, then*

$$C \leqslant \inf_{1 < s_1 \ldots s_n < p} \left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}} \ldots \left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}} A(s_1, \ldots, s_n). \tag{2.13}$$

Remark 2.3. For the case $n = 1$ Theorem 2.1 coincides with Theorem 4.4 in [4].

Proof. By applying Jensen's inequality to the left-hand side of (2.10) we obtain that

$$\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\Phi\left(A_K f(x_1,\dots,x_n)\right)\right]^q u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$=\left(\int_0^{b_1}\dots\int_0^{b_n}\Phi\left[\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)f(t_1,\dots,t_n)\,dt_1\dots dt_n\right]^q u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\Phi\left(f(t_1,\dots,t_n)\right)dt_1\dots dt_n\right]^q\right.$$
$$\left.\times\, u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}. \tag{2.14}$$

The proof will be completed if we can prove that

$$\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\Phi\left(f(t_1,\dots,t_n)\right)dt_1\dots dt_n\right]^q u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_0^{b_1}\dots\int_0^{b_n}\Phi^p\left(f(x_1,\dots,x_n)\right)v(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{p}}. \tag{2.15}$$

Define g so that $\Phi^p(f(x_1,\dots,x_n))\frac{v(x_1,\dots,x_n)}{x_1\dots x_n}=\Phi(g(x_1,\dots,x_n))$ and (2.15) can equivalently be rewritten as

$$L:=\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\Phi^{\frac{1}{p}}\left(g(t_1,\dots,t_n)\right)\right.\right.$$
$$\left.\left.\times\left(\frac{t_1\dots t_n}{v(t_1,\dots,t_n)}\right)^{\frac{1}{p}}dt_1\dots dt_n\right]^q u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_0^{b_1}\dots\int_0^{b_n}\Phi\left(g(x_1,\dots,x_n)\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{p}}. \tag{2.16}$$

By applying Hölder's inequality and Lemma 2.1 with p replaced by $\frac{q}{p}$ to the left-hand side of (2.16) and using the fact that $v(x_1,\dots,x_n)=v_1(x_1)\dots v_n(x_n)$ and $\frac{-p'}{p}=1-p'$ we find that

$$L=\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\Phi^{\frac{1}{p}}\left(g(t_1,\dots,t_n)\right)\right.\right.$$
$$\times V_1^{\frac{s_1-1}{p}}(t_1)\dots V_n^{\frac{s_n-1}{p}}(t_n)V_1^{\frac{-(s_1-1)}{p}}(t_1)\dots V_n^{\frac{-(s_n-1)}{p}}(t_n)v_1^{-\frac{1}{p}}(t_1)\dots v_n^{-\frac{1}{p}}(t_n)$$
$$\left.\left.\times\, t_1^{\frac{1}{p}}\dots t_n^{\frac{1}{p}}\,dt_1\dots dt_n\right]^q u(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\int_0^{x_1}\dots\int_0^{x_n}k^p(x_1,\dots,x_n,t_1,\dots,t_n)\Phi\left(g(t_1,\dots,t_n)\right)V_1^{s_1-1}(t_1)\dots V_n^{s_n-1}(t_n)\,dt_1\dots dt_n\right]^{\frac{q}{p}}\right.$$
$$\left.\times\left[\int_0^{x_1}\dots\int_0^{x_n}V_1^{\frac{-p'(s_1-1)}{p}}(t_1)\dots V_n^{\frac{-p'(s_n-1)}{p}}(t_n)v_1^{-\frac{p'}{p}}(t_1)\dots v_n^{-\frac{p'}{p}}(t_n)t_1^{\frac{p'}{p}}\dots t_n^{\frac{p'}{p}}\,dt_1\dots dt_n\right]^{\frac{q}{p'}}\frac{u(x_1,\dots,x_n)}{K^q(x_1,\dots,x_n)}\frac{dx_1\dots dx_n}{x_1\dots x_n}\right)^{\frac{1}{q}}$$
$$=\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\dots\left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}}\left(\int_0^{b_1}\dots\int_0^{b_n}\left[\int_0^{x_1}\dots\int_0^{x_n}k^p(x_1,\dots,x_n,t_1,\dots,t_n)\right.\right.$$

$$\times \Phi\big(g(t_1,\dots,t_n)\big)V_1^{s_1-1}(t_1)\dots V_n^{s_n-1}(t_n)\,dt_1\dots dt_n\Bigg]^{\frac{q}{p}} V_1^{\frac{q(p-s_1)}{p}}(x_1)\dots V_n^{\frac{q(p-s_n)}{p}}(x_n)\frac{u(x_1,\dots,x_n)}{K^q(x_1,\dots,x_n)}\frac{dx_1\dots dx_n}{x_1\dots x_n}\Bigg)^{\frac{1}{q}}$$

$$\leqslant \left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\dots\left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}}\Bigg(\int_0^{b_1}\dots\int_0^{b_n}\Phi\big(g(t_1,\dots,t_n)\big)V_1^{s_1-1}(t_1)\dots V_n^{s_n-1}(t_n)$$

$$\times\Bigg[\int_{t_1}^{b_1}\dots\int_{t_n}^{b_n}V_1^{\frac{q(p-s_1)}{p}}(x_1)\dots V_n^{\frac{q(p-s_n)}{p}}(x_n)u(x_1,\dots,x_n)\left(\frac{k(x_1\dots x_n,t_1,\dots,t_n)}{K(x_1,\dots,x_n)}\right)^q\frac{dx_1\dots dx_n}{x_1\dots x_n}\Bigg]^{\frac{p}{q}}dt_1\dots dt_n\Bigg)^{\frac{1}{p}}$$

$$\leqslant \left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\dots\left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}}A(s_1,\dots,s_n)\left(\int_0^{b_1}\dots\int_0^{b_n}\Phi\big(g(t_1,\dots,t_n)\big)\,dt_1\dots dt_n\right)^{\frac{1}{p}}.$$

Hence, (2.16) and, thus, (2.15) hold and the proof is complete. □

3. Concluding examples and remarks

By using our results in special cases and making suitable variable transformation as that in the introduction (showing that (1.1) and (1.2) are equivalent) we obtain directly some multidimensional Hardy-type inequalities. For example if the kernel $k\equiv 1$, then Proposition 2.1 with $\Phi(u)=u^p$ implies the following result

Example 3.1. Let $0\leqslant d_i\leqslant\infty$, $i=1,2,\dots,n$ $(n\in\mathbb{Z}_+)$.

(i) If $p>1$ or $p<0$, then,

$$\int_0^{d_1}\dots\int_0^{d_n}\left(\frac{1}{y_1\dots y_n}\int_0^{y_1}\dots\int_0^{y_n}g(s_1,\dots,s_n)\,ds_1\dots ds_n\right)^p dy_1\dots dy_n$$

$$\leqslant\left(\frac{p}{p-1}\right)^{pn}\int_0^{d_1}\dots\int_0^{d_n}g^p(y_1,\dots,y_n)\left(1-\left(\frac{y_1}{d_1}\right)^{\frac{p-1}{p}}\right)\dots\left(1-\left(\frac{y_n}{d_n}\right)^{\frac{p-1}{p}}\right)dy_1\dots dy_n$$

for each positive measurable function g.

(ii) If $0<p<1$, then

$$\int_0^{d_1}\dots\int_0^{d_n}\left(\frac{1}{y_1\dots y_n}\int_0^{y_1}\dots\int_0^{y_n}g(s_1,\dots,s_n)\,ds_1\dots ds_n\right)^p dy_1\dots dy_n$$

$$\geqslant\left(\frac{p}{1-p}\right)^{pn}\int_0^{d_1}\dots\int_0^{d_n}g^p(y_1,\dots,y_n)\left(1-\left(\frac{y_1}{d_1}\right)^{\frac{p-1}{p}}\right)\dots\left(1-\left(\frac{y_n}{d_n}\right)^{\frac{p-1}{p}}\right)dy_1\dots dy_n$$

for every positive measurable function g.

Remark 3.1. This result was also proved in [4].

Moreover, by choosing $\Phi(u)=\exp u$ and replacing f by $\ln f^p$ for any $p\in\mathbb{R}$ we obtain the following multidimensional version of Pólya–Knopp's inequality:

Example 3.2. The assumptions of Proposition 2.1 yields that

$$\int_0^{b_1}\dots\int_0^{b_n}u(x_1,\dots,x_n)\exp\left(\frac{1}{K(x_1,\dots,x_n)}\int_0^{x_1}\dots\int_0^{x_n}k(x_1,\dots,x_n,t_1,\dots,t_n)\ln f(t_1,\dots,t_n)\,dt_1\dots dt_n\right)^p\frac{dx_1\dots dx_n}{x_1\dots x_n}$$

$$\leqslant\int_0^{b_1}\dots\int_0^{b_n}v(x_1,\dots,x_n)f^p(x_1,\dots,x_n)\frac{dx_1\dots dx_n}{x_1\dots x_n}. \tag{3.1}$$

In particular, if $u(x_1,\dots,x_n)\equiv 1$, $k(x_1,\dots,x_n,t_1,\dots,t_n)\equiv 1$, and $f(t_1,\dots,t_n)$ is replaced by $\frac{g(t_1,\dots,t_n)}{t_1\dots t_n}$, then (3.1) reduces to Corollary 2.3 in [4].

Remark 3.2. Inequality (3.1) is a multidimensional generalization of *Pólya–Knopp type* inequality (4.2) in [4] and Example 3.1 is a (formal) generalization of Corollary 2.2 in [4].

Remark 3.3. For the case $k\equiv 1$ another proof of the multidimensional Minkowski inequality in Lemma 2.1 was presented and proved in the PhD thesis of A. Wedestig [10, Lemma 4.4]. However, our proof here is different and much simpler.

Remark 3.4. In the main result of this paper (Theorem 2.1) we have only obtained a sufficient condition for the multidimensional Hardy-type inequality (2.10) to hold for a general positive kernel. Even in the one-dimensional case we have necessary and sufficient conditions only for very special kernels (satisfying e.g. the Oinarov condition, see [5, p. 89]). We conjecture that also our condition (2.11) is necessary and sufficient if we impose additional properties on the kernel but we leave this as an open question.

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References

[1] G.H. Hardy, Note on a theorem of Hilbert, Math. Z. 6 (1920) 314–317.
[2] G.H. Hardy, Notes on some points in the integral calculus, LX. An inequality between integrals, Messenger of Math. 54 (1925) 150–156.
[3] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1959.
[4] S. Kaijser, L. Nikolova, L.-E. Persson, A. Wedestig, Hardy type inequalities via convexity, Math. Inequal. Appl. 8 (3) (2005) 403–417.
[5] A. Kufner, L.E. Persson, Weighted Inequalities of Hardy Type, World Scientific Publishing Co, Singapore/New Jersey/London/Hong Kong, 2003.
[6] A. Kufner, L. Maligranda, L.-E. Persson, The Hardy Inequality. About Its History and Some Related Results, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[7] A. Kufner, L. Maligranda, L.-E. Persson, The prehistory of the Hardy inequality, Amer. Math. Monthly 113 (2006) 715–732.
[8] J.A. Oguntuase, C.A. Okpoti, L.-E. Persson, F.K.A. Allotey, Weighted multidimensional Hardy type inequalities via Jensen's inequality, Proc. A. Razmadze Math. Inst. 144 (2007) 91–105.
[9] B. Opic, A. Kufner, Hardy-type Inequalities, Pitman Res. Notes in Math., vol. 219, Longman Scientific and Technical, Harlow, 1990.
[10] A. Wedestig, Weighted inequalities of Hardy-type and their limiting inequalities, PhD thesis, Department of Mathematics, Luleå University of Technology, Luleå, Sweden, 2003.