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POSITIVE PERIODIC SOLUTIONS FOR SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

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ABSTRACT. We use Krasnoselskii's fixed point theorem to prove the existence of positive periodic solutions of the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = c\frac{d}{dt}x(t-\tau(t)) + f(t, h(x(t)), g(x(t-\tau(t)))).$$

1. Introduction

In this work, we prove the existence of positive periodic solutions for the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t) = c\frac{d}{dt}x(t-\tau(t))+f(t, h(x(t)), g(x(t-\tau(t)))), \quad (1.1)$$

where p and q are positive continuous real-valued functions. The function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in its respective arguments. We are mainly motivated by the articles [4, 10, 11, 12, 14] and the references therein. In [12], the Krasnoselskii's fixed point theorem was used to establish the existence of positive periodic solutions for the first-order nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = r(t)x(t) + c\frac{d}{dt}x(t-\tau) - f(t, x(t-\tau)) \quad (1.2)$$

To show the existence of solutions, we transform (1.1) into an integral equation which is then expressed as a sum of two mappings, one is a contraction and the other is compact.

The rest of this article is organized as follows. In Section 2, we introduce some notation and state some preliminary results needed in later sections. Then we give the Green's function of (1.1), which plays an important role in this paper. Also, we present the inversion of (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [13]. In Section 3, we present our main results on existence.

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2. Preliminaries

For $T>0$, let P_T be the set of continuous scalar functions x that are periodic in t, with period T. Then $(P_T,\|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

In this paper we make the following assumptions.

$$p(t+T)=p(t),\quad q(t+T)=q(t),\quad \tau(t+T)=\tau(t), \tag{2.1}$$

with τ being scalar function, continuous, and $\tau(t)\geq\tau^*>0$. Also, we assume

$$\int_0^T p(s)ds>0,\quad \int_0^T q(s)ds>0. \tag{2.2}$$

We also assume that $f(t,h,g)$ is periodic in t with period T; that is,

$$f(t+T,h,g)=f(t,h,g). \tag{2.3}$$

Lemma 2.1 ([9]). *Suppose that* (2.1) *and* (2.2) *hold and*

$$\frac{R_1[\exp(\int_0^T p(u)du)-1]}{Q_1T}\geq 1, \tag{2.4}$$

where

$$R_1=\max_{t\in[0,T]}\Big|\int_t^{t+T}\frac{\exp(\int_t^s p(u)du)}{\exp(\int_0^T p(u)du)-1}q(s)ds\Big|,$$
$$Q_1=\Big(1+\exp\Big(\int_0^T p(u)du\Big)\Big)^2R_1^2.$$

Then there are continuous and T-periodic functions a and b such that $b(t)>0$, $\int_0^T a(u)du>0$, and

$$a(t)+b(t)=p(t),\quad \frac{d}{dt}b(t)+a(t)b(t)=q(t),\quad \text{for } t\in\mathbb{R}.$$

Lemma 2.2 ([14]). *Suppose the conditions of Lemma 2.1 hold and $\phi\in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t)=\int_t^{t+T}G(t,s)\phi(s)ds,$$

where

$$G(t,s)=\frac{\int_t^s\exp[\int_t^u b(v)dv+\int_u^s a(v)dv]du+\int_s^{t+T}\exp[\int_t^u b(v)dv+\int_u^{s+T}a(v)dv]du}{[\exp\big(\int_0^T a(u)du\big)-1][\exp\big(\int_0^T b(u)du\big)-1]}.$$

Corollary 2.3. [14] *Green's function G satisfies the following properties*

$$G(t,t+T)=G(t,t),\quad G(t+T,s+T)=G(t,s),$$
$$\frac{\partial}{\partial s}G(t,s)=a(s)G(t,s)-\frac{\exp\big(\int_t^s b(v)dv\big)}{\exp\big(\int_0^T b(v)dv\big)-1},$$

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + \frac{\exp\big(\int_t^s a(v)dv\big)}{\exp\big(\int_0^T a(v)dv\big)-1}.$$

We next state and prove the following lemma which will play an essential role in obtaining our results.

Lemma 2.4. *Suppose* (2.1)-(2.3) *and* (2.4) *hold. If $x \in P_T$, then x is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) &= \int_t^{t+T} cE(t,s)x(s-\tau(s))ds \\ &\quad + \int_t^{t+T} G(t,s)[-a(s)cx(s-\tau(s)) + f(s,h(x(s)),g(x(s-\tau(s))))]ds, \end{aligned} \tag{2.5}$$

where

$$E(t,s) = \frac{\exp(\int_t^s b(v)dv)}{\exp(\int_0^T b(v)dv)-1}. \tag{2.6}$$

Proof. Let $x \in P_T$ be a solution of (1.1). From Lemma 2.2, we have

$$x(t) = \int_t^{t+T} G(t,s)[c\frac{\partial}{\partial s}x(s-\tau(s)) + f(s,h(x(s)),g(x(s-\tau(s))))]ds. \tag{2.7}$$

Integrating by parts, we have

$$\begin{aligned} &\int_t^{t+T} cG(t,s)\frac{\partial}{\partial s}x(s-\tau(s))ds \\ &= -\int_t^{t+T} c[\frac{\partial}{\partial s}G(t,s)]x(s-\tau(s))ds \\ &= \int_t^{t+T} cx(s-\tau(s))[E(t,s)-a(s)G(t,s)]ds, \end{aligned} \tag{2.8}$$

where E is given by (2.6). Then substituting (2.8) in (2.7) completes the proof. $\square$

Lemma 2.5 ([14])**.** *Let $A = \int_0^T p(u)du$, $B = T^2\exp\big(\frac{1}{T}\int_0^T \ln(q(u))du\big)$. If*

$$A^2 \geq 4B, \tag{2.9}$$

then

$$\begin{aligned} \min\Big\{\int_0^T a(u)du, \int_0^T b(u)du\Big\} &\geq \frac{1}{2}(A - \sqrt{A^2-4B}) := l, \\ \max\Big\{\int_0^T a(u)du, \int_0^T b(u)du\Big\} &\leq \frac{1}{2}(A + \sqrt{A^2-4B}) := m. \end{aligned}$$

Corollary 2.6 ([14])**.** *Functions G and E satisfy*

$$\frac{T}{(e^m-1)^2} \leq G(t,s) \leq \frac{T\exp\big(\int_0^T p(u)du\big)}{(e^l-1)^2}, \quad |E(t,s)| \leq \frac{e^m}{e^l-1}.$$

To simplify notation, we introduce the constants

$$\beta = \frac{e^m}{e^l-1}, \quad \alpha = \frac{T\exp\big(\int_0^T p(u)du\big)}{(e^l-1)^2}, \quad \gamma = \frac{T}{(e^m-1)^2}. \tag{2.10}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic solutions to (1.1). For its proof we refer the reader to [13].

Theorem 2.7 (Krasnoselskii)**.** *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *$x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,*
(ii) *$\mathcal{A}$ is compact and continuous,*
(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. Main results

We present our existence results in this section by considering two cases; $c \geq 0$, $c \leq 0$. For some non-negative constant K and a positive constant L we define the set

$$\mathbb{D} = \{\varphi \in P_T : K \leq \varphi \leq L\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that there exist a positive constant σ such that

$$\sigma < E(t,s), \quad \text{for all } (t,s) \in [0,T] \times [0,T], \tag{3.1}$$

$$c \geq 0 \tag{3.2}$$

and for all $s \in \mathbb{R}$, $\mu \in \mathbb{D}$

$$\frac{K(1-\sigma cT)}{\gamma T} \leq f(s, h(\mu), g(\mu)) - ca(s)\mu \leq \frac{L(1-\beta cT)}{\alpha T}. \tag{3.3}$$

To apply Theorem 2.7, we construct two mappings in which one is a contraction and the other is completely continuous. Thus, we set the map $\mathcal{A} : \mathbb{D} \to P_T$

$$\begin{aligned}&(\mathcal{A}\varphi)(t)\\ &= \int_t^{t+T} G(t,s)[f(s, h(\varphi(s)), g(\varphi(s-\tau(s)))) - ca(s)\varphi(s-\tau(s))]ds.\end{aligned} \tag{3.4}$$

Similarly, we define the map $\mathcal{B} : \mathbb{D} \to P_T$ by

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds. \tag{3.5}$$

Lemma 3.1. *If $\mathcal{B}$ is given by* (3.5) *with*

$$c\beta T < 1, \tag{3.6}$$

then $\mathcal{B} : \mathbb{D} \to P_T$ is a contraction.

Proof. It is easy to see that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. Let $\varphi, \psi \in \mathbb{D}$ then

$$\|\mathcal{B}\varphi - \mathcal{B}\psi\| = \sup_{t\in[0,T]} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq c\beta T\|\varphi - \psi\|.$$

Hence $\mathcal{B} : P_T \to P_T$ is a contraction. □

Lemma 3.2. *Suppose that conditions* (2.1)-(2.3)*, and* (3.1)-(3.3),(3.6) *hold. Then $\mathcal{A} : P_T \to P_T$ is completely continuous on $\mathbb{D}$.*

Proof. Let $\mathcal{A}$ be defined by (3.4). It is easy to see that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. For $t \in [0, T]$ and for $\varphi \in \mathbb{D}$ we have that

$$\begin{aligned}|(\mathcal{A}\varphi)(t)| &\le |\int_t^{t+T} G(t,s)[f(s, h(\varphi(s)), g(\varphi(s-\tau(s)))) - ca(s)\varphi(s-\tau(s))]ds| \\ &\le T\alpha \frac{L(1-\beta cT)}{\alpha T} = L(1-\beta cT).\end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have

$$\|\mathcal{A}\varphi\| \le L(1-\beta cT).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. We next show that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi \in \mathbb{D}$. By using (2.1), (2.2) and (2.3) we obtain by taking the derivative in (3.4) that

$$\begin{aligned}\frac{d}{dt}(\mathcal{A}\varphi)(t) = \int_t^{t+T} &[-b(t)G(t,s) + \frac{\exp\big(\int_t^s a(v)dv\big)}{\exp\big(\int_0^T a(v)dv\big) - 1}] \\ &\times [-ca(s)\varphi(s-\tau(s)) + f(s, h(\varphi(s)), g(\varphi(s-\tau(s))))]ds.\end{aligned}$$

Consequently, by invoking (2.10), and (3.3), we obtain

$$|\frac{d}{dt}(\mathcal{A}\varphi)(t)| \le T(\|b\|\alpha + \beta)\frac{L(1-\beta cT)}{\alpha T} \le M,$$

for some positive constant M. Hence $(\mathcal{A}\varphi)$ is equicontinuous. Then by the Ascoli-Arzela theorem we obtain that $\mathcal{A}$ is a compact map. Due to the continuity of all the terms in (3.4), we have that $\mathcal{A}$ is continuous. This completes the proof. $\square$

Theorem 3.3. *Let α, β and γ be given by* (2.10). *Suppose that conditions* (2.1)-(2.4), (2.9),(3.2),(3.3) *and* (3.6) *hold, then Equation* (1.1) *has a positive periodic solution z satisfying $K \le z \le L$.*

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (3.4) and (3.5) we obtain

$$\begin{aligned}&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\ &= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s, h(\psi(s)), g(\psi(s-\tau(s)))) \\ &\quad - ca(s)\psi(s-\tau(s))]ds \\ &\le c\beta LT + \alpha\int_t^{t+T}[f(s, h(\psi(s)), g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds \\ &\le c\beta LT + \alpha T\frac{L(1-\beta cT)}{\alpha T} = L.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) \\ &= \int_t^{t+T} cE(t,s)\varphi(s-\tau(s))ds + \int_t^{t+T} G(t,s)[f(s, h(\psi(s)), g(\psi(s-\tau(s)))) \\ &\quad - ca(s)\psi(s-\tau(s))]ds \\ &\ge c\sigma KT + \gamma\int_t^{t+T}[f(s, h(\psi(s)), g(\psi(s-\tau(s)))) - ca(s)\psi(s-\tau(s))]ds\end{aligned}$$

$$\geq c\sigma KT + \gamma T\frac{K(1-\sigma cT)}{\gamma T} = K.$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Thus all the hypotheses of Theorem 2.7 are satisfied and therefore equation (1.1) has a periodic solution in $\mathbb{D}$. This completes the proof. $\square$

We next consider the case when $c \leq 0$. To this end we substitute conditions (3.2) and (3.3) with the following conditions respectively.

$$c \leq 0 \tag{3.7}$$

and for all $s \in \mathbb{R}, \mu \in \mathbb{D}$

$$\frac{K - c\beta LT}{\gamma T} \leq f(s, h(\mu), g(\mu)) - ca(s)\mu \leq \frac{L - c\sigma KT}{\alpha T}. \tag{3.8}$$

Theorem 3.4. *Let α, β and γ be given by* (2.10). *Suppose that conditions* (2.1)-(2.4)*,* (2.9),(3.6)*,* (3.7)*, and* (3.8) *hold, then* (1.1) *has a positive periodic solution z satisfying $K \leq z \leq L$.*

The proof follows along the lines of Theorem 3.3, and hence we omit it.

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