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Positive Solutions for a System of Periodic Neutral Delay Difference Equations

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Abstract

In this article we consider the existence of positive solutions of a system of periodic neutral difference equations. The main tool employed is the Krasnosel'skii's fixed point theorem for the sum of a completely continuous operator and a contraction.

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1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_-$ the negative integers, $\mathbb{Z}^+$ the non-negative integers, and $T \geq 1$ is an integer. In this paper we consider the system of neutral difference equations

$$\begin{aligned} x(n+1) \quad &= \quad A(n)x(n) + C(n)\Delta x(n-\tau(n)) + g(n, x(n-\tau(n))), \qquad (1.1)\\ &\quad\quad x(n) = x(n+T), \end{aligned}$$

where $A(n) = \text{diag}[a_1(n), a_2(n), ..., a_k(n)]$, a_j is T-periodic, $C(n) = \text{diag}[c_1(n), c_2(n), ..., c_k(n)]$, c_j is T-periodic, $\tau(n)$ is T-periodic, $g : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x and $g(n, x)$ is T-periodic in n and x, whenever x is T-periodic. Let P_T be the set of all real T-periodic sequences $\phi : \mathbb{Z} \to \mathbb{R}^k$. Endowed with the maximum norm $||\phi|| = \max_{\theta\in\mathbb{Z}} \sum_{j=1}^{k} |\phi_j(\theta)|$ where $\phi = (\phi_1, \phi_2, ..., \phi_k)^t$, P_T is a Banach space. Here t stands for the transpose.

The study of positive periodic solutions of differential and difference equations has gained the attention of many researchers in recent times: see [1]-[3],[6],[7] and references therein.

We are motivated by the work of Raffoul and the present author in [7] where the scalar difference equation

$$x(n+1) = a(n)x(n) + c\Delta x(n-\tau) + g(n, x(n-\tau)), \qquad (1.2)$$

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with a constant delay τ was considered.
In this research we generalize (1.2) to systems with functional delay.
Let $\mathbb{R}_+ = [0, +\infty)$, for each $x = (x_1, x_2, ..., x_k)^t \in \mathbb{R}^k$, the norm of x is defined as $|x| = \sum_{j=1}^{k} |x_j|$. $\mathbb{R}_+^k = \{(x_1, x_2, ..., x_k)^t \in \mathbb{R}^k : \; x_j \geq 0, \; j = 1, 2, ..., k\}$. Also, we denote $g = (g_1, g_2, ..., g_k)^t$, where t stands for transpose. We say that x is "positive" whenever $x \in \mathbb{R}_+^k$. In this paper we use Krasnosel'skii's fixed point theorem for the sum of a completely continuous operator and a contraction to obtain sufficient conditions for the existence of positive periodic solutions for (1.1).
In this paper we make the following assumptions.

(H1) There exist a constant $\sigma_j > 0$ such that $\sigma_j < c_j(n), \; j = 1, ..., k$, for all $n \in [0, T-1]$.

(H2) $0 < a_j(n) < 1$ for all $n \in [0, T-1]$, $j = 1, ..., k$.

(H3) There exist constants α_j, such that $||c_j|| \leq \alpha_j \leq 1, \; j = 1, 2, ..., k$.

The rest of the paper is organized as follows. In section 2, we introduce our notation in this paper and state without proof Krasnosel'skii's theorem. In section 3, we state and prove our main results.

2 Preliminaries

We begin this section by introducing some notations. Let

$$G_j(n, u) = \frac{\prod_{s=u+1}^{n+T-1} a_j(s)}{1 - \prod_{s=n}^{n+T-1} a_j(s)}, \; u \in [n, n+T-1]. \tag{2.1}$$

Note that the denominator in $G_j(n, u)$ is not zero since $0 < a_j(n) < 1$ for $n \in [0, T-1]$.

Define

$$G(n, u) = \operatorname{diag}[G_1(n, u), G_2(n, u), ..., G_k(n, u)]. \tag{2.2}$$

It is clear that $G(n, u) = G(n+T, u+T)$ for all $(n, u) \in \mathbb{Z}^2$. Also, let

$$q_j := \min\{G_j(n, u) : n \geq 0, \; u \leq T\} = G_j(n, n) > 0, \; j = 1, ..., k. \tag{2.3}$$

$$\begin{aligned} Q_j := \max\{G_j(n, u) : n \geq 0, \; u \leq T\} &= G_j(n, n+T-1) \\ &= G_j(0, T-1) > 0, j = 1, ..., k. \end{aligned} \tag{2.4}$$

Set $q = \min_{1 \leq j \leq k} q_j$ and $Q = \max_{1 \leq j \leq k} Q_j$. We next state below Krasnosel'skii's theorem and refer to [5] for the proof.

Theorem 2.1. *(Krasnosel'skii) Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that* *(i) A is completely continuous,*

(ii) B is a contraction mapping.

(iii) $x, y \in \mathbb{M}$*, implies* $Ax + By \in \mathbb{M}$.

Then there exists $z \in \mathbb{M}$ *with* $z = Az + Bz$.

For the next lemma we consider

$$x_j(n+1) = a_j(n)x_j(n) + c_j(n)\Delta x_j(n-\tau(n)) + g_j(n, x_j(n-\tau(n))),\ j = 1, ..., k. \tag{2.5}$$

Lemma 2.2. *Suppose (H2) holds. Then* $x_j(n) \in P_T$ *is a solution of* (2.5) *if and only if*

$$\begin{aligned} x_j(n) &= c_j(n-1)x_j(n-\tau(n)) + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u, x_j(u-\tau(u))) \\ &\quad -x_j(u-\tau(u))\phi_j(u)a_j(u)\Big]. \end{aligned} \tag{2.6}$$

where $\phi_j(u) = c_j(u) - c_j(u-1)$.

Proof. Rewrite (2.5) as

$$\Delta\Big[x_j(n)\prod_{s=0}^{n-1} a_j^{-1}(s)\Big] = \Big[c_j(n)\Delta x_j(n-\tau(n)) + g_j(n, x_j(n-\tau(n)))\Big]\prod_{s=0}^{n} a_j^{-1}(s). \tag{2.7}$$

Summing equation (2.7) from n to $n+T-1$ we obtain

$$\begin{aligned} \sum_{u=n}^{n+T-1}\Delta\Big[x_j(u)\prod_{s=0}^{u-1} a_j^{-1}(s)\Big] &= \sum_{u=n}^{n+T-1}\Big[c_j(u)\Delta x_j(u-\tau(u)) \\ &\quad + g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned}$$

Thus,

$$\begin{aligned} x(n+T)\prod_{s=0}^{n+T-1} a_j^{-1}(s) - x(n)\prod_{s=0}^{n-1} a_j^{-1}(s) & \\ &= \sum_{u=n}^{n+T-1}\Big[c_j(u)\Delta x_j(u-\tau(u)) \\ &\quad + g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned}$$

Since $x(n+T)=x(n)$, we obtain

$$x(n)\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big] = \sum_{u=n}^{n+T-1} \Big[c_j(u)\Delta x_j(u-\tau(u)) + g_j(u, x_j(u-\tau(u)))\Big] \prod_{s=0}^{u} a_j^{-1}(s). \tag{2.8}$$

But

$$\begin{aligned}\sum_{u=n}^{n+T-1} c_j(u)\Delta x_j(u-\tau(u)) \prod_{s=0}^{u} a_j^{-1}(s) &= c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big] \\ &\quad - \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Delta\Big[c_j(u-1)\prod_{s=0}^{u-1} a_j^{-1}(s)\Big] \\ &= c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big] - \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Big[c_j(u) \\ &\quad - c_j(u-1)a_j(u)\Big]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8) gives

$$x(n)\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big] = c_j(n-1)x_j(n-\tau(u))\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$$

$$- \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Big[c_j(u) - c_j(u-1)a_j(u)\Big]\prod_{s=0}^{u} a_j^{-1}(s)$$
$$+ \quad g_j(u, x_j(u-\tau(u)))\Big]\prod_{s=0}^{u} a_j^{-1}(s). \tag{2.10}$$

Dividing through by $\Big[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\Big]$ gives the desired result. □

3 Main Results

In this section we obtain sufficient conditions for the existence of positive periodic solutions for (1.1). For some nonnegative constant L and a positive constant J we define the set

$$\mathbb{M} = \Big\{\phi \in P_T \ : \ L \leq ||\phi|| \leq J, \text{ with } \frac{L}{k} \leq \phi_j \leq \frac{J}{k}, \ j = 1, 2, ..., k.\Big\}, \tag{3.1}$$

which is a closed convex and bounded subset of the Banach space P_T. We also assume that for all $u \in \mathbb{Z}$ and $\rho \in \mathbb{M}$,

$$\frac{(1-\sigma_j)L}{Tq_jk} \leq g_j(u, \rho_j, \rho_j) - \rho_j\phi_j(u)a_j(u) \leq \frac{(1-\alpha_j)J}{TQ_jk}. \tag{3.2}$$

Define a mapping $H : \mathbb{M} \to P_T$ by

$$(Hx)(n) \ = \ C(n-1)x(n-\tau(n)) + \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u, x(u), x(u-\tau(u))) - \Phi(u)A(u)x(u-\tau(u))\Big]$$

where $\Phi(u) = \text{diag}[\phi_1(u), ..., \phi_k(u)]$.
We denote

$$(Hx) = (H_1x_1, H_2x_2, ..., H_kx_k)^t. \tag{3.3}$$

It is clear that $(Hx)(n+T) = (Hx)(n)$. In order to apply Theorem 2.1 we will construct two mappings of which one is a contraction and the other is compact. Thus we define the map $D : \mathbb{M} \to P_T$ by

$$(D\varphi)(n) = C(n-1)\varphi(n-\tau(n)). \tag{3.4}$$

We also define the map $F : \mathbb{M} \to P_T$ by

$$(F\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u, \varphi(u), \varphi(u-\tau(u))) - \Phi(u)A(u)\varphi(u-\tau(u))\Big]. \tag{3.5}$$

Lemma 3.1. *Suppose (H3) hold. Then the operator D defined by* (3.4) *is a contraction.*

Proof. Let $\varphi, \psi \in \mathbb{M}$ and $\alpha = \max_{1\le j\le k} \alpha_j$. Then

$$||(D\varphi) - (D\psi)|| = \max_{n\in[0,T-1]} \sum_{j=1}^{k} |(D_j\varphi_j)(n) - (D_j\psi_j)(n)|$$

But,

$$\begin{aligned} |(D_j\varphi_j)(n) - (D_j\psi_j)(n)| &= |c_j(n-1)\varphi_j(n) - c_j(n-1)\psi_j(n)| \\ &\le \alpha_j||\varphi_j - \psi_j||. \end{aligned}$$

Thus,

$$\begin{aligned} ||(D\varphi) - (D\psi)|| &\le \sum_{j=1}^{k} \alpha_j||\varphi_j - \psi_j|| \\ &\le \alpha||\varphi - \psi||. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Suppose that (H1), (H2), (H3) and* (3.2) *hold. Then the operator F defined by* (3.5) *is completely continuous on $\mathbb{M}$.*

Proof. For $n \in [0, T-1]$ and for $\varphi \in \mathbb{M}$, we have by (3.2) that

$$\begin{aligned} |(F_j\varphi_j)(n)| &\le \Big| \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\varphi_j(u-\tau(u))) - \varphi_j(u-g(u))\phi_j(u)a_j(u)\Big]\Big| \\ &\le Q_jT\frac{(1-\alpha_j)J}{TQ_jk} \\ &\le \frac{(1-\alpha_j)J}{k}. \end{aligned}$$

Thus,

$$\begin{aligned} ||(F\varphi)|| &\le \sum_{j=1}^{k} \frac{(1-\alpha_j)J}{k} \\ &\le (1-\alpha^*)J, \end{aligned}$$

where $\alpha^* = \min_{1\le j\le k} \alpha_j$. It therefore follows that

$$||(F\varphi)|| \le J.$$

This shows that $F(\mathbb{M})$ is uniformly bounded. Due to the continuity of all terms, we have that F is continuous.
Next we show that F maps bounded subsets into compact sets. Let $S = \{ \varphi \in P_T : ||\varphi|| \leq \mu\}$ and $Q = \{ (F\varphi)(n) : \varphi \in S\}$, then S is a subset of $\mathbb{R}^{Tk}$ which is closed and bounded and thus compact. As F is continuous in φ, it maps compact sets into compact sets. Therefore $Q = F(S)$ is compact. This completes the proof. □

Theorem 3.3. *Suppose that (H1),(H2), (H3) and (3.2) hold. Also suppose that the hypothesis of Lemma 3.2 also hold. Then equation* (1.1) *has a positive periodic solution.*

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then we have that

$$\begin{aligned}
(D_j\varphi_j)(n) + (F_j\psi_j)(n) &= c_j(n-1)\varphi_j(n-\tau(n)) \\
&\quad + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\leq \frac{\alpha_j J}{k} + Q_j \sum_{u=n}^{n+T-1} \Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\leq \frac{\alpha_j J}{k} + \frac{Q_j T(1-\alpha_j)J}{TQ_j k} = \frac{J}{k}.
\end{aligned}$$

Thus,

$$||(D\varphi)(n) + (F\psi)(n)|| \leq \sum_{j=1}^{k} \frac{J}{k} = J.$$

On the other hand,

$$\begin{aligned}
(D_j\varphi_j)(n) + (F_j\psi_j)(n) &= c_j(n-1)\varphi_j(n-\tau(n)) \\
&\quad + \sum_{u=n}^{n+T-1} G_j(n,u)\Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\geq \frac{\sigma_j L}{k} + q_j \sum_{u=n}^{n+T-1} \Big[g_j(u,\psi_j(u),\psi_j(u-\tau(u))) \\
&\quad -\psi_j(u-\tau(u))\phi_j(u)a_j(u)\Big] \\
&\geq \frac{\sigma_j L}{k} + \frac{q_j T(1-\sigma_j)L}{Tq_j k} = \frac{L}{k}.
\end{aligned}$$

Thus,

$$||(D\varphi)(n) + (F\psi)(n)|| \geq \sum_{j=1}^{k} \frac{L}{k} = L.$$

This shows that $(D\varphi)(n) + (F\psi)(n) \in \mathbb{M}$. Therefore by Theorem 2.1 equation (1.1) has a positive periodic solution in $\mathbb{M}$. □

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